TSUKUBA J. MATH.
Vol. 16 No. 2 (1992), 279—293

STRONGLY INNER ACTIONS, COACTIONS, AND DUALITY THEOREMS

By

Margaret BEATTIE*

0. Introduction.

Let H be a Hopf algebra with bijective antipode S over a commutative ring k, and M a right H-module. Then End(M) is a right H-module algebra over k with H-action $(\varphi\leftarrow h)(m)=\sum_{(h)}\varphi(m\leftarrow\bar{S}(h_{(2)}))\leftarrow(h_{(1)})$. ($\bar{S}$ is the composition inverse of S.) This H-action on End(M) is strongly inner so that $H\#\mathrm{End}(M)\cong H\otimes\mathrm{End}(M)$. (Here the smash product is the right smash product of [6] or [12].) Similarly, if M is a finitely generated projective k-module and a left H-comodule, then End(M) is a left H-comodule algebra, the left H-coaction is strongly inner, and $\mathrm{End}(M)\#H\cong\mathrm{End}(M)\otimes H$.

In this paper, we exploit the above observation to examine some well-known duality results, along with some new examples, from the point of view that the duality involves an endomorphism ring and a strongly inner action or coaction.

1. Bialgebra actions, coactions and smash products.

Throughout, we work over a commutative ring k. Unless otherwise stated, all maps are k-linear, $\otimes$ means $\otimes_k$, Hom means Hom_k, algebra means a k-algebra with 1, etc. The word ring will mean a k-algebra, not necessarily with 1. For A an algebra, $\mathcal{U}(A)$ will denote the group of multiplicative units of A.

H will denote a bialgebra over k with comultiplication Δ, and counit ε. If H is a Hopf algebra, S will denote the antipode; if the antipode S is bijective, $\bar{S}$ will denote the composition inverse of S. We use Sweedler's sigma notation, i.e. we write $\Delta(h)=\sum h_{(1)}\otimes h_{(2)}$, $\Delta\otimes 1\cdot\Delta(h)=1\otimes\Delta\cdot\Delta(h)=\sum h_{(1)}\otimes h_{(2)}\otimes h_{(3)}$, and so on. (We will usually omit the summation index (h).) Our rings may have left and/or right H-actions; we will denote these by arrows on the left or

---

* Research supported by NSERC Grant OGP0009137.

Received September 17, 1991, Revised November 19, 1991.

right.

DEFINITION 1.1. (i) We call a ring A a left (right) H-module ring if A is a left (right) H-module such that $h\rightarrow(ab)=\sum(h_{(1)}\rightarrow a)(h_{(2)}\rightarrow b)$ $((ab)\leftarrow h=\sum(a\leftarrow h_{(1)})(b\leftarrow h_{(2)}))$. If A is a k-algebra, we also require that h acting on 1 be $\varepsilon(h)$ for all $h\in H$, and then A is called an H-module algebra, (i.e., A is an H-module and H measures A to A [23, p. 138].) We call an $H-H$-bimodule A an $H-H$-bimodule ring (algebra) if A is a left and right H-module ring (algebra).

(ii) We call a ring A a left (right) H-comodule ring if A is a left (right) H-comodule such that the H-comodule structure map preserves multiplication. Again we use Sweedler's sigma notation and write $a\rightarrow\sum a_{(-1)}\otimes a_{(0)}\in H\otimes A$ for a left H-comodule map and $a\rightarrow\sum a_{(0)}\otimes a_{(1)}\in A\otimes H$ for a right H-comodule structure map on A. If A is a k-algebra then the comodule structure map must map 1 to $1\otimes 1$, and A is called an H-comodule algebra. We call an $H-H$-bicomodule A an $H-H$-bicomodule ring (algebra) if A is a left and right H-comodule ring (algebra).

EXAMPLE 1.2. (i) The dual $H^*=\mathrm{Hom}(H, k)$ of H is an $H-H$-bimodule algebra. The left action of H on H^* is given by $(h\rightarrow f^*)(g)=f^*(gh)$ and the right action by $(f^*\leftarrow h)(g)=f^*(hg)$. If H is commutative, then clearly these left and right actions coincide.

(ii) H is itself an $H-H$-bicomodule algebra with comultiplication giving both H-comodule structure maps.

REMARK 1.3. (i) If A is a left H-module ring (right H-module ring, $H-H$-bimodule ring) without 1, then A can be embedded in a left H-module (right H-module, $H-H$-bimodule) algebra, denoted A^1, and A is a left (right, etc) H-module subring and ideal of A^1. This is done as follows.

Let $A^1=A\times k$. As usual, addition in A^1 is componentwise addition, and multiplication is defined by $(a, \alpha)(b, \beta)=(ab+\alpha b+\beta a, \alpha\beta)$, $a, b\in A$, $\alpha, \beta\in k$; then $(0, 1)$ is the multiplicative identity for A^1. The left H-action on A may be extended to a left H-module action on A^1 by $h\rightarrow(a, \alpha)=(h\rightarrow a, \varepsilon(h)\alpha)$. Then $h\rightarrow(0, 1)=(0, \varepsilon(h))=\varepsilon(h)(0, 1)$, and, it is easily checked that

$$\begin{aligned}\sum(h_{(1)}\rightarrow(a, \alpha))(h_{(2)}\rightarrow(b, \beta))&=(h\rightarrow(ab)+\beta(h\rightarrow a)+\alpha(h\rightarrow b), \varepsilon(h)\alpha\beta)\\&=h\rightarrow((a, \alpha)(b, \beta)),\end{aligned}$$

so A^1 is a left H-module algebra.

(ii) For A, A^1 as above, if A is a right (left, etc.) H-comodule ring without 1, then the H-coaction may be extended to A^1 so that A is a H-comodule subring. Define a right H-coaction on A^1 by $\gamma(a, \alpha)=\sum(a_{(0)}, 0)\otimes a_{(1)}+(0, \alpha)\otimes 1$. Clearly $(\gamma\otimes 1)\circ\gamma=(1\otimes\Delta)\circ\gamma$, $\gamma(0, 1)=(0, 1)\otimes 1$, and, for $a, b\in A$, $\alpha, \beta\in k$,

$$\begin{aligned}\gamma(a, \alpha)\gamma(b, \beta)&=\sum(a_{(0)}b_{(0)}, 0)\otimes a_{(1)}b_{(1)}+\sum(\beta a_{(0)}, 0)\otimes a_{(1)}\\&\quad+\sum(\alpha b_{(0)}, 0)\otimes b_{(1)}+(0, \alpha\beta)\otimes 1\\&=\gamma(ab+\alpha b+\beta a, \alpha\beta)=\gamma((a, \alpha)(b, \beta)),\end{aligned}$$

so that γ preserves multiplication, as required. □

Recall the definitions of left and right smash products.

DEFINITION 1.4. (i). [6], [24, p. 471] Let A be a left H-module ring and B a left H-comodule ring. The left smash product $A\#_H^{\mathcal{L}}B$ is defined to be the k-module $A\otimes B$ with multiplication defined by $(a\#_H^{\mathcal{L}}b)(c\#_H^{\mathcal{L}}d)=\sum a(b_{(-1)}\rightharpoonup c)\#_H^{\mathcal{L}}b_{(0)}d$. If $B=H$, then $A\#_H^{\mathcal{L}}H$ is the usual smash product $A\#H$ [23, p. 155].

