TSUKUBA J. MATH.
Vol. 16 No. 1 (1992), 235–256

THE L_2-WELLPOSED CAUCHY PROBLEM FOR SCHRÖDINGER TYPE EQUATIONS

By

Akio BABA

§1. Introduction.

We study the Cauchy problem for a Schrödinger type operator

$$P=P(x, t, D_x, D_t)=D_2+\frac{1}{2}\sum_{j=1}^{n}\Big(D_j-a_j(x, t)\Big)^2+c(x, t),$$

where $a(x, t)=(a_1(x, t), \cdots, a_n(x, t))$, $c(x, t)\in C_t^0([0, T_0]; \mathcal{B}^\infty(R^n))$, $(T_0>0)$, $D_t=-i\partial/\partial t$, $D_j=-i\partial/\partial x_j$ and $a_j(x, t)=a_j^R(x, t)+ia_j^I(x, t)$ ($a_j^R(x, t)$ and $a_j^I(x, t)$ are real valued functions in $R_x^n\times R_t$). Hence $\mathcal{B}^\infty(R^n)$ denotes the set of C^∞-functions whose derivatives of any order are all bounded and $g(x, t)\in C_t^k([0, T_0]; X)$ $(k=0, 1, 2, \cdots)$ means that the mapping: $[0, T_0]\ni t\to g(x, t)\in X$ is k-times continuously differentiable in the topology of X.

In this paper we give a sufficient condition for the Cauchy problem

$$(*)\qquad \begin{cases} P(x, t, D_x, D_t)u(x, t)=f(x, t) & (x, t)\in R^n\times[0, T] \quad (T>0), \\ u(x, 0)=u_0(x) & x\in R^n \end{cases}$$

to be L_2-wellposed.

We say that the Cauchy problem $(*)$ is L_2-wellposed if there exists $T>0$ such that for any initial data $u_0\in H_2$ and for any $f(x, t)\in C_t^0([0, T]; H_2)$ there exists a unique solution $u(x, t)\in C_t^1([0, T]; L_2)\cap C_t^0([0, T]; H_2)$ satisfying

$$\|u(\cdot, t)\|_{L_2}\leqq C(t)\Big\{\|u_0\|_{L_2}+\int_0^t\|f(\cdot, s)\|_{L_2}ds\Big\}\qquad t\in[0, T],$$

where $L_2=L_2(R^n)$, and H_2 is the Sobolev space which defined by $H_2=\{v\in\mathcal{S}'; \langle D_x\rangle^2v\in L_2(R^n)\}$. Here, $\mathcal{S}'$ is the dual space of the rapidly decreasing functions $\mathcal{S}$, and $\langle D_x\rangle^2$ is the pseudo-differential operator of symbol $\langle\xi\rangle^2$ $(\langle\xi\rangle=\sqrt{1+|\xi|^2})$.

When the coefficients $a_j(x)=a_j^R(x)+ia_j^I(x)$ $(j=1, \cdots, n)$ are independent of t, Mizohata [3], [4] proved that the condition

$$(C_0)\qquad \sup_{\rho\geqq 0,\, x\in R^n,\, \omega\in S^{n-1}}\left|\int_0^\rho\sum_{j=1}^{n}a_j^I(x-s\omega)\omega_j ds\right|<\infty$$

Received May 14, 1991. Revised October 8, 1991.

is necessary for the Cauchy problem $(*)$ to be L_2-wellposed. S^{n-1} denotes the unit sphere in R^n. He also proved (C_0) and the following condition

$$(C_\alpha) \qquad \begin{cases} |\alpha|\geqq 1, \\ \max_{1\leq j\leq n}\left\{\sup_{x\in R^n,\,\omega\in S^{n-1}}\int_0^\infty |D_x^\alpha a_j(x+s\omega)|\,ds\right\}<\infty, \end{cases}$$

are sufficient for the Cauchy problem $(*)$ to be L_2-wellposed. Takeuchi [5] proved that if there is $\varepsilon_0>0$ such that $a_j(x)=a_j^R(x)+ia_j^I(x)$ $(j=1, \cdots, n)$ satisfy

$$(T) \qquad \begin{cases} |D_x^\alpha a_j^I(x)|\leqq \dfrac{C_\alpha}{\langle x\rangle^{1+\varepsilon_0+|\alpha|}} & (all\ \alpha), \\ |D_x^\alpha a_j^R(x)|\leqq \dfrac{C_\alpha}{\langle x\rangle} & (|\alpha|\geqq 1), \end{cases}$$

for $x\in R^n$, then the Cauchy problem $(*)$ is L_2-wellposed.

In the present paper we give a sufficient condition for the Cauchy problem $(*)$ to be L_2-wellposed.

THEOREM. *Suppose that there are* ε_1, $\varepsilon_2>0$ *such that*

$$(S) \qquad \begin{cases} |D_x^\alpha a_j^I(x, t)|\leqq \begin{cases} \dfrac{C_0}{\langle x\rangle^{1+\varepsilon_1}} & (|\alpha|=0), \\ \dfrac{C_\alpha}{\langle x\rangle^{2+\varepsilon_1}} & (|\alpha|\geqq 1), \end{cases} \\ |D_x^\alpha a_j^R(x,t)|\leqq \dfrac{C_\alpha}{\langle x\rangle^{1+\varepsilon_2}} \qquad\qquad (|\alpha|\geqq 2), \end{cases}$$

for $(x, t)\in R^n\times[0, T_0]$ *and* $j=1, \cdots, n$. *Then the Cauchy problem* $(*)$ *is* L_2-*wellposed.*

We can see from the above Theorem that the condition for $a_j^R(x)$ in (T) is removed for $|\alpha|=1$.

To prove our Theorem, we reduce the operator P to an operator of which imaginary part of first order term vanishes identically, by use of pseudo-differential operator of type $S_{0,0}^0$. To do so, we need some properties of a solution of a characteristic equation for P. In §2 we investigate the characteristic equations, in §3 conjugate P by a pseudo-differential operator $e^{\Lambda}(x, D_x)$ of type $S_{0,0}^0$ and in §4 prove that the Cauchy problem $(*)$ for P with $a_j^I\equiv 0$ is L_2-wellposed.

§2. Properties of characteristic curve.

We consider a characteristic equation for P

$$\Lambda_t(x, \xi; t)+\xi\cdot\Lambda_x(x, \xi; t)+a_x^R(x, t, \xi)\cdot\Lambda_\xi(x, \xi; t)+a^I(x, t, \xi)=0, \tag{2.1}$$

where

$$a^R(x, t, \xi)=\sum_{j=1}^n a_j^R(x, t)\xi_j, \qquad a^I(x, t, \xi)=\sum_{j=1}^n a_j^I(x, t)\xi_j,$$

$$a_x^R(x, t, \xi)=\Big(\sum_{j=1}^n a_{j,x_1}^R(x, t)\xi_j, \cdots, \sum_{j=1}^n a_{j,x_n}^R(x, t)\xi_j\Big),$$

$$x\cdot\xi=\sum_{j=1}^n x_j\xi_j, \qquad (x, \xi\in R^n).$$

The characteristic curve $(x(t), \xi(t))=(x(y, \eta, t), \xi(y, \eta, t))$ for (2.1) is defined by

$$\begin{cases} \dot{x}=\xi, & x(0)=y, \\ \dot{\xi}=a_x^R(x, t, \xi), & \xi(0)=\eta, \end{cases} \tag{2.2}$$

where $\dot{x}=\partial x(t)/\partial t$, $\dot{\xi}=\partial\xi(t)/\partial t$.

