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HOMOGENEOUS LORENTZ MANIFOLDS WITH ISOTROPY SUBGROUP $U(2)$ OR $SO(2)$

Dedicated to Professor Tsunero Takahasi on his 60th birthday

By

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1. Introduction.

Let $(M, \langle , \rangle)$ be an n-dimensional connected Lorentz manifold with metric $\langle , \rangle$ of signature $(-, +, \cdots, +)$. In this note, we assume that an isometry group has *compact* isotropy subgroup at every point in M.

In [8], we showed that, if $n\geqq 6$, there is no r-dimensional isometry group for $(n-1)(n-2)/2+3\leqq r\leqq n(n-1)/2-1$, and we determined simply connected n-dimensional Lorentz manifolds admitting an isometry of dimension $(n-1)(n-2)/2+2$ for $n\geqq 6$. However, there exists a 5-dimensional Lorentz manifold admitting a 9 $(=(5-1)(5-2)/2+3=5(5-1)/2-1)$-dimensional isometry group (see Remark 1.3 in [8]). In § 3, we will determine simply connected 5-dimensional Lorentz manifolds admitting an isometry group of dimension 9. That is, we have the following Theorem A.

THEOREM A. *Let $(M, \langle , \rangle)$ be a simply connected 5-dimensional Lorentz manifold admitting a connected 9-dimensional isometry group G with compact isotropy subgroup at every point in M. Then $(M, \langle , \rangle)$ is one of the following:*

(1) *$(M, \langle , \rangle)$ is isometric to $(\boldsymbol{R}\times M_2, -dt^2+ds^2)$ where (M_2, ds^2) is a 2-dimensional simply connected complex space form;*

(2) *$(M, \langle , \rangle)$ is isometric to a simply connected 5-dimensional Lie group with a left-invariant Lorentz metric $\langle , \rangle$ and G is isomorphic to a semi-direct product $U(2)\rtimes M$;*

(3) *$(M, \langle , \rangle)$ is a principal fibre bundle, with a 1-dimensional structure group, over a 2-dimensional simply connected complex space form.*

In [6], [7], we determined n-dimensional Lorentz manifolds admitting an isometry group of dimension $n(n-1)/2+1$ (for $n\geqq 4$). In § 4, we will determine simply connected n-dimensional Lorentz manifolds admitting a connected isometry group of dimension $n(n-1)/2+1$ for $n=3$. This is, we have the follow-

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ing Theorem B.

THEOREM B. *Let* $(M, \langle\,,\rangle)$ *be a simply connected* 3-*dimensional Lorentz manifold admitting a connected* 4-*dimensional isometry group* G *with compact isotropy subgroup at every point in* M. *Then* $(M, \langle\,,\rangle)$ *is one of the following*:

(1) $(M, \langle\,,\rangle)$ *is isometric to* $(\boldsymbol{R}\times M_2, -dt^2+ds^2)$ *where* (M_2, ds^2) *is a simply connected* 2-*dimensional Riemannian space form*;

(2) $(M, \langle\,,\rangle)$ *is isometric to a simply connected* 3-*dimensional Lie group with a left-invariant Lorentz metric and* G *is isomorphic to a semi-direct product* $SO(2)\rtimes M$;

(3) $(M, \langle\,,\rangle)$ *is a principal fibre bundle, with a* 1-*dimensional structure group, over a simply connected* 2-*dimensional Riemannian space form.*

In [8], we determined simply connected n-dimensional Lorentz manifolds M admitting an isometry group of dimension $(n-1)(n-2)/2+2$ for $n\geqq 6$. In §4, we will determine simply connected n-dimensional Lorentz manifolds M admitting an isometry group of dimension $(n-1)(n-2)/2+2$ for $n=4$. That is, we will show the following Theorem C.