(ii) [6], [12, 1.3] Let B be a right H-module ring and A a right H-comodule ring. The right smash product $A\#_H^{\mathcal{R}}B$ is defined to be the k-module $A\otimes B$ with multiplication defined by $(a\#_H^{\mathcal{R}}b)(c\#_H^{\mathcal{R}}d)=\sum ac_{(0)}\#_H^{\mathcal{R}}(b\leftharpoonup c_{(1)})d$.

REMARK 1.5. Note that the smash products in Definition 1.4 make sense for H a coalgebra with an associative multiplication and a coassociative multiplication-preserving comultiplication. For example, H could be the semigroup ring kS where S is a semigroup without identity; see Examples 1.6(ii), 2.10 and 3.8.

For A a right H-comodule algebra, a smash product structure denoted $\#(H, A)$ on the homomorphism ring $\mathrm{Hom}(H, A)$, is defined in [13, p. 1166] and a modified form is used to prove the duality results in [14].

If the meaning is clear from the context, we write $a\#b$ for $a\#_H^{\mathcal{L}}b$ or $a\#_H^{\mathcal{R}}b$. Next we give some examples of right smash products.

EXAMPLE 1.6. (i) Let G be a group and $H=kG$, the group ring. Let A be a right kG-comodule ring (i.e. A is G-graded) and let P be the subring of $\mathrm{Hom}(kG, k)$ generated by the projection maps p_g, $g\in G$, i.e. $p_g(h)=\delta_{g,h}$. If G is finite, then $\sum_{g\in G}p_g$ is a multiplicative identity for P, and P is a subalgebra of H^*; otherwise P is a subring of H^* without a 1. The group G acts as automorphisms of P on the left and right by $g\rightharpoonup p_h=p_{hg^{-1}}$ and $p_h\leftharpoonup g=p_{g^{-1}h}$

respectively, making P an $H-H$-bimodule subring of H^* (with the $H-H$-bimodule structure of Example 1.2(i)). For A a k-algebra, the right smash product $A\#^{\mathcal{R}}_{kG}P$ is the "generalized smash product" $A\#G^*$ of [3]; the right smash product $A\#^{\mathcal{R}}_{kG}P^1$ is the smash product $\tilde{A}\#G$ of [22].

(ii) The construction of the right smash product in (i) can be extended to rings graded by a semigroup S. First let's take S to be cancellative and let $P\subseteqq \mathrm{Hom}(kS, k)$ be the subring generated by the projections as before. Then for $s, t\in S$, $p_s\leftarrow t$ is 0 if there is no $w\in S$ with $tw=s$ or the projection p_w if such a (unique) w exists. Thus P is a right kS-module ring and if A is S-graded, we may form $A\#^{\mathcal{R}}_{kS}P$.

If S is not cancellative, then $p_s\leftarrow t$ may not be in P. Let $\tilde{P}$ be the smallest right kS-module subring of $\mathrm{Hom}(kS, k)$ containing P. Then for A graded by S, $A\#^{\mathcal{R}}_{kS}\tilde{P}$ is defined; the elements of $A\#^{\mathcal{R}}_{kS}\tilde{P}$ of the form $\sum a_i\# p_{s_i}$ form a subring of $A\#^{\mathcal{R}}_{kS}\tilde{P}$.

Abrams' smash product for a ring graded by a category [1, Definition 2.1] is a subring of the subring above.

(iii) Suppose T is a right G-set, and A is a G-graded ring. Let P_T be the ring generated by orthogonal idempotents p_t, $t\in T$. Then P_T is a right kG-module ring via $p_t\leftarrow g=p_{t\cdot g}$, and, for A a G-graded ring, we may form the smash product $A\#^{\mathcal{R}}_{kG}P_T$. If A has a 1, and T is finite, this is [20, 2.11]; if T is infinite, then $A\#^{\mathcal{R}}_{kG}P^1_T$ is the smash product $A\#T$ of [21]. Later on, we will use this construction for H a subgroup of G and G/H the set of left cosets of H in G. Then if $(g_i : i\in I)$ is a set of coset representatives, $P_{G/H}$ is generated by the idempotents $p_{[g_i]}$ and $p_{[g_i]}\leftarrow h=p_{[h^{-1}g_i]}$.

(iv) Let H be a Hopf algebra and U a Hopf subalgebra of H^0, the finite dual of H. Suppose A is a left H-module algebra which is U-locally finite, so that A is a right U-comodule algebra and $h\rightarrow a=\sum a_{(0)}a_{(1)}(h)$ for all $h\in H$ [9, p. 157]. Let B be a left H-comodule algebra; thus B is a right U-module algebra by $b\leftarrow m=\sum m(b_{(-1)})b_{(0)}$. Then by [6, Lemma 1.9], $A\#^{\mathcal{L}}_{H}B=A\#^{\mathcal{R}}_{U}B$. □

The proofs of the next two lemmas are straightforward, and therefore omitted.

LEMMA 1.7. (cf. [6, Lemmas 1.5, 1.7]) *Suppose A is an $H-H$-bimodule ring and B is a left (right) H-comodule ring. Then $A\#^{\mathcal{L}}_{H}B$ ($B\#^{\mathcal{R}}_{H}A$) is a right (left) H-module ring with H-action induced by the H-action on A, i.e. $(a\#^{\mathcal{L}}_{H}b)\leftarrow h=a\leftarrow h\#^{\mathcal{L}}_{H}b$ ($h\rightarrow(b\#^{\mathcal{R}}_{H}a)=b\#^{\mathcal{R}}_{H}h\rightarrow a$).*

Similarly, if A is a left (right) H-module ring, B an $H-H$-bicomodule ring, then $A\#^{\mathcal{L}}_H B$ ($B\#^{\mathcal{R}}_H A$) is a right (left) H-comodule ring with the H-comodule structure induced by that on B. □

LEMMA 1.8. (cf. [6, Lemma 1.8]). *Suppose A is a right H-comodule ring, B is an $H-H$-bimodule ring, and C is a left H-comodule ring. Then the map taking $(a\#b)\#c$ to $a\#(b\#c)$ is a natural isomorphism from $(A\#^{\mathcal{R}}_H B)\#^{\mathcal{L}}_H C$ to $A\#^{\mathcal{R}}_H(B\#^{\mathcal{L}}_H C)$ where the smash products $(A\#^{\mathcal{R}}_H B)$ and $(B\#^{\mathcal{L}}_H C)$ have the left and right H-module structure described in Lemma 1.7.*

Similarly, if A is a left H-module ring, B a $H-H$-bicomodule ring, and C a right H-module ring, then $(A\#^{\mathcal{L}}_H B)\#^{\mathcal{R}}_H C$ is naturally isomorphic to $A\#^{\mathcal{L}}_H(B\#^{\mathcal{R}}_H C)$.
□.

2. Inner actions and duality for right smash products.

Let A be a right H-comodule ring and L an $H-H$-bimodule ring; we want to consider $(A\#^{\mathcal{R}}_H L)\#^{\mathcal{L}}_H H=A\#^{\mathcal{R}}_H(L\#^{\mathcal{L}}_H H)$ and discuss some situations where $A\#^{\mathcal{R}}_H(L\#^{\mathcal{L}}_H H)$ is ring isomorphic to $A\otimes(L\#^{\mathcal{L}}_H H)$.