Put

$$\tilde{\Lambda}(y, \eta; t)=-\int_0^t a^I(x(s), s, \xi(s))ds.$$

Then we can prove the following

LEMMA 2.1. *Assume that the conditions (S) in Theorem are valid. Then the solution of (2.2) satisfies the following properties.*

(i) $$x(y, \eta, t), \qquad \xi(y, \eta, t)\in C^\infty(R^{2n}\times[0, T_0]).$$

(ii) *There is a constant C such that*

$$C^{-1}|t\eta|\leqq|x(y, \eta, t)-y|\leqq C|t\eta|,$$
$$C^{-1}|\eta|\leqq|\xi(y, \eta, t)|\leqq C|\eta|, \qquad (y, \eta, t)\in R^{2n}\times[0, T_0].$$

(iii) *There are $T>0$ $(T\leqq T_0)$ and $C>0$ such that*

$$\int_0^t \frac{|\xi(y, \eta, s)|}{\langle x(y, \eta, s)\rangle^{1+\varepsilon_2}}ds\leqq C, \qquad (y, \eta, t)\in R^{2n}\times[0, T].$$

(iv) *There is $T>0$ $(T\leqq T_0)$ such that for $|\alpha+\beta|\geqq 1$*

$$x_{(\beta)}^{(\alpha)}(y, \eta, t), \qquad \xi_{(\beta)}^{(\alpha)}(y, \eta, t)\in\mathcal{B}_t^0([0, T]; S_{0,0}^0)\cap\mathcal{B}_t^1([0, T]; S_{0,0}^1),$$

where $x_{(\beta)}^{(\alpha)}(y, \eta, t)=\partial_\eta^\alpha D_\eta^\beta x(y, \eta, t)$, $\xi_{(\beta)}^{(\alpha)}(y, \eta, t)=\partial_\eta^\alpha D_\eta^\beta\xi(y, \eta, t)$, $\mathcal{B}_t^k([0, T]; X)$ *is the set of symbols of which k-th derivatives are bounded in X for $t\in[0, T]$ and $S_{\rho,\delta}^m$ is the symbol class of pseudo-differential operator which defined by* $S_{\rho,\delta}^m=\{p(x, \xi)\in C^\infty(R^{2n});\ |p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$ *in* $R^{2n}\}$, *for some ρ and δ between* 0 *and* 1.

(v) *There exist* $T>0$ $(T\leqq T_0)$ *and* $C>0$ *such that*

$$C^{-1}\leqq\left|\frac{\partial(x(y,\eta,t),\xi(y,\eta,t))}{\partial(y,\eta)}\right|\leqq C,\qquad (y,\eta,t)\in R^{2n}\times[0,T].$$

(vi) *There is* $T>0$ $(T\leqq T_0)$ *such that*

$$\tilde{\Lambda}(y,\eta,t)\in\mathcal{B}_t^0([0,T];S_{0,0}^0)\cap\mathcal{B}_t^1([0,T];S_{0,0}^1).$$

PROOF. (i) This fact is well known (see [1], Chapter 1, Theorem 4.2).

(ii) From (2.2) we have

$$\begin{aligned}2|\xi(t)|\frac{d}{dt}|\xi(t)|=\frac{d}{dt}|\xi(t)|^2&\\ &\leqq 2|\xi(t)||\dot{\xi}(t)|\\ &\leqq 2|\xi(t)|^2|a_x^R(x(t),t)|,\end{aligned}$$

which shows

$$\begin{aligned}|\xi(t)|&\leqq|\xi(0)|\exp\left[\int_0^t|a_x^R(x(s),s)|ds\right]\\ &\leqq C|\eta|.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\frac{d}{dt}|\xi(t)|^2&\geqq -2|\xi(t)||\dot{\xi}(t)|\\ &\geqq -2|\xi(t)|^2|a_x^R(x(t),t)|\end{aligned}$$

and

$$\begin{aligned}|\xi(t)|&\geqq|\xi(0)|\exp\left[-\int_0^t|a_x^R(x(s),s)|ds\right]\\ &\geqq C'|\eta|.\end{aligned}$$

Therefore, we can get

$$C|\eta|\geqq|\xi(t)|\geqq C^{-1}|\eta|.$$

From (2.2) and the mean value theorem, there exists a θ $(0<\theta<1)$ such that

$$\begin{aligned}x_j(t)&=y_j+\int_0^t\xi_j(y,\eta,s)ds\\ &=y_j+t\xi_j(y,\eta,\theta t),\end{aligned}$$

for $j=1,\cdots,n$. Therefore we get

$$C^{-1}|t\eta|\leqq|x(t)-y|\leqq C|t\eta|.$$

(iii) From (ii) we have

$$\sup_{y\in R^n,\eta\in R^n}\int_0^t \frac{|\xi(y,\eta,s)|}{\langle x(s)\rangle^{1+\varepsilon_2}}ds$$
$$\leqq \sup_{y\in R^n,|\eta|\geqq 1}\int_0^t \frac{|\xi(y,\eta,s)|}{\langle x(s)\rangle^{1+\varepsilon_2}}ds+\sup_{y\in R^n,|\eta|\leqq 1}\int_0^t \frac{|\xi(y,\eta,s)|}{\langle x(s)\rangle^{1+\varepsilon_2}}ds$$
$$\leqq n\max_{1\leqq i\leqq n}\left\{\sup_{y\in R^n,|\eta_i|\geqq|\eta|/\sqrt{n}}\int_0^t \frac{|\xi(y,\eta,s)|}{\langle x(s)\rangle^{1+\varepsilon_2}}ds\right\}+C\,.$$

When $|\eta_i|/|\eta|\geqq 1/\sqrt{n}$, we have

$$\left|\frac{\xi_i}{|\eta|}-\frac{\eta_i}{|\eta|}\right|=\left|\frac{\int_0^t a_{x_i}^R(x(s),s,\xi(s))ds}{|\eta|}\right|$$
$$\leqq CT\leqq\frac{1}{2\sqrt{n}}$$

for $T\leqq 1/2\sqrt{n}C$. Hence we obtain $|\xi_i|\geqq|\eta|/2\sqrt{n}$. Put $\sigma=\int_0^s\xi_i(\tau)d\tau$.

Since $x(y,\eta,s)=y+\int_0^s\xi(\tau)d\tau$, we have

$$\int_0^t\frac{|\xi(y,\eta,s)|}{\langle x(y,\eta,s)\rangle^{1+\varepsilon_2}}ds\leqq\int_0^t\frac{C|\eta|}{\left\langle y+\int_0^s\xi(\tau)d\tau\right\rangle^{1+\varepsilon_2}}ds$$
$$\leqq\int_{-\infty}^{\infty}\frac{C}{\langle y_i+\sigma\rangle^{1+\varepsilon_2}}\frac{|\eta|}{|\xi_i(s)|}d\sigma$$
$$\leqq\int_{-\infty}^{\infty}\frac{C'}{\langle y_i+\sigma\rangle^{1+\varepsilon_2}}d\sigma\leqq C''\,.$$

This completes the proof of (iii).

(iv) Taking the derivative of (2.2) with respect to y,

$$\text{(2.3)}\qquad\begin{cases}\dot{x}_y=\xi_y\,,\\ \dot{\xi}_y=a_{xx}^R(x,t,\xi)\times x_y+a_x^R(x,t)\times\xi_y\,,\end{cases}$$

where

$$\xi_y=\begin{pmatrix}\partial\xi_1(t)/\partial y_1 & \cdots & \partial\xi_1(t)/\partial y_n\\ \vdots & \ddots & \vdots\\ \partial\xi_n(t)/\partial y_1 & \cdots & \partial\xi_n(t)/\partial y_n\end{pmatrix},$$

$$a_{xx}^R(x,t,\xi)\times x_y$$
$$=\begin{pmatrix}a_{x_1x_1}^R(x,t,\xi) & \cdots & a_{x_1x_n}^R(x,t,\xi)\\ \vdots & \ddots & \vdots\\ a_{x_nx_1}^R(x,t,\xi) & \cdots & a_{x_nx_n}^R(x,t,\xi)\end{pmatrix}\begin{pmatrix}\partial x_1(t)/\partial y_1 & \cdots & \partial x_1(t)/\partial y_n\\ \vdots & \ddots & \vdots\\ \partial x_n(t)/\partial y_1 & \cdots & \partial x_n(t)/\partial y_n\end{pmatrix}$$

and

$$a_x^R(x, t)\times\xi_y=\begin{pmatrix} a_{1x_1}^R(x, t) & \cdots & a_{nx_1}^R(x, t) \\ \vdots & \ddots & \vdots \\ a_{1x_n}^R(x, t) & \cdots & a_{nx_n}^R(x, t) \end{pmatrix} \begin{pmatrix} \partial\xi_1(t)/\partial y_1 & \cdots & \partial\xi_1(t)/\partial y_n \\ \vdots & \ddots & \vdots \\ \partial\xi_n(t)/\partial y_1 & \cdots & \partial\xi_n(t)/\partial y_n \end{pmatrix}.$$