THEORFM C. *Let* $(M, \langle\,,\rangle)$ *be a simply connected* 4-*dimensional Lorentz manifold admitting a connected* 5-*dimensional isometry group* G *with compact isotropy subgroup at every point in* M. *Then* $(M, \langle\,,\rangle)$ *is one of the following*:

(1) $(M, \langle\,,\rangle)$ *is isometric to* $(M_1\times M_2, ds_1^2+ds_2^2)$ *where* (M_1, ds_1^2) *is a simply connected* 2-*dimensional Lie group with a left-invariant Lorentz metric* ds_1^2 *and* (M_2, ds_2^2) *is a simply connected* 2-*dimensional Riemannian space form*;

(2) $(M, \langle\,,\rangle)$ *is isometric to a simply connected* 4-*dimensional Lie group with a left-invariant Lorentz metric and* G *is a semi-direct product* $SO(2)\rtimes M$;

(3) $(M, \langle\,,\rangle)$ *is a principal fibre bundle, with a* 2-*dimensional abelian structure group, over a simply connected* 2-*dimensional Riemannian space form.*

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2. Preliminaries.

Let $(M, \langle\,,\rangle)$ be a connected Lorentz manifold with metric $\langle\,,\rangle$ of signature $(-, +, \cdots, +)$ and let G be a connected isometry group acting on M such that the isotropy subgroup H at $o\in M$ is compact. Then the linear isotropy subgroup $\tilde{H}=\{dh\,;\,h\in H\}$ is a closed subgroup of $O(1, n-1)=\{A\in GL(n, \boldsymbol{R})\,;\,{}^tASA=S\}$ where S is the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

(I_{n-1} is the unit matrix of degree $n-1$). So $\tilde{H}$ is conjugate to a closed subgroup of $O(1)\times O(n-1)$.

PROPOSITION 2.1. *Let* $(M, \langle\,,\rangle)$ *be a simply connected* 5*-dimensional Lorentz manifold admitting a* 9*-dimensional isometry group* G *with compact isotropy subgroup at every point in* M. *Then* G *acts on* M *transitively and the linear isotropy subgroup is conjugate to* $1\times U(2)$.

PROOF. Suppose that G does not act on M transitively. Then $\dim G(o)\leqq 4 (o\in M)$. Hence the dimension of the isotropy subgroup H at o is not less than 5. On the other hand, since H is compact, the linear isotropy subgroup is isomorphic to a subgroup of $O(1)\times O(4)$, so $\dim H\leqq 4(4-1)/2=6$. Thus $5\leqq \dim H\leqq 6$. Then we have $\dim H=6$(c.f., [2], [9]), so that we have $\dim G(o)=3$, which contradicts Lemma 1.2 in [8]. Therefore G is transitive on M.

Since M is simply connected, H is connected and the linear isotropy subgroup $\tilde{H}$ is isomorphic to a subgroup of $1\times SO(4)$. Since $\dim H=\dim G-\dim M=4$, $\tilde{H}$ is conjugate to $1\times U(2)$ (c.f., [9]). ∎

By the same way as the proof of Proposition 2.1, we have

PROPOSITION 2.2. *Let* $(M, \langle\,,\rangle)$ *be a simply connected* 4(*resp.* 3)*-dimensional Lorentz manifold admitting a* 5(*resp.* 4)*-dimensional isometry group with compact isotropy subgroup at every point in* M. *Then* G *acts on* M *transitively and the linear isotropy subgroup is conjugate to* $I_2\times SO(2)$ (*resp.* $1\times SO(2)$).

In view of Propositions 2.1 and 2.2, we consider homogeneous Lorentz manifolds $G/H=M$ (H is the isotropy subgroup of G at some point $o\in M$). We denote Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Since H is compact, there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}\ , \qquad [\mathfrak{h}, \mathfrak{m}]\subseteqq\mathfrak{m}\,.$$

Let $\pi: G\to G/H=M$ be the natural projection. We identify the tangent space T_oM and $\mathfrak{m}$ by $d\pi$. The Lorentz inner product on T_oM induces the Lorentz inner product $\langle\,,\rangle_{\mathfrak{m}}$ on $\mathfrak{m}$ so that $d\pi: \mathfrak{m}\to T_oM$ is a linear isometry.