DEFINITION 2.1. [11, p. 52] Let R be an H-module ring which is a subring of an algebra T. The H-action on R is called T-inner (or just inner if $T=R$) if there exists a convolution invertible u in $\mathrm{Hom}(H, T)$ with convolution inverse v such that for all $r\in R$, $h\in H$, $h\rightarrow r$ or $r\leftarrow h$ is $\sum u(h_{(1)})rv(h_{(2)})$. If the action is a left action and $v\in \mathrm{Alg}(H, T)$, we call the action left strongly T-inner; if the action is a right action and $v\in \mathrm{Alg}(H, T)$, the action is called right strongly T-inner.

EXAMPLES 2.2. (i) Let H be a Hopf algebra, and A an H-module k-Azumaya algebra. If k is a field, then the H-action on A is inner [18, 3.1]. If k is a semilocal ring or a von Neumann regular ring, and H is finitely generated projective over k, then the H-action on A is inner [7, Corollary 2.5].

(ii) If M is a right H-module, and $\mathrm{End}(M)$ has the H-action induced by M mentioned in the introduction, then it is easily checked that the H-action on $\mathrm{End}(M)$ is strongly inner.

(iiii) Note that various examples of inner and strongly inner actions are given in [8, Section 1].

The next theorem is a slight generalization of [8, Proposition 1.19] or [6, Lemma 1.11] which provides some insight into the role of strongly inner actions in some duality theorems for right smash products. Our applications will involve right strongly inner actions, but of course parallel results hold for left actions.

THEOREM 2.3. *Let A be a right H-comodule ring and B a right H-module ring which is a subring of an algebra C such that the action of H on B is right strongly C-inner, implemented by $u\in\mathrm{Hom}(H, C)$ with convolution inverse v. Then $A\#^{\mathcal{R}}_{H}B$ is isomorphic to a subring of $A\otimes C$, and if $u(h)b$ and $v(h)b$ are in B for all $h\in H$, $b\in B$, (for example if B is a left ideal of C), then $A\#^{\mathcal{R}}_{H}B\cong A\otimes B$. If B has a 1, then this last condition says that $u, v\in\mathrm{Hom}(H, B)$.*

PROOF. Map $A\#^{\mathcal{R}}_{H}B$ to $A\otimes C$ by $\phi(a\#b)=\sum a_{(0)}\otimes v(a_{(1)})b$. Then $A\#^{\mathcal{R}}_{H}B$ is isomorphic to $\phi(A\#^{\mathcal{R}}_{H}B)\subseteq A\otimes C$; the inverse map from $\phi(A\#^{\mathcal{R}}_{H}B)$ to $A\#^{\mathcal{R}}_{H}B$ is given by $a\otimes b\rightarrow\sum a_{(0)}\#u(a_{(1)})b$. Clearly if A and B are algebras, $1\#1$ maps to $1\otimes 1$. $\square$

Before stating a useful corollary to Theorem 2.3, we give an immediate application of the theorem.

EXAMPLE 2.4. Let H be a commutative Hopf algebra, let L be an H-module ring and define an $H-H$-bimodule ring structure on L by $h\rightarrow m=m\leftarrow h$ for all $h\in H$, $m\in L$. Let A be a right H-comodule ring and consider $(A\#^{\mathcal{R}}_{H}L)\#^{\mathcal{L}}_{H}H\cong A\#^{\mathcal{R}}_{H}(L\#^{\mathcal{L}}_{H}H)$.

If L does not have a 1, identify L with the $H-H$-bimodule subring $L\times\{0\}$ in L^1, and $L\#^{\mathcal{L}}_{H}H$ with the subring $(L\times\{0\})\#^{\mathcal{L}}_{H}H\cong L\#^{\mathcal{L}}_{H}H$ of $L^1\#^{\mathcal{L}}_{H}H$.

Now the right H-action on $L\#^{\mathcal{L}}_{H}H$ is right strongly $L^1\#^{\mathcal{L}}_{H}H$-inner (cf. [8, Example 1.9]). For, let $u: H\rightarrow L^1\#^{\mathcal{L}}_{H}H$ be defined by $u(h)=1\#h$, where we write 1 for the multiplicative identity (0, 1) in L^1. Then v, the convolution inverse of u, is given by $v(h)=1\#S(h)$. Since H is commutative, u and v are both algebra maps, and, for $h, g\in H$, $m\in L$,

$$\sum(1\#h_{(1)})(m\#g)(1\#S(h_{(2)}))=\sum h_{(1)}\rightarrow m\#h_{(2)}gS(h_{(3)})=m\leftarrow h\#g\,.$$

Thus, by Theorem 2.3, if A is a right H-comodule algebra, $(A\#^{\mathcal{R}}_{H}L)\#^{\mathcal{L}}_{H}H\cong A\otimes(L\#^{\mathcal{L}}_{H}H)$, and $(A\#^{\mathcal{R}}_{H}L^1)\#^{\mathcal{L}}_{H}H\cong A\otimes(L^1\#^{\mathcal{L}}_{H}H)$.

For example, L could be any $H-H$-bimodule subring of H^* with the H-action described in Example 1.2 (i). In particular, if $H=kG$, G an abelian group, L might be P or P^1 of Example 1.6(i). Or, suppose U is a cocommutative pointed Hopf algebra over a field k such that $G=G(U)$, the group of grouplikes, is abelian, and suppose A is a G-graded k-algebra. If $L=U^1$ is the irreducible component of U containing 1 [23, 8.1], then $(A\#_{kG}^{\mathcal{R}}L)\#_{kG}^{\mathcal{L}}kG\cong A\#_{kG}^{\mathcal{R}}(L\#_{kG}^{\mathcal{L}}kG)\cong A\otimes_{kG}^{\mathcal{R}}U$ where the G-action on U is given by $g\rightharpoonup h=h\leftharpoonup g=ghg^{-1}$. □

Corollary 2.5. *Let H be a Hopf algebra with bijective antipode S, let A be a right H-comodule ring and L an $H-H$-bimodule ring. Let λ be the natural ring homomorphism from $L\#_H^{\mathcal{L}}H$ to* End(L) *defined by $\lambda(m\#h)(t)=m(h\rightharpoonup t)$, and suppose λ is an embedding. Let ρ be the map from H to* End(L) *induced by the right H-action on L, i.e. $\rho(h)(t)=t\leftharpoonup h$, and suppose that for all $h\in H$, $\rho(h)\lambda(L\#_H^{\mathcal{L}}H)\subseteq\lambda(L\#_H^{\mathcal{L}}H)$. Then $A\#_H^{\mathcal{R}}(L\#_H^{\mathcal{L}}H)\cong A\otimes(L\#_H^{\mathcal{L}}H)$.*