Let

$$\rho(t)=\sqrt{|x_y(t)|^2+|\xi_y(t)|^2},$$

where $|x_y(t)|=\{\sum_{i,j=1}^n|\partial x_i(t)/\partial y_j|^2\}^{1/2}$, $|\xi_y(t)|=\{\sum_{i,j=1}^n|\partial\xi_i(t)/\partial y_j|^2\}^{1/2}$.

Then we have

$$\begin{aligned}\frac{d}{dt}\rho(t)^2&=\frac{d}{dt}|x_y(t)|^2+\frac{d}{dt}|\xi_y(t)|^2\\ &\leqq 2|x_y||\dot{x}_y(t)|+2|\xi_y||\dot{\xi}_y(t)|\\ &\leqq 2|x_y||\xi_y(t)|+2|\xi_y|(|a_{xx}^R(x, t, \xi)||x_y|+|a_x^R(x, t)\times\xi_y|)\\ &\leqq 2\rho(t)^2\{1+|a_{xx}^R(x, t, \xi)|+|a_x^R(x, t)|\}\\ &\leqq 2C\rho(t)^2\{1+\alpha(t)\}.\end{aligned}$$

where $\alpha(t)=|a_{xx}^R(x(t), t, \xi(t))|$.

Therefore

$$\frac{d}{dt}\rho(t)\leqq C\rho(t)(1+\alpha(t)).$$

Since $\rho(0)=(|x_y(0)|^2+|\xi_y(0)|^2)^{1/2}=\sqrt{n}$, we get

$$\rho(t)\leqq\rho(0)\exp\left[C\int_0^t\alpha(s)ds\right]=\sqrt{n}\exp\left[C\int_0^t\alpha(s)ds\right].$$

By (iii) and (S) we have

$$\int_0^t\alpha(s)ds=\int_0^t\frac{C|\xi(y, \eta, s)|}{\langle x(y, \eta, s)\rangle^{1+\varepsilon_2}}ds\leqq C'.$$

Thus we have $\rho(t)\leqq C$ which yields $|x_y|$, $|\xi_y|\leqq C$. Similarly, $|x_\eta|$, $|\xi_\eta|\leqq C$.

For $l\geqq 1$ we suppose

$$|x_{(\beta)}^{(\alpha)}(y, \eta, t)|, \qquad |\xi_{(\beta)}^{(\alpha)}(y, \eta, t)|\leqq C$$

for $1\leqq|\alpha+\beta|\leqq l$. It follows from (2.2)

$$\partial_y\partial_\eta^\alpha D_y^\beta\dot{x}(y, \eta, t)=\partial_y\partial_\eta^\alpha D_y^\beta\xi(y, \eta, t).$$

We put $f(y, \eta, t)=a_{xx}^R(x(y, \eta, t), t, \xi(y, \eta, t))$ and $g(y, \eta, t)=a_x^R(x(y, \eta, t), t)$. Then we have

$$\partial_y \partial_\eta^\alpha D_y^\beta \dot{\xi}(y,\eta,t)=\partial_\eta^\alpha D_y^\beta(f\times x_y+g\times\xi_y)$$
$$=\sum_{\substack{\alpha^1+\alpha^2=\alpha\\ \beta^2+\beta^2=\beta}} C_{\beta^1\beta^2}^{\alpha^1\alpha^2} f_{(\beta^1)}^{(\alpha^1)}(y,\eta,t)\times\partial_\eta^{\alpha^2}D_y^{\beta^2}x_y(y,\eta,t)$$
$$+\sum_{\substack{\alpha^3+\alpha^4=\alpha\\ \beta^3+\beta^4=\beta}} C_{\beta^3\beta^4}^{\alpha^3\alpha^4} g_{(\beta^3)}^{(\alpha^3)}(y,\eta,t)\times\partial_\eta^{\alpha^4}D_y^{\beta^4}\xi_y(y,\eta,t),$$

where

$$|f_{(\beta)}^{(\alpha)}(y,\eta,t)|\leqq C_{\alpha\beta}\frac{1+|\xi|}{\langle x(t)\rangle^{1+\varepsilon_2}},$$

$$|g_{(\beta)}^{(\alpha)}(y,\eta,t)|\leqq C_{\alpha\beta}\frac{1}{\langle x(t)\rangle^{1+\varepsilon_2}}.$$

Let

$$\tilde{\rho}(t)=\{|\partial_y x_{(\beta)}^{(\alpha)}(y,\eta,t)|^2+|\partial_y\xi_{(\beta)}^{(\alpha)}(y,\eta,t)|^2\}^{1/2}.$$

Then we have

$$\frac{d}{dt}\tilde{\rho}(t)^2=\frac{d}{dt}|\partial_y x_{(\beta)}^{(\alpha)}(y,\eta,t)|^2+\frac{d}{dt}|\partial_y\xi_{(\beta)}^{(\alpha)}(y,\eta,t)|^2$$
$$\leqq 2|\partial_y x_{(\beta)}^{(\alpha)}(y,\eta,t)|\,|\partial_y \dot{x}_{(\beta)}^{(\alpha)}(y,\eta,t)|+2|\partial_y\xi_{(\beta)}^{(\alpha)}(y,\eta,t)|\,|\partial_y\dot{\xi}_{(\beta)}^{(\alpha)}(y,\eta,t)|$$
$$\leqq 2\tilde{\rho}(t)^2+2\tilde{\rho}(t)\{C\alpha(t)+C'\alpha(t)\tilde{\rho}(t)\}.$$

Consequently,

$$\tilde{\rho}(t)\leqq\int_0^t(C\alpha(s))ds+\int_0^t(1+C'\alpha(s))\tilde{\rho}(s)ds$$
$$\leqq C+\int_0^t(1+C'\alpha(s))\tilde{\rho}(s)ds.$$

Therefore, by Gronwall's inequality we have

$$\tilde{\rho}(t)\leqq C\exp\left[\int_0^t(1+C'\alpha(s))ds\right]\leqq\tilde{C},$$

which implies

$$|\partial_y x_{(\beta)}^{(\alpha)}(y,\eta,t)|,\ |\partial_y\xi_{(\beta)}^{(\alpha)}(y,\eta,t)|\leqq C.$$

Similarly, we obtain

$$|\partial_\eta x_{(\beta)}^{(\alpha)}(y,\eta,t)|,\ |\partial_\eta\xi_{(\beta)}^{(\alpha)}(y,\eta,t)|\leqq C.$$

Thus $x_{(\beta)}^{(\alpha)}(y,\eta,t)$ and $\xi_{(\beta)}^{(\alpha)}(y,\eta,t)$ are in $\mathcal{B}_t^0([0,T];S_{0,0}^0)$ and (2.2) implies that $x_{(\beta)}^{(\alpha)}(y,\eta,t)$ and $\xi_{(\beta)}^{(\alpha)}(y,\eta,t)$ are in $\mathcal{B}_t^1([0,T];S_{0,0}^1)$. This completes the proof of (iv).