3. Proof of Theorem A.

Let $(M, \langle\ ,\ \rangle)$ be a simply connected 5-dimensional Lorentz manifold admitting a connected isometry group G of dimension 9. By the Proposition 2.1, M is a simply connected homogeneous Lorentz manifold G/H and the linear isotropy subgroup is conjugate to $1\times U(2)$. Then $Ad(H)$ acts on $\mathfrak{m}$ as $1\times U(2)$, so there exists a 1-dimensional subspace $\mathfrak{m}_1$ and a 4-dimensional subspace $\mathfrak{m}_2$ of $\mathfrak{m}$ such that

$$\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2$$

and $Ad(H)=id.$ on $\mathfrak{m}_1$ (so, $[\mathfrak{h}, \mathfrak{m}_1]=\{0\}$), $Ad(H)=U(2)$ on $\mathfrak{m}_2$. Since $U(2)$ acts on $\mathfrak{m}_2$ irreducibly and contains $-I_2$, we have Lemma 3.1 by using Schur's Lemma.

LEMMA 3.1. *$\mathfrak{m}_2$ is spacelike and $\mathfrak{m}_1$ is perpendicular to $\mathfrak{m}_2$ (so, $\mathfrak{m}_1$ is timelike).*

Let p_0, p_1 and p_2 be orthogonal projections from $\mathfrak{g}$ to $\mathfrak{h}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. Since $\mathfrak{h}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are $Ad(H)$-invariant, we see

$$p_iAd(h)=Ad(h)p_i \qquad (i=0, 1, 2) \tag{3.1}$$

for any $h\in H$. Since there exists $E\in\mathfrak{h}$ such that

$$Ad(\exp tE)=\begin{pmatrix}\cos tI_2 & -\sin tI_2\\ \sin tI_2 & \cos tI_2\end{pmatrix}$$

on $\mathfrak{m}_2$, we have

$$[E, X]=JX \tag{3.2}$$

for any $X\in\mathfrak{m}_2$, where J is an almost complex structure on $\mathfrak{m}_2$.

REMARK 3.2. E belongs to the center of $\mathfrak{h}$.

LEMMA 3.3. $[\mathfrak{m}_1, \mathfrak{m}_2]\subseteq\mathfrak{m}_2$. *More precisely, there exist linear maps L_1, L_2: $\mathfrak{m}_1\to\boldsymbol{R}$ such that*

$$[A, X]=L_1(A)X+L_2(A)JX$$

for any $A\in\mathfrak{m}_1$ and any $X\in\mathfrak{m}_2$.

PROOF. For any fixed $A\in\mathfrak{m}_1$, we define a linear map f_A: $\mathfrak{m}_2\to\mathfrak{g}$ by $f_A(X)=[A, X]$ $(X\in\mathfrak{m}_2)$. Since $Ad(H)=id.$ on $\mathfrak{m}_1$, we have

$$f_AAd(h)=Ad(h)f_A \tag{3.3}$$

for any $h \in H$. By (3.1) and (3.3), we have

$$(p_i f_A) Ad(h) = Ad(h)(p_i f_A) \qquad (i=0, 1, 2) \tag{3.4}$$

for any $h \in H$.

Step 1. We claim $p_1[\mathfrak{m}_1, \mathfrak{m}_2]=\{0\}$. Since $\ker(p_1 f_A)$ is $Ad(H)$-invariant by (3.4) and $Ad(H)$ acts on $\mathfrak{m}_2$ irreducibly, we have $\ker(p_1 f_A)=\{0\}$ or $\mathfrak{m}_2$. On the other hand, there exist a non-zero $X \in \mathfrak{m}_2$ and $h \in H$ such that $Ad(h)X - X \neq 0$. We have $p_1 f_A(Ad(h)X - X)=0$, which implies that $Ad(h)X - X \in \ker(p_1 f_A)$. Therefore we have $\ker(p_1 f_A)=\mathfrak{m}_2$, that is, $p_1[A, \mathfrak{m}_2]=\{0\}$. Since A is arbitrary, we have $p_1[\mathfrak{m}_1, \mathfrak{m}_2]=\}0\}$.