Proof. Identify $L\#_H^{\mathcal{L}}H$ with its image under λ and give $\lambda(L\#_H^{\mathcal{L}}H)$ a right H-module structure via λ. Note that $\rho(hg)=\rho(g)\rho(h)$, so that $\rho\circ\bar{S}$ is an algebra map. Now the right H-action on $\lambda(L\#_H^{\mathcal{L}}H)$ is right strongly End(L)-inner since $\lambda(m\#h)\leftharpoonup g=\lambda(m\leftharpoonup g\#h)=\sum\rho(g_{(1)})\lambda(m\#h)\rho(\bar{S}(g_{(2)}))$ ([6] or [8, Proposition 5.13]). The statement then follows from Theorem 2.3. □

Example 2.6. Note that the duality theorem of Blattner and Montgomery [9, Theorem 2.1] or [19, Theorem 5], and its generalization to Hopf algebras over Dedekind rings [10, Theorem 5] can be viewed as cases of Corollary 2.5. For here λ is an embedding [9, Proposition 2.2] or [19, Lemma 5], [10, Proposition 6], and the RL-condition guarantees the rest. (See also [6] for details of this point of view.) □

Example 2.7. Let G be a group, let H be the group ring kG, and let $P\subseteq H^*$ be the ring of projections described in Example 1.6(i). For A a right kG-comodule algebra, it is shown in [4] using the Morita theory of [2], or by direct computation in [5] for A a right kG-comodule ring, that $(A\#_{kG}^{\mathcal{R}}P)\#_{kG}^{\mathcal{L}}kG\cong M_G^{\mathrm{fin}}(A)$, the ring of matrices over A with rows and columns indexed by G and with finitely many nonzero entries. Actually this duality result follows easily from Corollary 2.5.

Identify End(P) with $M_G^{\mathrm{col}}(k)$, the ring of column finite matrices over k with rows and columns indexed by G; elements of P are viewed as column matrices. Then $\lambda(p_g\#h)=E_{g,gh}$, the matrix with 1 in the (g, gh)-th position and zeroes elsewhere, and it is easy to verify that λ is a ring isomorphism from $P\#_{kG}^{\mathcal{L}}kG$ onto $M_G^{\mathrm{fin}}(k)$. Since $\lambda(P\#_{kG}^{\mathcal{L}}kG)=M_G^{\mathrm{fin}}(k)$ is a left ideal of $M_G^{\mathrm{col}}(k)$, Corollary 2.5 applies, and $A\#_H^{\mathcal{R}}(P\#_H^{\mathcal{L}}H)\cong A\otimes(P\#_H^{\mathcal{L}}H)\cong A\otimes M_G^{\mathrm{fin}}(k)\cong M_G^{\mathrm{fin}}(A)$.

Note that the isomorphism in [5] from $(A\#_{kG}^{\mathcal{R}}P)\#_{kG}^{\mathcal{L}}kG$ to $M_G^{\mathrm{fin}}(A)$, namely $(a\#p_g)\#h\to\sum_{t\in G}a_{tg^{-1}}E_{t,gh}$ is precisely the map taking $a\#(p_g\#h)$ to $\sum a_{(0)}\otimes\rho(S(a_{(1)}))\lambda(p_g\#h)$. $\square$

EXAMPLE 2.8. For P and G as in Example 2.7, let N be a normal subgroup of G, and let A be an N-graded ring. Consider $(A\#_{kG}^{\mathcal{R}}P)\#_{kG}^{\mathcal{L}}kN=(A\#_{kN}^{\mathcal{R}}P)\#_{kN}^{\mathcal{L}}kN\cong A\#_{kN}^{\mathcal{R}}(P\#_{kN}^{\mathcal{L}}kN)$. Then $\lambda(P\#_{kN}^{\mathcal{L}}kN)\subseteq\mathrm{End}(P)$ is the subring T of $M_G^{\mathrm{fin}}(k)$ with nonzero entries only in the (g, gn)-slots, $g\in G$, $n\in N$. For $m\in N$, $\rho(m^{-1})\in\mathrm{End}(P)$ is the matrix $(E_{mt,t})_{t\in G}$ (we use this notation to mean the matrix with a 1 in the mt-row and t-column for all $t\in G$, zeroes elsewhere). T is not a left ideal of $M_G^{\mathrm{col}}(k)$ but for any $m\in N$, $(E_{mt,t})_{t\in G}E_{g,gn}=E_{mg,gn}\in T$ since $(mg)^{-1}gn=g^{-1}m^{-1}gn\in N$. Therefore $A\#_{kN}^{\mathcal{R}}(P\#_{kN}^{\mathcal{L}}kN)\cong A\otimes T$. $\square$

The next example discusses duality for the G-set G/H where H is a (not necessarily normal) subgroup of G and G/H is the set of cosets g_iH, $i\in I$, of H in G. As in Example 1.6(iii), $P_{G/H}$ is the ring generated by idempotents $p_{[g_i]}$, $i\in I$. For G finite, this example yields [20, Corollary 2.18]. Note that [20, 2.18] is also a corollary of the smash data approach of [15]. (See Remark 3.10·)

EXAMPLE 2.9. Let G be a group with subgroup H, and let P and P_H denote the rings generated by the projections in $\mathrm{Hom}(kG, k)$ and $\mathrm{Hom}(kH, k)$ respectively. Let A be a G-graded ring; we will show that $(A\#_{kG}^{\mathcal{R}}P)\#_{kH}^{\mathcal{L}}kH\cong M_H^{\mathrm{fin}}(A\#_{kG}^{\mathcal{R}}P_{G/H})$.

Recall from Example 1.6(iii) that kG acts on $P_{G/H}$ on the right by $p_{[g_i]}\leftarrow g=p_{[g^{-1}g_i]}$ and consider $A\#_{kG}^{\mathcal{R}}P_{G/H}$. Now $A\#_{kG}^{\mathcal{R}}P_{G/H}$ is a right kH-comodule ring under the map γ defined by

$$\gamma(a\#p_{[g_i]})=\sum_{k\in I,\,m\in H}a_{g_kmg_i^{-1}}\#p_{[g_i]}\otimes m\,.$$

It is easily checked that γ gives a right comodule structure; we show that γ preserves multiplication. For $a\#p_{[g_i]}$, $b\#p_{[g_j]}\in A\#_{kG}^{\mathcal{R}}P_{G/H}$, we obtain,

$$(a\#p_{[g_i]})(b\#p_{[g_j]})=\sum_{t\in G}ab_t\#p_{[t^{-1}g_i]}p_{[g_j]}=\sum_{n\in H}ab_{g_ing_j^{-1}}\#p_{[g_j]}\,,$$

since $p_{[t^{-1}g_i]}p_{[g_j]}$ is nonzero if and only if $g_j=t^{-1}g_in$ for some $n\in H$. Then

$$\begin{aligned}\gamma(\textstyle\sum_{n\in H}ab_{g_ing_j^{-1}}\#p_{[g_j]})&=\textstyle\sum_{k\in I,\,m,\,n\in H}(ab_{g_ing_j^{-1}})_{g_kmg_j^{-1}}\#p_{[g_j]}\otimes m\\&=\textstyle\sum_{k\in I,\,m,\,n\in H}a_{g_kmn^{-1}g_i^{-1}}b_{g_ing_j^{-1}}\#p_{[g_j]}\otimes m.\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma(a\#p_{[g_i]})\gamma(b\#p_{[g_j]})&\\ =&(\textstyle\sum_{k\in I,s\in H}a_{g_k s g_i^{-1}}\#p_{[g_i]}\otimes s)(\sum_{l\in I,h\in H}b_{g_l h g_j^{-1}}\#p_{[g_j]}\otimes h)\\ =&\textstyle\sum_{k,l\in I,h,s\in H}a_{g_k s g_i^{-1}}b_{g_l h g_j^{-1}}\#p_{[g_j h^{-1}g_l^{-1}g_i]}p_{[g_j]}\otimes sh\\ =&\textstyle\sum_{k\in I,h,s\in H}a_{g_k s g_i^{-1}}b_{g_i h g_j^{-1}}\#p_{[g_i]}\otimes sh\\ =&\textstyle\sum_{k\in I,h,m\in H}a_{g_k m h^{-1}g_j}b_{g_i h g_j^{-1}}\#p_{[g_j]}\otimes m\,,\end{aligned}$$

so that $A\#^{\mathcal{R}}_{kG}P_{G/H}$ is a right kH-comodule ring.