(v) Let

$$X(t)=\begin{pmatrix}\partial x(t)/\partial y & \partial x(t)/\partial\eta\\ \partial\xi(t)/\partial y & \partial\xi(t)/\partial\eta\end{pmatrix},$$

where

$$X(0)=\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then taking the derivative of (2.2) with respect to y and η we have

$$\dot{X}(t)=\begin{pmatrix} 0 & I \\ a_{xx}^R(x(t), t, \xi) & a_x^R(x(t), t) \end{pmatrix} X(t) \equiv A(t)X(t),$$

where $\dot{X}(t)=\partial X(t)/\partial t$.

On the other hand,

$$\int_0^t |A(s)|\,ds \leqq C\int_0^t (1+\alpha(s))ds \leqq C_1,$$

where $|A(t)|=\{\sum_{i,j=1}^{2n}(a_{ij}(t))^2\}^{1/2}$ and $a_{ij}(t)$ $(i, j=1, \cdots, 2n)$ are components of $A(t)$.

Consequently,

$$\begin{aligned} \frac{d}{dt}|X(t)|^2 &= 2X(t)\dot{X}(t) \\ &\geqq -2|A(t)|\,|X(t)|^2, \end{aligned}$$

and

$$\begin{aligned} |X(t)| &\geqq |X(0)| \exp\left[-\int_0^t |A(s)|\,ds\right] \\ &\geqq C|X(0)|=C'. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{d}{dt}|X(t)|^2 &= 2X(t)\dot{X}(t) \\ &\leqq 2|A(t)|\,|X(t)|^2, \end{aligned}$$

and

$$\begin{aligned} |X(t)| &\leqq |X(0)| \exp\left[\int_0^t |A(s)|\,ds\right] \\ &\leqq C|X(0)|=C''. \end{aligned}$$

This completes the proof of (v).

(iv) We have

$$\tilde{A}_{\{\beta\}}^{\{\alpha\}}(y, \eta, t) = \sum_{j=1}^{n} \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ \beta^1+\beta^2=\beta}} C_{\beta^1\beta^2}^{\alpha^1\alpha^2} \int_0^t \partial_\eta^{\alpha^1} D_y^{\beta^1} a_j^I(x(y, \eta, s), s)\xi_{j\{\beta^2\}}^{\{\alpha^2\}}(y, \eta, s)ds,$$

where

$$|\partial_\eta^{\alpha^1} D_y^{\beta^1} a^I(x(y, \eta, s), s)| \leqq \frac{C_{\alpha\beta}}{\langle x(y, \eta, s)\rangle^{1+\varepsilon_1}}.$$

Then

$$|\tilde{\Lambda}^{\{\alpha\}}_{\{\beta\}}(y, \eta, t)| \leqq C_{\alpha\beta}\int_0^t (\alpha(s)+1)ds \leqq C\,.$$

It implies (vi).

Let $y(x, \xi, t)$, $\eta(x, \xi, t)$ be the inverse functions of $x(y, \eta, t)$, $\xi(y, \eta, t)$ and put

(2.4) $$\Lambda(x, \xi; t) = \tilde{\Lambda}(y(x, \xi, t), \eta(x, \xi, t); t).$$

LEMMA 2.2. *Assume that the condition (S) in Theorem is valid. Then there is $T>0$ $(T \leqq T_0)$ such that*

(i) $y(x, \xi, t)$, $\eta(x, \xi, t) \in C^\infty(R^{2n} \times [0, T])$

(ii) $y^{\{\alpha\}}_{\{\beta\}}(x, \xi, t)$, $\eta^{\{\alpha\}}_{\{\beta\}}(x, \xi, t) \in \mathscr{B}^0_t([0, T]; S^0_{0,0}) \cap \mathscr{B}^1_t([0, T]; S^1_{0,0})$, $(|\alpha+\beta| \geqq 1)$

(iii) $|\partial^\alpha_\xi D^\beta_x x(y(x, \xi, t), \eta(x, \xi, t), s)| \leqq C_\alpha |t-s|$, $(|\alpha| \geqq 1$, *all* $\beta)$

(v) $$\begin{cases} \Lambda(x, \xi, t) \in \mathscr{B}^0_t([0, T]; S^0_{0,0}) \cap \mathscr{B}^1_t([0, T]; S^1_{0,0}) \\ |\Lambda^{\{\alpha\}}_{\{\beta\}}(x, \xi, t)| \leqq \begin{cases} C_{\alpha\beta} t, & (|\alpha| \geqq 1) \\ C_{0\beta}, & (|\alpha|=0). \end{cases} \end{cases}$$

i.e. $\Lambda(x, \xi; t) \in S^0_{0,0}$.

PROOF. (i) It is well known the proof of the differentiability of solutions with respect to parameters (see [1], Chapter 1, Theorem 4.2).

(ii) Set

$$F = \begin{pmatrix} x(y, \eta) - x \\ \xi(y, \eta) - \xi \end{pmatrix}.$$

Then,

$$\frac{\partial F}{\partial(y, \eta)} = \frac{\partial(x, \xi)}{\partial(y, \eta)}, \qquad \frac{\partial F}{\partial(x, \xi)} = -I\,.$$

By the implicit function theorem,

$$\frac{\partial(y, \eta)}{\partial(x, \xi)} = -\left(\frac{\partial F}{\partial(y, \eta)}\right)^{-1} \times \left(\frac{\partial F}{\partial(x, \xi)}\right) = \left(\frac{\partial(x, \xi)}{\partial(y, \eta)}\right)^{-1}.$$

Thus the conclusions of (ii) follow from (iv) and (v) of Lemma 2.1.

(iii) Since $x(y(x, \xi, t), \eta(x, \xi, t), t) = x$, we have for $|\alpha| \geqq 1$,

$$\begin{aligned}
&|\partial_\xi^\alpha D_x^\beta x(y(x,\xi,t),\eta(x,\xi,t),s)| \\
&=\left|\partial_\xi^\alpha D_x^\beta\left\{x+\int_t^s \dot{x}(y(x,\xi,t),\eta(x,\xi,t),\tau)d\tau\right\}\right| \\
&=\left|\partial_\xi^\alpha D_x^\beta\left\{\int_t^s \xi(y(x,\xi,t),\eta(x,\xi,t),\tau)d\tau\right\}\right| \\
&\leqq C_{\alpha\beta}|t-s|.
\end{aligned}$$

(iv) The estimate (iii) and (S) imply

$$\begin{aligned}
&|D_x^\beta\partial_\xi^\alpha \Lambda(x,\xi,t)| \\
&=|D_x^\beta\partial_\xi^\alpha \tilde{\Lambda}(y(x,\xi,t),\eta(x,\xi,t),t)| \\
&=\left|D_x^\beta\partial_\xi^\alpha\int_0^t a^I(x(y(x,\xi,t),\eta(x,\xi,t),s),s)\cdot\xi(y(x,\xi,t),\eta(x,\xi,t),s)ds\right|
\end{aligned}$$

$$\begin{cases}
\leqq C_{\alpha\beta}\left\{\int_0^t \frac{|(t-s)\xi(s)|}{\langle x(s)\rangle^{2+\varepsilon_1}}ds+\int_0^t\frac{ds}{\langle x(s)\rangle^{1+\varepsilon_1}}\right\}\leqq C_{\alpha\beta}t & if\ |\alpha|\geqq 1, \\
\leqq C_{\alpha\beta}\left\{\int_0^t \frac{|\xi(s)|}{\langle x(s)\rangle^{2+\varepsilon_1}}ds+\int_0^t\frac{ds}{\langle x(s)\rangle^{1+\varepsilon_1}}\right\}\leqq C_{0\beta} & if\ |\alpha|=0.
\end{cases}$$

This implies (iv).

§3. Transform by e^{Λ}.