Step 2. We claim $p_0[\mathfrak{m}_1, \mathfrak{m}_2]=\{0\}$. By the same procedure as step 1, we have $\ker(p_0 f_A)=\{0\}$ or $\mathfrak{m}_2$. Assume $\ker(p_0 f_A)=\{0\}$. Since $p_0 f_A : \mathfrak{m}_2 \to \mathfrak{h}$ is injective, we have $\dim(p_0 f_A(\mathfrak{m}_2))=4=\dim \mathfrak{h}$, so we have $[A, \mathfrak{m}_2]=\mathfrak{h}$. On the other hand, $[A, \mathfrak{m}_2]$ is spaned by $[A, X]$'s ($X \in \mathfrak{m}_2$) and we have

$$\begin{aligned}[A, X] &= [A, -J^2 X] = -[A, [E, JX]] \\ &= -[E, [A, JX]] = 0,\end{aligned}$$

because E belongs to the center of $\mathfrak{h}$. Thus we have $[A, \mathfrak{m}_2]=\{0\}$, which is a contradiction. Therefore, we have $p_0[A, \mathfrak{m}_2]=\{0\}$. Since A is arbitrary, we have $p_0[\mathfrak{m}_1, \mathfrak{m}_2]=\{0\}$.

Step 3. f_A is a linear map from $\mathfrak{m}_2$ to $\mathfrak{m}_2$ by step 1 and step 2, and f_A commutes with $Ad(h)$ for any $h \in H$, so by Schur's Lemma, there exist linear maps $L_1, L_2 : \mathfrak{m}_1 \to \boldsymbol{R}$ such that

$$f_A(X) = L_1(A)X + L_2(A)JX \qquad (X \in \mathfrak{m}_2). \quad \blacksquare$$

For a non-zero $A_1 \in \mathfrak{m}_1$, set $\mathfrak{m}_1' = \boldsymbol{R}\{A_1 - L_2(A_1)E\}$. Since E belongs to the center of $\mathfrak{h}$, we have $[\mathfrak{m}_1', \mathfrak{m}_2]=\{0\}$. It is trivial that

$$[A, X] = L_1(p_2(A))X \qquad (A \in \mathfrak{m}_1', X \in \mathfrak{m}_2).$$

Thus we have a new decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{m}' \oplus \mathfrak{h},$$

(where $\mathfrak{m}' = \mathfrak{m}_1' \oplus \mathfrak{m}_2$), according which we define a Lorentz inner product on $\mathfrak{m}'$ as in § 2. Then $\mathfrak{m}_2$ is spacelike and perpendicular to $\mathfrak{m}_1'$, and we have Lemma 3.3'.

LEMMA 3.3'. $[\mathfrak{m}_1', \mathfrak{m}_2] \subseteqq \mathfrak{m}_2$. *More precisely, there exists a linear map* L_1': $\mathfrak{m}_1' \to \boldsymbol{R}$ *such that*

$$[A, X] = L_1'(A)X \qquad (A \in \mathfrak{m}_1', X \in \mathfrak{m}_2).$$

We use notations $\mathfrak{m}$, $\mathfrak{m}_1$ and L instead of $\mathfrak{m}'$, $\mathfrak{m}_1'$ and L_1' respectively.

LEMMA 3.4. (1) *If* $L\neq 0$, *then* $[\mathfrak{m}_2, \mathfrak{m}_2]=0$.
(2) *If* $L=0$, *then* $[\mathfrak{m}_2, \mathfrak{m}_2]\subseteqq\mathfrak{h}\oplus\mathfrak{m}_1$.