Thus, since P_H is a $kH-kH$-bimodule ring, by Lemma 1.7 it makes sense to form the left kH-module ring $(A\#^{\mathcal{R}}_{kG}P_{G/H})\#^{\mathcal{R}}_{kH}P_H$, and we show that the left kH-module rings $(A\#^{\mathcal{R}}_{kG}P_{G/H})\#^{\mathcal{R}}_{kH}P_H$ and $A\#^{\mathcal{R}}_{kG}P$ are isomorphic. Let Ω map $A\#^{\mathcal{R}}_{kG}P$ to $(A\#^{\mathcal{R}}_{kG}P_{G/H})\#^{\mathcal{R}}_{kH}P_H$ by $\Omega(a\#p_{g_i n})=(a\#p_{[g_i]})\#p_n$ for $i\in I$. This map is clearly a bijection; it remains to prove that Ω preserves multiplication and the left kH-module structure.

Now,

$$\begin{aligned}\Omega(a\#p_{g_i n})\Omega(b\#p_{g_j m})&=((a\#p_{[g_i]})\#p_n)((b\#p_{[g_j]})\#p_m)\\ &=\textstyle\sum_{k\in I,h\in H}(a\#p_{[g_i]})(b_{g_k h g_j^{-1}}\#p_{[g_j]})\#p_{h^{-1}n}p_m\\ &=\textstyle\sum_{k\in I}(a\#p_{[g_i]})(b_{g_k n m^{-1}g_j^{-1}}\#p_{[g_j]})\#p_m\\ &=\textstyle\sum_{k\in I}(ab_{g_k n m^{-1}g_j^{-1}}\#p_{[g_j m n^{-1}g_k^{-1}g_i]}p_{[g_j]})\#p_m\\ &=(ab_{g_i n m^{-1}g_j^{-1}}\#p_{[g_j]})\#p_m\\ &=\Omega((a\#p_{g_i n})(b\#p_{g_j m}))\,.\end{aligned}$$

Also Ω is a left kH-module map since

$$\Omega(m\rightarrow(a\#p_{g_i n}))=\Omega(a\#p_{g_i n m^{-1}})=(a\#p_{[g_i]})\#p_{nm^{-1}}=(a\#p_{[g_i]})\#m\rightarrow p_n\,.$$

Now we have that

$$\begin{aligned}(A\#^{\mathcal{R}}_{kG}P)\#^{\mathcal{L}}_{kH}kH&\cong((A\#^{\mathcal{R}}_{kG}P_{G/H})\#^{\mathcal{R}}_{kH}P_H)\#^{\mathcal{L}}_{kH}kH\\ &\cong(A\#^{\mathcal{R}}_{kG}P_{G/H})\#^{\mathcal{R}}_{kH}(P_H\#^{\mathcal{L}}_{kH}kH)\quad\text{by Lemma 1.8}\\ &\cong M^{\text{fin}}_H(A\#^{\mathcal{R}}_{kG}P_{G/H})\quad\text{by Example 2.7.}\end{aligned}$$

If A is graded by H rather than G, and H is normal then we are back in the situation of Example 2.8. Here $A\#^{\mathcal{R}}_{kG}P_{G/H}\cong A\otimes P_{G/H}$ and $M^{\text{fin}}_H(A\otimes P_{G/H})\cong A\otimes M^{\text{fin}}_H(P_{G/H})\cong A\otimes T$, where T is the ring of matrices described in Example 2.8. The isomorphism $\Omega: M^{\text{fin}}_H(P_{G/H})\rightarrow T$ is given by $\Omega(p_{[g_i]}E_{n,h})=E_{g_i n,g_i h}$. □

EXAMPLE 2.10. Now let S be a cancellative semigroup, let P be as in Example 1.6(ii) and λ as in Example 2.7. We will show that for a ring A graded by S, although $A\#_{kS}^{\mathcal{R}}(P\#_{kS}^{\mathcal{L}}kS)$ is not isomorphic to $A\otimes\lambda(P\#_{kS}^{\mathcal{L}}kS)$, it can be viewed as a set of matrices with entries from A, with multiplication different from the usual multiplication in $M_S^{\mathrm{col}}(A)$.

Since S is cancellative, it is still true that the map λ from $P\#_{kS}^{\mathcal{L}}kS$ to $M_S^{\mathrm{fin}}(k)\subseteq M_S^{\mathrm{col}}(k)\cong\mathrm{End}(P)$ is an embedding. Also the right S-action on $\lambda(P\#_{kS}^{\mathcal{L}}kS)$ still appears to be $M_S^{\mathrm{col}}(k)$-strongly inner. For $p_g\leftarrow m$ is p_w if S contains a (unique) element w with $g=mw$; otherwise $p_g\leftarrow m=0$. Define u, v in $\mathrm{Hom}(kS, M_S^{\mathrm{col}}(k))$ by $u(m)=(E_{w,mw})_{w\in S}$ and $v(m)=(E_{mt,t})_{t\in S}$. Then if $mw=g$ for some $w\in S$, $u(m)\lambda(p_g\#h)v(m)=E_{w,wh}$ and the product is 0 otherwise, so that $u(m)\lambda(p_g\#h)v(m)=\lambda(p_g\#h)\leftarrow m$. Also note that $v(ms)=v(m)v(s)$. The problem is that although $u(m)v(m)$ is the identity in $M_S^{\mathrm{col}}(k)$, $v(m)u(m)=(E_{mw,mw})_{w\in S}$ which is not the identity.

However, $A\#_{kS}^{\mathcal{R}}(P\#_{kS}^{\mathcal{L}}kS)$ can still be viewed as a set of matrices over A but with a different multiplication from the usual matrix multiplication. Consider the subset of $M_S^{\mathrm{fin}}(A)$ of matrices where elements from A_s must lie in the sg-th row and gt-th column for some $g, t\in S$. Such a matrix with one non-zero entry a_s will be written $a_sE_{sg,gt}$. Define a multiplication on this set of matrices by

$$(a_sE_{sg,gt})\circ(b_vE_{vm,mw})=a_sb_vE_{sg,gt}((E_{vht,vht})_{h\in G})E_{vm,mw}$$
$$=\begin{cases} a_sb_vE_{sg,mw}=a_sb_vE_{svh,htw} & \text{if } g=vh \text{ and } ht=m \text{ for some } h\in S\\ 0 \quad \text{otherwise}\end{cases}$$

It is straightforward to check that this multiplication is associative and that $A\#_{kS}^{\mathcal{R}}(P\#_{kS}^{\mathcal{L}}kS)$ is isomorphic to this ring of matrices via $a_s\#(p_g\#t)\rightarrow a_sE_{sg,gt}$. $\square$

3. Inner coactions and duality

In this final section, we consider inner coactions and related duality theorems.