Let $\sigma(K)(x,\xi;t)=e^{\Lambda(x,\xi;t)}$ and $\sigma(\tilde{K})(x,\xi;t)=e^{-\Lambda(x,\xi;t)}$. Where $\Lambda(x,\xi;t)$ is given in (2.4). Then, we have

$$\begin{aligned}
\sigma(\tilde{K}\circ K)(x,\xi;t)&=Os-\iint e^{-iy\cdot\eta}e^{-\Lambda(x,\xi+\eta;t)}e^{\Lambda(x+y,\xi;t)}dy\bar{d}\eta\,, \\
&=1+\int_0^1\sum_{|\gamma|=1}Os-\iint e^{-iy\cdot\eta}\{-D_\eta^\gamma\Lambda(x,\xi+\eta;t)\}e^{-\Lambda(x,\xi+\eta;t)} \\
&\quad\times\partial_x^\gamma\Lambda(x+\theta y,\xi;t)e^{\Lambda(x+\theta y,\xi;t)}dy\bar{d}\eta d\theta \\
&\equiv 1+\sigma(R)(x,\xi;t),
\end{aligned}$$

where

$$\begin{aligned}
\sigma(R)(x,\xi;t)=\int_0^1\sum_{|\gamma|=1}Os-\iint e^{-iy\cdot\eta}\{-D_\eta^\gamma\Lambda(x,\xi+\eta;t)\}e^{-\Lambda(x,\xi+\eta;t)} \\
\times\partial_x^\gamma\Lambda(x+\theta y,\xi;t)e^{\Lambda(x+\theta y,\xi;t)}dy\bar{d}\eta d\theta\,.
\end{aligned}$$

Here

$$\bar{d}\eta=(2\pi)^{-n}d\eta=(2\pi)^{-n}d\eta_1\cdots d\eta_n\,,$$

and

$$Os-\iint e^{-iy\cdot\eta}a(y,\eta)dy\bar{d}\eta=\lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon\eta,\varepsilon y)a(y,\eta)dy\bar{d}\eta\,,$$

for $\chi \in \mathcal{S}$ in R^{2n} such that $\chi(0, 0)=1$. Then we can get the following Proposition 3.1 from Lemma 2.2 (iv).

PROPOSITION 3.1. *Assume that the same condition (S) as in Theorem is valid. Then, for any α, β ($|\alpha| \geqq 1$) we have*

$$|r_{(\beta)}^{(\alpha)}(x, \xi; t)| \leqq C_{\alpha\beta} t, \qquad (t \in [0, T]), \tag{3.1}$$

where $r(x, \xi; t)=\sigma(R)(x, \xi; t)$.

From (3.1) and the Calderón-Vaillancourt theorem, we obtain $\|R(x, D_x; t)\|_{L_2} < 1$ for $t \in [0, T]$, if we take $T>0$ sufficiently small. Hence we can define

$$Q(x, D_x; t)=\sum_{j=0}^{\infty}(-R(x, D_x; t))^j \tag{3.2}$$

which converges in the sense of L_2 norm. Moreover by virtue of estimates of the symbols of multiple products of pseudo-differential operators we can show that (3.2) is convergent in the symbol class $S_{0,0}^0$.

PROPOSITION 3.2 (Kumano-go [2]). *Let $q_j(x, \xi)$ $(j=1, \cdots, \nu+1)$ be in $S_{0,0}^0$. Define for $\nu \geqq 0$*

$$\begin{aligned} p_{\nu+1}(x, \xi) &\\ = Os-&\iint \exp\Big(-i\sum_{j=1}^{\nu} y^j \cdot \eta^j\Big) \cdot \prod_{j=1}^{\nu+1} q_j(x+\bar{y}^{j-1}, \xi+\eta^j) dy^1 \cdots dy^\nu đ\eta^1 \cdots đ\eta^\nu, \end{aligned} \tag{3.3}$$

where $\bar{y}^0=0$, $\bar{y}^j=y^1+\cdots+y^j$, $(j=1, \cdots, \nu)$, $\eta^{\nu+1}=0$ and $y^j, \eta^j \in R^n$. Then there is $C>0$ independent of ν such that

$$|p_{\nu+1}(x, \xi)| \leqq C^{\nu+1} \prod_{j=1}^{\nu+1} |q_j|_{n_0}^{(0)}, \qquad n_0=2[n/2+1], \tag{3.4}$$

for $x, \xi \in R^n$.

Define $p_{\nu+1}(x, D_x; t)=(-R(x, D_x; t))^{\nu+1}$. Then it's symbol $p_{\nu+1}(x, \xi; t)$ is given by (3.3) with $q_j=-r(x, \xi; t)$ $(j=1, \cdots, \nu+1)$. Moreover $p_{\nu+1}{}_{(\beta)}^{(\alpha)}(x, \xi; t)$ is given by (3.3) with $q_j=r_{(\beta^j)}^{(\alpha^j)}(x, \xi; t)$ $(\sum\alpha^j=\alpha, \sum\beta^j=\beta)$. Therefore we have by virtue of (3.4)

$$|p_{\nu+1}{}_{(\beta)}^{(\alpha)}(x, \xi; t)| \leqq \begin{cases} C^{\nu+1}(|r|_{|\alpha+\beta|+n_0}^{(0)})^{\nu+1} & \text{if } |\alpha+\beta|>\nu, \\ C^{\nu+1}(|r|_{|\alpha+\beta|+n_0}^{(0)})^{|\alpha+\beta|}(|r|_{n_0}^{(0)})^{\nu+1-|\alpha+\beta|} & \text{if } |\alpha+\beta| \leqq \nu \end{cases}$$

for $x, \xi \in R^n$.

From (3.1) we have $|r|_{n_0}^{(0)} \leqq C_0 t$ $(t \in [0, T])$. Hence

$$\left|\sum_{\nu=0}^{\infty} p_{\nu+1}{}^{(\alpha)}_{(\beta)}(x,\xi;t)\right| \leqq \sum_{\nu=0}^{|\alpha+\beta|} |p_{\nu+1}{}^{(\alpha)}_{(\beta)}| + \sum_{\nu=|\alpha+\beta|+1}^{\infty} |p_{\nu+1}{}^{(\alpha)}_{(\beta)}|$$

$$\leqq C_{\alpha\beta}\Big(1+\sum_{\nu+1}^{\infty}(C_0 t)^{\nu}\Big) \leqq C'_{\alpha\beta}, \qquad t\in[0,T]$$

if T is sufficiently small, which implies

$$Q(x,\xi;t)=\sum_{\nu=0}^{\infty} p_{\nu+1}(x,\xi;t)+1 \in \mathscr{B}_t^0([0,T];S^0_{0,0}) \cap \mathscr{B}_t^1([0,T];S^1_{0,0}).$$

Now we can construct the inverse K^{-1} of $K(x,D_x;t)$ as follows

$$K(x,D_x;t)^{-1}=(1+R(x,D_x;t))^{-1}\tilde{K}(x,D_x;t),$$

where $\tilde{K}=e^{-\Lambda(x,\xi;t)}$. The symbols of $K^{-1}(x,D_x;t)$ is in $\mathscr{B}_t^0([0,T];S^0_{0,0}) \cap \mathscr{B}_t^1([0,T];S^1_{0,0})$.