PROOF. For any $Z, W\in\mathfrak{m}_2$, we have

$$Jp_2[Z, W]=p_2[JZ, W]+p_2[Z, JW]$$

by the equality

$$[E, [Z, W]]=[[E, Z], W]+[Z, [E, W]] .$$

Therefore, for a basis X, JX, Y, JY of $\mathfrak{m}_2$, we have

$$p_2[X, JX]=p_2[Y, JY]=0$$

$$p_2[X, Y]+p_2[JX, JY]=0$$

$$p_2[X, JY]+p_2[Y, JX]=0 ,$$

so $\dim p_2[\mathfrak{m}_2, \mathfrak{m}_2]\leqq 2$. On the other hand, $p_2[\mathfrak{m}_2, \mathfrak{m}_2]$ is $Ad(H)$-invariant subspace of $\mathfrak{m}_2$ by (3.1). Thus we have $p_2[\mathfrak{m}_2, \mathfrak{m}_2]=\{0\}$, that is, $[\mathfrak{m}_2, \mathfrak{m}_2]\subseteqq\mathfrak{h}\oplus\mathfrak{m}_1$. Thus, for any $X, Y\in\mathfrak{m}_2$, we can set $[X, Y]=U+A(U\in\mathfrak{h}, A\in\mathfrak{m}_1)$. Then, for $B\in\mathfrak{m}_1$, we have $2L(B)[X, Y]=[B, A]=0$. If $L\neq 0$, then $[X, Y]=0$. ∎

LEMMA 3.5. *If* $L=0$, *then* $\mathfrak{m}_1=\mathfrak{z}(\mathfrak{g})$ *where* $\mathfrak{z}(\mathfrak{g})$ *is a center of* $\mathfrak{g}$.

PROOF. Since $[\mathfrak{m}_1, \mathfrak{h}]=\{0\}=[\mathfrak{m}_1, \mathfrak{m}_2]$, it is trivial that $\mathfrak{m}_1\subseteqq\mathfrak{z}(\mathfrak{g})$.

Let Z be any vector in $\mathfrak{z}(\mathfrak{g})$. For any $X\in\mathfrak{m}_2$, we have $[p_0(Z), X]+[p_2(Z), X]=0$. Since $[p_0(Z), X]\in\mathfrak{m}_2$ and $[p_2(Z), X]\in\mathfrak{h}\oplus\mathfrak{m}_1$, we have $[p_0(Z), X]=0$, which implies $p_0(Z)=0$. We have $p_2(Z)=0$ by equalities $0=[E, Z]=Jp_2(Z)$. Therefore we have $Z\in\mathfrak{z}(\mathfrak{m}_1)$. ∎

By the above argument, we have following possibilities;

(i) $[\mathfrak{m}_1, \mathfrak{m}_2]=\mathfrak{m}_2$ $\quad[\mathfrak{m}_2, \mathfrak{m}_2]=\{0\}$;

(ii) $[\mathfrak{m}_2, \mathfrak{m}_2]=\{0\}$, $\quad[\mathfrak{m}_2, \mathfrak{m}_2]\subseteqq\mathfrak{h}$;

(iii) $[\mathfrak{m}_1, \mathfrak{m}_2]=\{0\}$, $\quad[\mathfrak{m}_2, \mathfrak{m}_2]\subseteqq\mathfrak{z}(\mathfrak{g})=\mathfrak{m}_1$;

(iv) $[\mathfrak{m}_1, \mathfrak{m}_2]=\{0\}$, $\quad p_0[\mathfrak{m}_2, \mathfrak{m}_2]\neq\{0\}$, $\quad p_1[\mathfrak{m}_2, \mathfrak{m}_2]\neq\{0\}$.

Case (ii). By the same way as in the proof of the Theorem B in [7], we have the space (1).