DEFINITION 3.1. (cf. [8, Definition 2.2]) Let R be a left (right) H-comodule ring such that $H\otimes R$ $(R\otimes H)$ is a subring of a k-algebra T. The H-coaction on R is called T-inner (or just inner if $T=H\otimes R$ or $R\otimes H$) if there is $w\in\mathcal{U}(T)$ such that for all $r\in R$,

(3.1.1) $$\sum r_{(-1)}\otimes r_{(0)}=w(1\otimes r)w^{-1} \qquad (\sum r_{(0)}\otimes r_{(1)}=w(r\otimes 1)w^{-1}).$$

The coaction is called strongly left (right) T-inner if, for all $y\in R$ such that $w(1\otimes y)=\sum h_i\otimes r_i\in H\otimes R$ $(w(y\otimes 1)=\sum r_i\otimes h_i\in R\otimes H)$, (3.1.2) $\sum\Delta(h_i)\otimes r_i=\sum h_i\otimes w(1\otimes r_i)$ $\quad(\sum r_i\otimes\Delta(h_i)=\sum w(r_i\otimes 1)\otimes h_i)$.

REMARK 3.2. If H is finitely generated projective over k, then H-coactions correspond to H^*-actions, and an H-coaction is (strongly) inner if and only if the corresponding H^*-action is (strongly) inner [8, Proposition 2.6]. □

EXAMPLE 3.3. Let H be a Hopf algebra with bijective antipode and M a finitely generated projective left H-comodule. Since M is finitely generated projective, we may identify $\mathrm{Hom}(M, H\otimes M)$ with $H\otimes\mathrm{End}(M)$; let $\Omega: H\otimes\mathrm{End}(M)\to\mathrm{Hom}(M, H\otimes M)$ be the natural isomorphism with inverse $\Phi: \mathrm{Hom}(M, H\otimes M)\to H\otimes\mathrm{End}(M)$.

It is shown in [17, Section 2] that $\mathrm{End}(M)$ is a left H-comodule algebra with the H-comodule structure map χ from $\mathrm{End}(M)$ to $H\otimes\mathrm{End}(M)$ given by

$$\chi(f)(m)=\sum(f(m_{(0)}))_{(-1)}\bar{S}(m_{(-1)})\otimes f(m_{(0)})_{(0)},$$

[17, Propositions 2.5, 2.6, 2.11 and Lemma 2.7]. (Note that the structures in [17] are on the right; the arguments for left comodule structures are essentially the same.)

We show that this is a strongly inner coaction. Let α_M be the comodule structure map for M and let $\Phi(\alpha_M)=\sum_i h_i\otimes\varphi_i$, and $\Phi(\bar{S}\otimes 1\cdot\alpha_M)=\sum_j g_j\otimes\phi_j\in H\otimes\mathrm{End}(M)$. Then, for $f\in\mathrm{End}(M)$, $m\in M$,

$$\begin{aligned}\sum h_i g_j\otimes\varphi_i\cdot f\cdot\phi_j(m)&=\sum h_i\bar{S}(m_{(-1)})\otimes\varphi_i(f(m_{(0)}))\\&=\sum(f(m_{(0)}))_{(-1)}\bar{S}(m_{(-1)})\otimes f(m_{(0)})_{(0)}=\chi(f)(m).\end{aligned}$$

Note that $\Phi(\alpha_M)$ and $\Phi(\bar{S}\otimes 1\cdot\alpha_M)$ are multiplicative inverses in $H\otimes\mathrm{End}(M)$ and also that $\sum\Delta(h_i)\otimes\varphi_i=\sum h_i\otimes h_k\otimes\varphi_k\varphi_i$, so that this coaction is left strongly inner.

If H is finitely generated projective, this H-coaction corresponds to the usual strongly inner right H^*-action on $\mathrm{End}(M)$ implemented by $u: H^*\to\mathrm{End}(M)$, where $u(h^*)(m)=\sum h^*(m_{(-1)})m_{(0)}$, $v(g^*)(m)=\sum g^*(S(m_{(-1)}))m_{(0)}$ (cf. Example 2.2(ii)).

Finally note that this coaction is a special case of an H-coaction on $\mathrm{Hom}(M, N)$, M, N, H-comodules, M finitely generated projective over k [17, p. 572]; in [25], this structure is generalized to define a Hopf algebra analogue to $\mathrm{HOM}(M, N)$, the graded homomorphisms from M to N, M, N G-graded modules. □

The next theorem is the coaction analogue to Theorem 2.3. We give the result only for left strongly inner coactions, but of course an analogous result holds for right coactions.

THEOREM 3.4. *Let A be a left H-module ring and B a left H-comodule ring such that $H\otimes B$ is a subring of an algebra C. Suppose the left H-coaction of B is strongly C-inner implemented by $c\in\mathcal{U}(C)$. Furthermore, suppose $c(1\otimes b)$ and $c^{-1}(1\otimes b)$ are in $H\otimes B$ for all $b\in B$. Then $A\#_H^f B\cong A\otimes B$.*

PROOF. Map $A\otimes B$ to $A\#_H^f B$ by $\phi(a\otimes b)=\sum h_i\rightarrow a\#b_i$ where $c(1\otimes b)=\sum h_i\otimes b_i$. Then, if $c(1\otimes e)=\sum h_k\otimes b_k$,

$$\begin{aligned}\phi(a\otimes b)\phi(d\otimes e)&=\sum(h_i\rightarrow a\#b_i)(h_k\rightarrow d\#b_k)\\&=\sum(h_i\rightarrow a)(b_{i(-1)}h_k\rightarrow d)\#b_{i(0)}b_k\,.\end{aligned}$$

Now, for all i,

$$\begin{aligned}\textstyle\sum_{k,(b_i)}b_{i(-1)}h_k\otimes b_{i(0)}b_k&=c(1\otimes b_i)c^{-1}(\textstyle\sum_k h_k\otimes b_k)=c(1\otimes b_i)c^{-1}c(1\otimes e)\\&=c(1\otimes b_i)(1\otimes e)=\textstyle\sum_m h(i)_m\otimes b(i)_m e\quad\text{where } c(1\otimes b_i)=\textstyle\sum_m h(i)_m\otimes b(i)_m\,.\end{aligned}$$

So,

$$\begin{aligned}\phi(a\otimes b)\phi(d\otimes e)&=\textstyle\sum_{i,m}(h_i\rightarrow a)(h(i)_m\rightarrow d)\#b(i)_m e\\&=\textstyle\sum h_{i(1)}\rightarrow a)(h_{i(2)}\rightarrow d)\#b_i e\qquad\text{by (3.1.2)}\\&=\phi(ad\otimes be)\,.\end{aligned}$$

The map ϕ is bijective with inverse ψ, $\psi(a\#b)=\sum g_j\rightarrow a\otimes e_j$ where $c^{-1}(1\otimes b)=\sum g_j\otimes e_j$. For then $\psi\cdot\phi(a\otimes b)=\sum_{i,j}g(i)_j h_i\rightarrow a\otimes e(i)_j$ where $c^{-1}(1\otimes b_i)=\sum g(i)_j\otimes e(i)_j$. Since $1\otimes b=c^{-1}c(1\otimes b)=\sum_i c^{-1}(1\otimes b_i)(h_i\otimes 1)=\sum_{i,j}g(i)_j h_i\otimes e(i)_j$, $\psi\cdot\phi$ is the identity on $A\#_H^f B$. Similarly $\phi\cdot\psi$ is the identity on $A\otimes B$.