Put $u(x,t)=Kv(x,t)$. We have

$$Pu(x,t)=P\circ Kv(x,t)$$

$$=\int e^{ix\cdot\xi}e^{\Lambda(x,\xi;t)}\Big\{D_t\Lambda+\sum_{j=1}^n \xi_j(D_j\Lambda-ia_j^I)+D_t+\frac{1}{2}\sum_{j=1}^n(\xi_j-a_j^R)^2$$

$$+c(x)+\frac{1}{2}\sum_{j=1}^n(D_j\Lambda-a_j)^2+\frac{1}{2}\sum_{j=1}^n D_j(D_j\Lambda-a_j)-\frac{1}{2}\sum_{j=1}^n(a_j^R)^2\Big\}$$

$$\times\hat{v}(\xi,t)\bar{d}\xi.$$

On the other hand,

$$K\circ\Big\{D_t+\frac{1}{2}\sum_{j=1}^n(D_j-a_j^R)^2\Big\}v(x,t)$$

$$=K\circ\Big\{D_t+\frac{1}{2}\sum_{j=1}^n(D_j^2-2a_j^RD_j+(a_j^R)^2-D_ja_j^R)\Big\}v(x,t)$$

$$=\int e^{ix\cdot\xi+\Lambda(x,\xi;t)}\Big(D_t+\frac{1}{2}\sum_{j=1}^n\xi_j^2\Big)\hat{v}(\xi,t)\bar{d}\xi$$

$$-\sum_{j=1}^n K\circ(a_j^R(x,t)D_j)v(x,t)+\frac{1}{2}\sum_{j=1}^n K\circ((a_j^R)^2-D_ja_j^R)v(x,t),$$

and

$$\sigma(K\circ a_j^RD_j)(x,\xi;t)$$

$$=\sum_{j=1}^n Os-\iint e^{-iy\cdot\eta}e^{\Lambda(x,\xi+\eta;t)}a_j^R(x+y,t)\xi_j\,dy\,\bar{d}\eta$$

$$=\sum_{j=1}^n e^{\Lambda(x,\xi;t)}a_j^R(x,t)\xi_j+\sum_{j=1}^n\sum_{|\gamma|=1}(e^{\Lambda(x,\xi;t)})^{(\gamma)}(a_j^R(x,t)\xi_j)_{(\gamma)}+b(x,\xi;t),$$

where

$$b(x, \xi ; t)$$

$$=2\sum_{j=1}^{n}\sum_{|\gamma|=2}\int_0^1\frac{1-\theta}{\gamma !}\left\{Os-\iint e^{-iy\cdot\eta}(e^{\Lambda(x,\xi+\eta;t)})^{(\gamma)}a^R_{j(\gamma)}(x+\theta y, t)\xi_j dy đ\eta\right\}d\theta ,$$

$$\sigma\Big(\sum_{j=1}^{n}K\circ(a_j^R)^2\Big)(x, \xi ; t)=\sum_{j=1}^{n} Os-\iint e^{-iy\cdot\eta}e^{\Lambda(x,\xi+\eta;t)}(a_j^R(x+y, t))^2 dy đ\eta$$

$$=\sum_{j=1}^{n} e^{\Lambda(x,\xi;t)}(a_j^R(x, t))^2+c_1(x, \xi ; t)$$

and

$$c_1(x, \xi ; t)=\sum_{j=1}^{n}\sum_{|\gamma|=1}\int_0^1 Os-\iint e^{-iy\cdot\eta}(e^{\Lambda(x,\xi+\eta;t)})^{(\gamma)}(a_j^R(x+\theta y, t))^2_{(\gamma)}dy đ\eta d\theta .$$

Therefore we have

$$K^{-1}\circ P\circ Kv(x, t)=\left\{D_t+\frac{1}{2}\sum_{j=1}^{n}(D_j-a_j^R)^2\right\}v(x, t)+\tilde{c}(x, D_x ; t)v(x, t),$$

where

(3.5) $\tilde{c}(x, D_x ; t)v(x, t)$

$$=K^{-1}\circ\int e^{ix\cdot\xi+\Lambda(x,\xi;t)}(-i)\{\Lambda_t(x, \xi ; t)+\xi\cdot\Lambda_x(x, \xi ; t)+a_x^R(x, t, \xi)$$

$$\cdot\Lambda_\xi(x, \xi ; t)+a^I(x, t, \xi)\}\hat{v}(\xi, t)đ\xi$$

$$+K^{-1}\circ b(x, D_x ; t)v(x, t)$$

$$+\frac{1}{2}\sum_{j=1}^{n}D_j a_j^R(x, t)v(x, t)-\frac{1}{2}K^{-1}\circ c_1(x, D_x ; t)v(x, t)$$

$$+\sum_{j=1}^{n}K^{-1}\circ\int e^{ix\cdot\xi+\Lambda(x,\xi;t)}\left\{c(x)+\frac{1}{2}(D_j\Lambda-a_j)^2+\frac{1}{2}D_j(D_j\Lambda-a_j)-\frac{1}{2}(a_j^R)^2\right\}$$

$$\times\hat{v}(\xi, t)đ\xi .$$

LEMMA 3.3. *Assume that the condition* (S) *in Theorem is valid. Then* $\tilde{c}(x, \xi ; t)$ *is in* $\mathscr{B}_t^0([0, T] ; S^0_{0,0})\cap\mathscr{B}_t^1([0, T] ; S^1_{0,0})$.

PROOF. Note that the first term of the right side of (3.5) vanishes because of (2.1) and it is evident that the terms in (3.5) except $b(x, \xi ; t)$ are in $\mathscr{B}_t^0([0, T] ; S^0_{0,0})\cap\mathscr{B}_t^1([0, T] ; S^1_{0,0})$. Hence it suffices to prove that $b(x, \xi ; t)$ is in $\mathscr{B}_t^0([0, T] ; S^0_{0,0})\cap\mathscr{B}_t^1([0, T] ; S^1_{0,0})$. By Lemma 2.1 (iii), Lemma 2.2 (iii), (iv) and $e^{-y\cdot\eta}=\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}e^{-y\cdot\eta}=\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}e^{-y\cdot\eta}$, where $\langle D_\eta\rangle^2=1-\Delta_\eta$, $\langle D_y\rangle^2=1-\Delta_y$, we have

$$
\begin{aligned}
&|b_{(\beta)}^{(\alpha)}(x, \xi; t)| \\
&=\left|2\sum_{|\gamma|=2}\frac{1-\theta}{\gamma!}\sum_{\substack{\alpha'\leq\alpha\\ \beta'\leq\beta}}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\right. \\
&\quad\times\int_0^1\Big\{Os-\iint\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}e^{-iy\cdot\eta}\partial_\xi^{\alpha-\alpha'}D_x^{\beta-\beta'}(e^{A(x,\xi+\eta;t)})^{(\gamma)} \\
&\quad\left.\times\partial_\xi^{\alpha'}D_x^{\beta'}a_{(\gamma)}^R(x+\theta y, t)\cdot\xi dy\bar{d}\eta\Big\}d\theta\right| \\
&\leqq C_{\alpha\beta}\int_0^1\iint\langle y\rangle^{-2l}\langle\eta\rangle^{-2l}\Big\{\int_0^t\frac{(t-s)|\xi(s)+\eta(s)|}{\langle x(s)\rangle^{2+\varepsilon_1}}ds+\int_0^t\frac{ds}{\langle x(s)\rangle^{1+\varepsilon_1}}\Big\} \\
&\quad\times\frac{|\xi|+1}{\langle x+\theta y\rangle^{1+\varepsilon_2}}dy\bar{d}\eta d\theta \\
&\leqq C'_{\alpha\beta}\iint\langle y\rangle^{-2l+2}\langle\eta\rangle^{-2l+1}dy\bar{d}\eta\int_0^t\frac{|\xi(s)|}{\langle x(s)\rangle^{2+\varepsilon_1}}ds\times\frac{(t-s)(|\xi|+1)}{\langle x\rangle^{1+\varepsilon_2}} \\
&\leqq\tilde{C}_{\alpha\beta}.
\end{aligned}
$$

Here we take $l=[n/2+2]$ and we use the inequality

$$
\begin{aligned}
\frac{(t-s)(|\xi|+1)}{\langle x(s)\rangle\langle x\rangle^{1+\varepsilon_2}}&\leqq\frac{\langle(t-s)\xi\rangle}{\left\langle x+\int_t^s\xi(y, \eta, \tau)d\tau\right\rangle\langle x\rangle} \\
&\leqq\frac{C\langle(t-s)\xi\rangle}{\langle(t-s)\xi(y, \eta, \theta(t-s)+s)\rangle} \\
&\leqq C,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\langle x+\theta y\rangle^{1+\varepsilon_2}\langle y\rangle^2}&\leqq\frac{1}{\langle x+\theta y\rangle^{1+\varepsilon_2}\langle y\rangle^{1+\varepsilon_2}} \\
&\leqq\frac{1}{\langle x+\theta y\rangle^{1+\varepsilon_2}\langle\theta y\rangle^{1+\varepsilon_2}} \\
&\leqq\frac{1}{\langle x\rangle^{1+\varepsilon_2}},
\end{aligned}
$$

and

$$
|D_x^\beta\partial_\xi^\alpha A(x, \xi, t)|\leqq C_{\alpha\beta}\Big\{\int_0^t\frac{|(t-s)\xi(s)|}{\langle x(s)\rangle^{2+\varepsilon_1}}ds+\int_0^t\frac{ds}{\langle x(s)\rangle^{1+\varepsilon_1}}\Big\},
$$

for any α ($|\alpha|\geqq 1$) (see the proof of Lemma 2.2 (iv)). This completes the proof.