Case (i) *and* (iii). $\mathfrak{m}_1\oplus\mathfrak{m}_2$ is an ideal in $\mathfrak{g}$. Let K be a connected Lie subgroup of G whose Lie algebra is $\mathfrak{m}$. Then K is a closed normal subgroup of

G. Since the dimension of the isotropy subgroup of K at $o\in M$ is equal to $\dim(K\cap H)=\dim(\mathfrak{m}\cap\mathfrak{h})=0$, we have $\dim K(o)=\dim M$. Therefore $K(o)$ is open in M. Since K is a normal subgroup of G, each K-orbit is open in M. By the connectedness of M, we have $K(o)=M$. Thus M is isometric to the Lie group K with a left invariant Lorentz metric. Since the sequence

$$1\longrightarrow H\longrightarrow G\longrightarrow G/H=K(o)\longrightarrow 1$$

is exact and there exists a cross section $s: K(o)\to G$ such that $\pi s=id$. G is a semi-direct product of $H=U(2)$ and $M=K(o)$. Thus we have space (2).

Case (iv). Let C be a Lie subgroup of G whose Lie algebra is $\mathfrak{z}(\mathfrak{g})=\mathfrak{m}_1$. Then C is a closed, commutative and normal subgroup of G, and acts on M freely (because $C\cap H=\{1\}$). Therefore, each C-orbit is a 1-dimensional closed submanifold and timelike (because $\mathfrak{m}_1$ is timelike).

LEMMA 3.6. *The orbit space M/C has a differentiable manifold structure.*

PROOF. Since H is compact and C is closed, C acts on M properly (c.f., [5], [11]). Then M/C is a Hausdorff space and satisfies the second countable axiom (c.f., [3]). Since each C-orbit $C(x)$ of $x\in M$ is timelike, there exists an open set V in $\boldsymbol{R}^4$ such that a normal exponential map $\exp_x^{\perp}: V\to S=\exp_x^{\perp}(V)$ is a diffeomorphism and $\langle T_xS, T_xC(x)\rangle=0$. Then M/C has a differentiable manifold structure (c.f., [3]). ∎

By the same procedure as in the proof of Theorem 30.2 in [3], we have

LEMMA 3.7. *$C\to M\to M/C$ is a principal fibre bundle with a structure group C.*

We introduce a Riemannian metric h on M/C so that $p: M\to M/C$ is a semi-Riemannian submersion as follows: Let $S(y)$ be a neighborhood of $y=p(\bar{y})$ in M/C and $\chi_{S(y)}$ be a local cross section from $S(y)$ to M. We define a Riemannian metric $h_{S(y)}$ on $S(y)$ by

$$h_{S(y)}(X, Y)=\langle d\chi_{S(y)}(X), d\chi_{S(y)}(Y)\rangle(\chi_{S(y)})$$

for any vector fields X and Y on M/C. Since $\chi_{S(y)}(x)$ and $\chi_{S(z)}(x)$ belong to the same C-orbit for $x\in S(y)\cap S(z)$, there exists $c\in C$ such that $c\chi_{S(y)}(x)=\chi_{S(z)}(x)$. Therefore we have

$$h_{S(z)}(X, Y)(x)=h_{S(y)}(X, Y)(x).$$

Thus $\{h_{S(y)}\}$ defines a Riemannian metric on M/C.

G/C is an isometry group acting on M/C effectively and transitively, and the isotropy subgroup is $H/C=H=U(2)$. So M/C is a simply connected 2-dimensional complex space form (c.f., [4]).

Thus M is a principal fibre bundle with an abelian structure group C of dimension 1, over a simply connected 2-dimensional complex space form. We complete the proof of the Theorem A.

REMARK 3.8. When $L\neq 0$, the space (2) in the Theorem A is isometric to the Lie group G_5 in [10] and G is a semi-direct product $U(2)\rtimes G_5$.

REMARK 3.9. By the similar way as the proof of the Theorem A, we have the following. Let $(M, \langle\,,\rangle)$ be a simply connected 6-dimensional Lorentz manifold on which a connected isometry group G acts transitively. If the linear isotopy subgroup H at $o\in M$ acts on T_oM as $I_2\times U(2)$, then $(M, \langle\,,\rangle)$ is one of the following:

(1) $(M, \langle\,,\rangle)$ is isometric to $(N_1\times M_2,\ dt^2+ds^2)$ where $(M_1,\ dt^2)$ is a simply connected 2-dimensional Lie group with a left-invariant Lorentz metric dt^2 and $(M_2,\ ds^2)$ is a 2-dimensional simply connected complex space form;

(2) $(M, \langle\,,\rangle)$ is isometric to a simply connected 6-dimensional Lie group with a left-invariant Lorentz metric and G is isomorphic to a semi-direct product $U(2)\rtimes M$;

(3) $(M, \langle\,,\rangle)$ is a principal fibre bundle, with a 2-dimensional abelian structure group, over a 2-dimensional simply connected complex space form.