Note that if B has a 1, then $c\in\mathcal{U}(H\otimes B)$, say $c=\sum h_i\otimes c_i$. Then $u=\sum\varepsilon(h_i)c_i\in\mathcal{U}(B)$, and (3.1.2) implies that $u^2=u$, so that $u=1$. Then if A also has a 1, $\phi(1\otimes 1)=1\#1$. □

REMARK 3.5. In view of Example 3.3, one wonders if a situation parallel to that of the Blattner-Montgomery duality theorem (see Example 2.6) holds here. Let k be a field, H a Hopf algebra and L a Hopf subalgebra of H^0; suppose H and L have bijective antipodes. The smash product $H\#_H^{\#}L$ is defined (in fact by Example 1.6(iv), $H\#_H^{\#}L\cong H\#_L^f L$) and the map λ: $H\#_H^{\#}L\rightarrow\mathrm{End}(H)$ given by $\lambda(h\#m)(g)=h(m\rightarrow g)=\sum hg_{(1)}m(g_{(2)})$ is an embedding [9, Proposition 2.2]. Give $\mathrm{Im}(\lambda)\subseteq\mathrm{End}(H)$ a left H-comodule algebra structure via λ, i.e., $\sum\lambda(h\#m)_{(-1)}\otimes\lambda(h\#m)_{(0)}=\sum h_{(1)}\otimes\lambda(h_{(2)}\#m)$. (If H is finite dimensional, this is the comodule structure from Example 3.3.)

For k a field, the natural map Ω: $H\otimes\mathrm{End}(H)\rightarrow\mathrm{Hom}(H, H\otimes H)$ given by

$\Omega(\sum h_j\otimes\xi_j)(h)=\sum h_j\otimes\xi_j(h)$ is an embedding. Then $\Omega(H\otimes\lambda(H\#_H^{\mathcal{R}}L))$ is a subring of $\mathrm{Hom}(H, H\otimes H)$ where multiplication in $\mathrm{Hom}(H, H\otimes H)$ is given by $\alpha*\beta(h)=(m\cdot tw\otimes 1)(1\otimes\alpha)\beta(h)$, for $\alpha, \beta\in\mathrm{Hom}(H, H\otimes H)$, $h\in H$. The identity in $\mathrm{Hom}(H, H\otimes H)$ is the map taking h to $1\otimes h$. Now it is easily checked that Δ is a unit in $\mathrm{Hom}(H, H\otimes H)$ with inverse $(\bar{S}\otimes 1)\Delta$. Also for $1\otimes\lambda(h\#m)\in H\otimes\lambda(H\#_H^{\mathcal{R}}L)\subseteq\mathrm{Hom}(H, H\otimes H)$,

$$\Delta*(1\otimes\lambda)(h\#m))*((\bar{S}\otimes 1)\Delta)(g)=\sum h_{(1)}\otimes\lambda(h_{(2)}\#m)(g),$$

so that the left comodule structure on $\lambda(H\#_H^{\mathcal{R}}L)$ is $\mathrm{Hom}(H, H\otimes H)$-inner. In particular, if $L=k$ and we identify H with $\lambda(H\#k)\subseteq\mathrm{End}(H)$, we see that the H-coaction Δ is $\mathrm{Hom}(H, H\otimes H)$-inner even though by [8, Example 2.4], the coaction is outer if H is not trivial.

A restriction parallel to the RL-condition of [9] in this case would seem to be that $\Delta\in H\otimes\lambda(H\#_H^{\mathcal{R}}L)$. This is a rather strong condition. For suppose there are $s_i, t_i\in H$, $m_i\in L$ such that for all $h\in H$,

$$\Delta(h)=\sum h_{(1)}\otimes h_{(2)}=\sum s_i\otimes\lambda(t_i\#m_i)(h)=\sum s_i\otimes t_ih_{(1)}m_i(h_{(2)}).$$

Then $h=\sum h_{(1)}\varepsilon(h_{(2)})=\sum(s_i\varepsilon(t_i))m_i(h)$ for all h, so that the elements $s_i\varepsilon(t_i)$ span H, i.e. H is finite dimensional over k.

However, even if Δ does not lie in $H\otimes\lambda(H\#_H^{\mathcal{R}}L)$, Theorem 3.4 can be used to produce an example analogous to Example 2.7. □

EXAMPLE 3.6. [4], [5] Let G, $H=kG$, and P be as in Example 2.7, and let A be a ring on which G acts on the left as a group of automorphisms. By Lemma 1.8, $(A\#_{kG}^{\mathcal{L}}kG)\#_{kG}^{\mathcal{R}}P\cong A\#_{kG}^{\mathcal{L}}(kG\#_{kG}^{\mathcal{R}}P)$; we show that Theorem 3.4 implies [5, Theorem 3.1], namely that $A\#_{kG}^{\mathcal{L}}(kG\#_{kG}^{\mathcal{R}}P)\cong A\otimes(kG\#_{kG}^{\mathcal{R}}P)\cong M_G^{\mathrm{fin}}(A)$.

Identify $kG\#_{kG}^{\mathcal{R}}P$ with $M_G^{\mathrm{fin}}(k)$ by mapping $g\#p_h$ to $E_{gh,h}$. Let $M\in M_G^{\mathrm{col}}(kG)\supseteq kG\otimes M_G^{\mathrm{fin}}(k)$ be the matrix with g in the g-th row and column and zeroes elsewhere; we write $M=(gE_{g,g})_{g\in G}$. Then $M^{-1}=(h^{-1}E_{h,h})_{h\in G}$. Now, for $E_{st,t}\in M_G^{\mathrm{fin}}(k)$, $M(1\otimes E_{st,t})M^{-1}=s\otimes E_{st,t}=s\otimes\lambda(s\#p_t)$. Also for all $E_{s,t}\in M_G^{\mathrm{fin}}(k)$, $M(1\otimes E_{s,t})$ and $M^{-1}(1\otimes E_{s,t})$ lie in $kG\otimes M_G^{\mathrm{fin}}(k)$, and $(\Delta\otimes 1)M(1\otimes E_{s,t})=s\otimes s\otimes E_{s,t}=s\otimes M(1\otimes E_{s,t})$, so that the kG-coaction of $M_G^{\mathrm{fin}}(k)$ is strongly $M_G^{\mathrm{col}}(kG)$-inner, and Theorem 3.4 applies. Thus $A\#_{kG}^{\mathcal{L}}(kG\#_{kG}^{\mathcal{R}}P)\cong A\otimes M_G^{\mathrm{fin}}(k)\cong M_G^{\mathrm{fin}}(A)$. □