By Lemma 3.3 we have $\sigma(\tilde{c})(x, \xi; t)\in S_{0,0}^0$. Finally put $u(x, t)=Kv(x, t)$, then we can transform the Cauchy problem (*) to the problem

$$
(**)\qquad\begin{cases} K^{-1}\circ P\circ Kv(x, t)=K^{-1}f, \\ v(x, 0)=v_0(x)\qquad(=u_0(x)), \end{cases}
$$

where

$$K^{-1}\circ P\circ K(x, D_x; t)=D_t+\sum_{j=1}^{n}(D_j-a_j^R(x, t))^2+\tilde{c}(x, D_x; t),$$

and

$$\tilde{c}(x, \xi; t)\in \mathscr{B}_t^0([0, T]; S_{0,0}^0)\cap \mathscr{B}_t^1([0, T]; S_{0,0}^1).$$

§ 4. The existence theorem for Schrödinger type operators of real valued coefficients.

Denote

$$P(t)=\sum_{j=1}^{n}(D_j-a_j(x, t))^2+c(x, D_x; t).$$

Assume that $a_j(x, t)$ are real valued and $c(x, \xi; t)$ is in $\mathscr{B}_t^0([0, T]; S_{0,0}^0)\cap \mathscr{B}_t^1([0, T]; S_{0,0}^1)$, then $P(t)$ satisfies

$$\sigma(P^*(t))-\sigma(P(t))\in \mathscr{B}_t^0([0, T]; S_{0,0}^0)\cap \mathscr{B}_t^1([0, T]; S_{0,0}^1), \tag{4.1}$$

where $P^*(t)$ is the adjoint operator of $P(t)$ which is defined as $(P(t)u, v)_{L_2}=(u, P^*(t)v)_{L_2}$ for any $u, v\in \mathcal{S}$. We consider the Cauchy problem

$$\begin{cases} \left(\dfrac{d}{dt}-iP(t)\right)u(x, t)=f(x, t) & x\in R^n,\ t\in[0, T], \\ u(x, 0)=u_0(x) & x\in R^n. \end{cases} \tag{4.2}$$

Then we obtain the following.

THEOREM 4.1. *Assume that $a_j(x, t)$ are real valued functions in $C_t^0([0, T]; \mathscr{B}^\infty(R^n))$ and $c(x, \xi; t)$ in $\mathscr{B}_t^0([0, T]; S_{0,0}^0)\cap \mathscr{B}_t^1([0, T]; S_{0,0}^1)$. Then for any $u_0\in H_2(R^n)$ and any $f(x, t)\in C_t^0([0, T]; H_2)$ there exists a unique sulution $u(x, t)$ of* (4.2) *which belongs to $C_t^0([0, T]; H_2)\cap C_t^1([0, T]; L_2)$ and satisfies*

$$\|u(t)\|_{H_k}\leqq C\left(\|u_0\|_{H_k}+\int_0^t\|f(\tau)\|_{H_k}d\tau\right) \tag{4.3}$$

for $t\in[0, T]$ and $k=0, 1, 2$.

Following the idea of Kumano-go [1], [2], we shall prove the above theorem. We need several lemmas. First, we define $\{\zeta_\nu(\xi)\}_{\nu=1}^\infty$ as

$$\zeta_\nu(\xi)=\left(\nu\sin\frac{\xi_1}{\nu}, \cdots, \nu\sin\frac{\xi_n}{\nu}\right) \tag{4.4}$$

and $P_\nu(t)=p_\nu(x, D_x; t)$ as

$$p_\nu(x, \xi; t)=p_1(x, \zeta_\nu(\xi); t)+c(x, \xi; t), \tag{4.5}$$

where $p_1(x, D_x; t)=\sum_{j=1}^n (D_j - a_j(x, t))^2$.

Then, we consider the following Cauchy problem

$$
(4.6) \qquad \begin{cases} L_\nu u_\nu = \partial_t u_\nu - iP_\nu(t) u_\nu = f(t) \qquad (t \in [0, T]), \\ u_\nu|_{t=0} = u_0 . \end{cases}
$$

We define the series of weight function $\{\lambda_\nu(\xi)\}_{\nu=1}^\infty$ as

$$
(4.7) \qquad \lambda_\nu(\xi) = \langle \zeta_\nu(\xi) \rangle = \left\{ 1 + \sum_{j=1}^n \left(\nu \sin \frac{\xi_j}{\nu} \right)^2 \right\}^{1/2},
$$

then we have

$$
(4.8) \qquad \begin{cases} \text{i)} \quad 1 \leqq \lambda_\nu(\xi) \leqq \min(\langle \xi \rangle, \sqrt{1+n\nu^2}), \\ \text{ii)} \quad |\partial_\xi^\alpha \lambda_\nu(\xi)| \leqq A_\alpha \lambda_\nu(\xi)^{1-|\alpha|}, \\ \text{iii)} \quad \lambda_\nu(\xi) \longrightarrow \langle \xi \rangle \qquad (\nu \to \infty) \quad \text{on } R_\xi^n, \\ \qquad \qquad \text{(uniform convergence in a compact set)}, \end{cases}
$$

In fact $\zeta_\nu(\xi)$ satisfies

$$
(4.9) \qquad \begin{cases} \text{i)} \quad |\zeta_\nu(\xi)| \leqq \min(|\xi|, \sqrt{n}\nu), \\ \text{ii)} \quad |\partial_\xi^\alpha \zeta_\nu(\xi)| \leqq A'_\alpha \lambda_\nu(\xi)^{1-|\alpha|}, \\ \text{iii)} \quad \zeta_\nu(\xi) \longrightarrow \xi \qquad (\nu \to \infty), \\ \qquad \qquad \text{(uniform convergence in a compact set)}. \end{cases}
$$

Denote by $S_{\lambda_\nu}^m$ the set of symbols $p(x, \xi) \in C^\infty(R^{2n})$ satisfying

$$
|p_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_{\alpha\beta} \lambda_\nu(\xi)^{m-|\alpha|}
$$

for any multi-index α, β. Then we get the following lemma.

LEMMA 4.1. *For $p(x, \xi) \in S^m$ put $p_\nu(x, \xi) = p(x, \zeta_\nu(\xi))$. Then $p(x, \xi) \in S_{\lambda_\nu}^m$, and for any α, β there is constant $A_{\alpha,\beta}$ which is independent of ν and p, we have*

$$
(4.10) \qquad \begin{cases} |p_{\nu(\beta)}^{(\alpha)}(x, \xi)| \leqq (A_{\alpha,\beta} |p|_{|\alpha+\beta|}^{(m)}) \lambda_\nu(\xi)^{m-|\alpha|}, \\ p_\nu(x, \xi) \longrightarrow p(x, \xi) \ \textit{(uniformly)} \ (\nu \to \infty) \ \textit{at} \ R_x^n \times K_\xi, \end{cases}
$$

where K_ξ is an arbitrary compact set of R_ξ^n.