4. Proofs of theTheorem B and Theorem C.

Let $(M, \langle\,,\rangle)$ be a simply connected n-dimensional Lorentz manifold admitting a connected isometry group of dimension $n(n-1)/2+1$ (resp. $(n-1)(n-2)/2+2$) for $n=3$ (resp. $n=4$). By the Proposition 2.2, M is a simply connected homogeneous Lorentz manifold G/H and the linear isotropy subgroup is conjugate to $I_{n-2}\times SO(2)$. Then $Ad(H)$ acts on $\mathfrak{m}$ as $I_{n-2}\times SO(2)$, so there exist an $(n-2)$-dimensional subspace $\mathfrak{m}_1$ and a 2-dimensional subspace $\mathfrak{m}_2$ of $\mathfrak{m}$ such that $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2$, $Ad(H)=I_{n-2}$ on $\mathfrak{m}_1$ and $Ad(H)=SO(2)$ on $\mathfrak{m}_2$. By the same way as the proof of Lemma 3.1, we have

LEMMA 4.1. *$\mathfrak{m}_2$ is spacelike and perpendicular to $\mathfrak{m}_1$ (therefore, $\mathfrak{m}_1$ is timelike).*

Let p_0, p_1 and p_2 be orthogonal projection from $\mathfrak{g}$ to $\mathfrak{h}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. Then by the same reason as in § 3, we have

$$p_i Ad(h) = Ad(h) p_i \qquad (i=0, 1, 2) \tag{4.1}$$

for any $h \in H$. Since there exists $E \in \mathfrak{h}$ such that

$$Ad(\exp tE) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

on $\mathfrak{m}_2$. We have

$$[E, X] = JX \qquad (X \in \mathfrak{m}_2) \tag{4.2}$$

where J is an almost complex structure on $\mathfrak{m}_2$.

LEMMA 4.2. *There exists linear maps $L_1, L_2 : \mathfrak{m}_1 \to \boldsymbol{R}$ such that*

$$[A, X] = L_1(A)X + L_2(A)JX$$

for any $A \in \mathfrak{m}_1$ and any $X \in \mathfrak{m}_2$.

PROOF. For any fixed $A \in \mathfrak{m}_1$, we define a linear map $f_A : \mathfrak{m}_2 \to \mathfrak{g}$ by $f_A(X) = [A, X] (X \in \mathfrak{m}_2)$. By the same procedure as in the proof of Lemma 3.3, we have $\ker(p_0 f_A) = \{0\}$ or $\mathfrak{m}_2$. Suppose that $\ker(p_0 f_A) = \{0\}$. Then $p_0 f_A : \mathfrak{m}_2 \to \mathfrak{h}$ is injective, so $\dim \mathfrak{h} \geqq 2$ which contradicts the fact that $\dim \mathfrak{h} = 1$. Since A is arbitrary, we have $p_0[\mathfrak{m}_1, \mathfrak{m}_2] = \{0\}$. We can show $p_1[\mathfrak{m}_1, \mathfrak{m}_2] = \{0\}$ by the same way as in the proof of Lemma 3.3. Therefore we have Lemma 4.2 by Schur's Lemma. ∎

Let $A_1, \cdots, A_{n-2}$ be a basis of $\mathfrak{m}_1$ such that $L_2(A_j) = 0$ $(j \neq 1)$. Set $\mathfrak{m}_1' = \boldsymbol{R}\{A_1 - L_2(A_1)E, A_2, \cdots, A_{n-2}\}$. Then we have a new decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}'$ (where $\mathfrak{m}' = \mathfrak{m}_1' \oplus \mathfrak{m}_2$) of $\mathfrak{g}$ and we have

$$[A'X] = L_1(p_1(A')X \qquad (A' \in \mathfrak{m}_1', X \in \mathfrak{m}_2). \tag{4.3}$$