EXAMPLE 3.7. The argument in Example 3.6 can be repeated for $(A\#_{kG}^{\mathcal{L}}kH)\#_{kG}^{\mathcal{R}}P\cong A\#_{kG}^{\mathcal{L}}(kH\#_{kG}^{\mathcal{R}}P)$ where H is any subgroup of G. Then $A\#_{kG}^{\mathcal{L}}(kH\#_{kG}^{\mathcal{R}}P)\cong A\otimes\lambda(kH\#_{kG}^{\mathcal{R}}P)$ is the subring of $M_G^{\mathrm{fin}}(A)$ such that the (h, g)-th slot may contain a nonzero entry only if $hg^{-1}\in H$. □

EXAMPLE 3.8. Example 3.6 can be generalized to cancellative semigroups. Let S be a cancellative semigroup, let kS be the semigroup ring and let P be as in Example 1.6(ii). Suppose A is a ring such that there is a semigroup homomorphism σ from S to $\mathrm{Aut}(A)$; let G be the smallest subgroup of $\mathrm{Aut}(A)$ containing the image of σ. Then A is a left kG-module ring, kS is a left kG-comodule ring via σ, and, by a straightforward generalization of Lemma 1.8. $(A\#_{kG}^{\mathcal{L}}kS)\#_{kS}^{\mathcal{R}}P \cong A\#_{kG}^{\mathcal{L}}(kS\#_{kG}^{\mathcal{R}}P)$.

Now map $kS\#_{kG}^{\mathcal{R}}P$ to $M_S^{\mathrm{fin}}(k)$ by mapping $s\#p_t$ to $E_{st,t}$ as usual; this map may no longer be onto as it was in Example 3.6 but since S is cancellative, it is still an embedding. Now the kS-coaction is strongly $M_S^{\mathrm{col}}(kG)$-inner implemented by $M=(\sigma(s)E_{s,s})_{s\in S}$, $M^{-1}=(\sigma(t)^{-1}E_{t,t})_{t\in S}$, and an argument as in Example 3.6 shows that $A\#_{kG}^{\mathcal{L}}(kS\#_{kS}^{\mathcal{R}}P)\cong A\otimes(kS\#_{kS}^{\mathcal{R}}P)$. □

REMARK 3.9. Abrams duality theorem [1, Theorem 2.4] is a refinement of the situation in Example 3.8. The semigroup arising from taking the morphisms of a category and adjoining a zero element is usually not cancellative. However, if the set of nonzero elments has the cancellation property, then by considering only matrices with zero entries in the row and column indexed by 0, and by taking σ from the nonzero elements of the semigroup to $\mathrm{Aut}(A)$, an argument similar to that above yields [1, Theorem 2.4]. □

REMARK 3.10. Finally we note that although we were able to describe [20, 2.18] in terms of smash products and inner actions, we seem unable to similarly describe [20, 2.20] in terms of coactions. For, even for $A=R\#_{kG}^{\mathcal{L}}kG$, the ring A does not seem to decompose as a ring into a smash product of the form $(R\#kH)\#k[G/H]$, H a (not necessarily normal) subgroup of G, since $kG\cong kH\otimes k[G/H]$ as coalgebras but not as rings. In [15], Koppinen is able to generalize both of these results from [20] using quintuples called smash data which include a bialgebra H, a right H-comodule algebra B and a left H-module coalgebra C, so that a smash product $\#(H;C,B)$ is defined (see Remark 1.5).

We also note that other duality theorems involving the G-set G/H have been proved by Liu Shaoxue [16]. □

References

[1] G.D. Abrams, Smash products for a class of semigroup-graded rings, preprint.
[2] P.N. Anh and L. Marki, Morita equivalence for rings without identity, Tsukuba J. Math. **11** (1987), 1-16.
[3] M. Beattie, A generalization of the smash product of a graded ring, J. Pure & Appl. Algebra **52** (1988), 219-226.

[4] M. Beattie, Duality theorems for rings with actions or coactions, J. Algebra **115** (1988), 303-312.
[5] M. Beattie, Duality theorems for group actions and gradings, in Ring Theory Proceedings, Granada 1986, Springer Lecture Notes in Mathematics **1328** (1988), 28-33.
[6] M. Beattie, On the Blattner-Montgomery duality theorem for Hopf algebras, in Contemporary Mathematics, **124**, 23-28, Amer. Math. Soc., Providence, 1992.
[7] M. Beattie and K.-H. Ulbrich, A Skolem-Noether theorem for Hopf algebra actions, Comm. in Algebra **18** (1990), 3713-3724.
[8] R.J. Blattner, M. Cohen and S. Montgomery, Crossed products and inner actions of Hopf algebras, Trans. Amer. Math. Soc. **298** (1986), 671-711.
[9] R.J. Blattner and S. Montgomery, A duality theorem for Hopf module algebras, J. Algebra **95** (1985), 153-172.
[10] Chen Cao-yu and Warren Nichols, A duality theorem for Hopf module algebras over Dedekind rings, Communications in Algebra **13** (1990), 3209-3221.
[11] M. Cohen, Smash products, inner actions and quotient rings, Pacific J. Math. **125** (1986), 45-66.
[12] Y. Doi, For unifying Hopf modules, J. Algebra, to appear.
[13] Y. Doi, Cleft comodule algebra and Hopf modules, Communications in Algebra **12** (1984), 1155-1169.
[14] M. Koppinen, A duality theorem for crossed products of Hopf algebras, J. Algebra, **145** (1992), 153-174.
[15] M. Koppinen, Variations on the smash product with applications to group graded rings, preprint.
[16] Liu, Shaoxue, private communication.
[17] F.W. Long, The Brauer group of dimodule algebras, J. Algebra **30** (1974), 559-601.
[18] A. Masuoka, Coalgebra actions on Azumaya algebras, Tsukuba J. Math. **14** (1990), 107-112.
[19] S. Montgomery, Duality for actions and coactions of groups, in Contemporary Mathematics, Vol. 43, 191-207, Amer. Math. Soc., Providence, 1985.
[20] C. Năstăsescu, S. Raianu and F. Van Oystaeyen, Modules graded by G-sets, Math. Z. 203, 1990, 605-627.
[21] C. Năstăsescu, F. Van Oystaeyen and Liu Shaoxue, Graded modules over G-sets II, Math. Z. **207** (1991), 341-358.
[22] D. Quinn, Group-graded rings and duality, Trans. Amer. Math. Soc. **292** (1985), 155-167.
[23] M.E. Sweedler, "Hopf Algebras", Benjamin, New York, 1969.
[24] M. Takeuchi, $\mathrm{Ext}_{ad}(\mathrm{Sp}\, R, \mu^A) \cong Br(A/k)$, J. Algebra **67** (1980), 436-475.
[25] K.-H. Ulbrich, Smash products and comodules of linear maps, Tsukuba J. Math. **14** (1990), 371-378.

Department of Mathematics and Computer Science
Mount Allison University
Sackville, N.B., Canada EOA 3C0
email: mbeattie @ mta. ca