By the same procedure as in § 3, $\mathfrak{m}_2$ is spacelike and perpendicular to $\mathfrak{m}_1'$ and we have

LEMMA 4.2′. *There exists a linear map $L_1' : \mathfrak{m}_1' \to \boldsymbol{R}$ such that*

$$[A', X] = L_1'(A')X \qquad (A' \in \mathfrak{m}_1', X \in \mathfrak{m}_2).$$

We use the notation $\mathfrak{m}_1$, $\mathfrak{m}$ and L instead of $\mathfrak{m}_1'$, $\mathfrak{m}'$ and L_1' respectively.

LEMMA 4.3. $[\mathfrak{m}_1, \mathfrak{m}_1] \subseteqq \ker(L)$ *for* $n=4$.

PROOF. For any $A, B \in \mathfrak{m}_1$, we have

$$Ad(h)p_2[A, B]=p_2[Ad(h)A, Ad(h)B]$$
$$=p_2[A, B],$$

for any $h\in H$. Since $Ad(H)$ acts on $\mathfrak{m}_2$ irreducibly, we have $p_2[A, B]=0$. Thus we can set $[A, B]=\alpha E+C$ (for some $\alpha\in\boldsymbol{R}$ and for some $C\in\mathfrak{m}_1$). For any $X\in\mathfrak{m}_2$, we have

$$[X, [A, B]]=[[X, A], B]+[A, [X, B]]$$
$$=L(A)L(B)X-L(A)L(B)X=0$$

Thus we have

$$0=[X, [A, B]]=[X, \alpha E+C]=-\alpha JX-L(C)X.$$

Since X and JX are linearly independent, we have $\alpha=0$ and $L(C)=0$, so we have $[A, B]=C$ where $C\in\ker(L)$. ∎

LEMMA 4.4. (1) *If* $L\neq 0$, *then* $[\mathfrak{m}_2, \mathfrak{m}_2]=\{0\}$ (*resp.* $[\mathfrak{m}_2, \mathfrak{m}_2]\subseteq\ker(L)$) *for* $n=3$ (*resp.* $n=4$).

(2) *If* $L=0$, *then* $[\mathfrak{m}_2, \mathfrak{m}_2]\subseteq\mathfrak{h}\oplus\mathfrak{z}(\mathfrak{m}_1)$, *where* $\mathfrak{z}(\mathfrak{m}_1)$ *is a center of* $\mathfrak{m}_1$ *in* $\mathfrak{m}_1$.

.PROOF. For any $X, Y\in\mathfrak{m}_2$, we can set $[X, Y]=U+A(U\in\mathfrak{h}, A\in\mathfrak{m}_1)$ by the same way as the proof of Lemma 3.4. Then for $B\in\mathfrak{m}_1$, we have $2L(B)[X, Y]=[A, B]$. If $L\neq 0$, then $[X, Y]=[B, A]/2L(B)$ for a nonzero B, so we have (1). If $L=0$, then $[B, A]=0$ for any $B\in\mathfrak{m}_1$, so $A\in\mathfrak{z}(\mathfrak{m}_1)$. ∎

By the same way as the proof of Lemma 3.5, we have

LEMMA 4.5. *If* $L=0$, *then* $\mathfrak{z}(\mathfrak{m}_1)=\mathfrak{z}(\mathfrak{g})$ *where* $\mathfrak{z}(\mathfrak{g})$ *is a center of* $\mathfrak{g}$.

REMARK 4.6. If $n=4$, then $\dim\mathfrak{m}_1=2$, so $\mathfrak{z}(\mathfrak{m}_1)=\{0\}$ or $\mathfrak{m}_1$.

By the same way as in §3, we have Theorem B and Theorem C.

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