Denote $H_{\lambda_\nu, s} = \{u \in \mathcal{S}' ; \lambda_\nu(\xi)^s \hat{u} \in L_2\}$.

LEMMA 4.2. *$P = p(x, D_x) \in S_{\lambda_\nu}^m$ is a continuous mapping from the Sobolev space $H_{\lambda, s+m}$ to $H_{\lambda, s}$ and for a constant $C_{s,m}$ and $l = l(s, m)$ we have*

(4.11) $$\|Pu\|_{\lambda_\nu,s} \leqq (C_{s,m}|p|_l^{(m)})\|u\|_{\lambda_\nu,s+m}, \qquad u \in H_{\lambda_\nu,s+m}.$$

Especially for $m=0$ and $s=0$ we have

(4.12) $$\|Pu\|_{L_2} \leqq (C|p|_l^{(0)})\|u\|_{L_2}, \qquad u \in L_2(R^n).$$

LEMMA 4.3. *For $c(x, \xi; t) \in \mathscr{B}_t^0([0, T]; S_{0,0}^m)$, $q_{j\nu}(\xi) \in S_{\lambda_\nu}^{l_j}$ $(j=1, 2)$ and $l_1+l_2 \geqq 0$ we have*

$$\sigma(q_{1\nu} \circ c \circ q_{2\nu})(x, \xi; t) \in \mathscr{B}_t^0([0, T]; S_{0,0}^{m+l_1+l_2}).$$

PROOF. We take $2l > \tilde{l}_1 + n$ $(\tilde{l}_1 = \max(l_1, 0))$, then by Lemma 4.1 and $\lambda_\nu(\xi+\eta)^{\pm} \leqq C\langle\eta\rangle\lambda_\nu(\xi)^{\pm}$ we have

$$\begin{aligned}
&|(q_{1\nu} \circ c \circ q_{2\nu})_{(\beta)}^{(\alpha)}(x, \xi; t)| \\
&= \left| D_x^\beta \partial_\xi^\alpha \left\{ Os-\iint e^{-iy\cdot\eta} q_{1\nu}(\xi+\eta) c(x+y, \xi; t) q_{2\nu}(\xi) dy d\!\!\!{}^-\eta \right\} \right| \\
&\leqq \sum C_\beta^{\alpha^1,\alpha^2,\alpha^3} \left| Os-\iint \langle\eta\rangle^{-2l} \langle D_y\rangle^{2l} \{\langle y\rangle^{-2l} \langle D_\eta\rangle^{2l} e^{-iy\cdot\eta}\} \right. \\
&\qquad \left. \times q_{1\nu}^{(\alpha^1)}(\xi+\eta) c_{(\beta)}^{(\alpha^2)}(x+y, \xi; t) q_{2\nu}^{(\alpha^3)}(\xi) dy d\!\!\!{}^-\eta \right| \\
&\leqq C \times Os-\iint e^{-iy\cdot\eta} \langle y\rangle^{-2l} \langle\eta\rangle^{-2l} \lambda_\nu(\xi+\eta)^{l_1} \langle\xi\rangle^m \lambda_\nu(\xi)^{l_2} dy d\!\!\!{}^-\eta \\
&\leqq C' \langle\xi\rangle^m \lambda_\nu(\xi)^{l_1+l_2} \iint \langle y\rangle^{-2l} \langle\eta\rangle^{-2l+\tilde{l}_1} dy d\!\!\!{}^-\eta \\
&\leqq C'' \langle\xi\rangle^{m+l_1+l_2},
\end{aligned}$$

where $\alpha^1+\alpha^2+\alpha^3=\alpha$.

LEMMA 4.4. *For any $u_0 \in L_2$ and any $f(t) \in C_t^0([0, T]; L_2)$ there exists a solution $u_\nu(t) \in C_t^1([0, T]; L_2)$ of (4.6) which satisfies the energy ineqalities*

(4.13) $$\|u_\nu(t)\| \leqq e^{\gamma t}\|u_0\| + \int_0^t e^{\gamma(t-\tau)}\|f(t)\| d\tau \qquad (t \in [0, T])$$

(4.14) $$\|\Lambda_\nu^j u_\nu(t)\| \leqq e^{\gamma_1 t}\|\Lambda_\nu^j u_0\| + \int_0^t e^{\gamma_1(t-\tau)}\|\Lambda_\nu^j f(\tau)\| d\tau \qquad (t \in [0, T];\ j=1, 2, 3, \cdots)$$

(4.15) $$\left\|\frac{d}{dt}\Lambda_\nu^j u_\nu(t)\right\| \leqq C_T\{\|\Lambda_\nu^{j+2}u_0\| + \max_{[0,T]}\|\Lambda_\nu^{j+2}f(\tau)\|\} \qquad (t \in [0, T];\ j=0, 1, 2, \cdots)$$

(4.16) $$\begin{aligned}&\|\Lambda_\nu^j(u_\nu(t)-u_\nu(t'))\| \leqq C_T'|t-t'|\{\|\Lambda_\nu^{j+2}u_0\| + \max_{[0,T]}\|\Lambda_\nu^{j+2}f(\tau)\|\} \\ &(t, t' \in [0, T];\ j=0, 1, 2, \cdots),\end{aligned}$$

where γ, γ_1, C_T, C_T' are constants which are independent of ν, and $\Lambda_\nu = \lambda_\nu(D_x)$,

$\|\cdot\|=\|\cdot\|_{L_2}$.

PROOF OF LEMMA 4.4. I) If we fix ν arbitrarily, we have $p_\nu(x, \xi; t)\in \mathcal{B}_t^k([0, T]; \mathcal{B}^\infty(R^{2n}_{x,\xi}))$ $(k=1, 2)$. We note $\mathcal{B}^\infty(R^{2n}_{x,\xi})=S^0_\lambda$ $(\lambda=1)$, and use Lemma 4.2. Then $P_\nu(t)$ is an L_2-bounded operator uniformly with respect to t. Therefore $u_\nu(t)$ to be a unique solution of the integral equation

$$u_\nu(t)=u_0+i\int_0^t P_\nu(\tau)u_\nu(\tau)d\tau+\int_0^t f(\tau)d\tau\,, \tag{4.17}$$

which can be solved as follows;

$$u_\nu(t)=u_0+\sum_{k=1}^\infty i^k\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}}P_\nu(\tau_1)\cdots P_\nu(\tau_k)u_0d\tau_k\cdots d\tau_1+\int_0^t f(\tau)d\tau$$

$$+\sum_{k=2}^n i^k\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}}P_\nu(\tau_1)\cdots P_\nu(\tau_{k-1})f(\tau_k)d\tau_k\cdots d\tau_1 \qquad (\tau_0=t).$$

Then we have $u_\nu(t)\in C_t^1([0, T]; L_2)$.

II) From (4.6) we have

$$\frac{d}{dt}\|u_\nu\|^2=2Re\Big(\frac{d}{dt}u_\nu,\ u_\nu\Big)$$
$$=Re(i(P_\nu-P_\nu^*)u_\nu,\ u_\nu)+2Re(f,\ u_\nu).$$

By the way, from the definitions of (4.1) and (4.5) we have

$$J_\nu\equiv\sigma(P_{1\nu})-\sigma(P_{1\nu}^*)\in C_t^0([0, T]; S^0_{\lambda_\nu}). \tag{4.18}$$

Thus by Lemma 4.2 and the Calderón-Vaillancourt theorem, for a constant $\gamma>0$ which independent of ν, t we have

$$\frac{d}{dt}\|u_\nu(t)\|^2\leqq 2\gamma\|u_\nu(t)\|^2+2\|f(t)\|\|u_\nu(t)\| \qquad (t\in[0, T];\ \nu=1, 2, \cdots)$$

and

$$\frac{d}{dt}\|u_\nu(t)\|\leqq\gamma\|u_\nu(t)\|+\|f(t)\|\,.$$

Therefore we get (4.13). Moreover from (4.6) we have

$$\frac{d}{dt}\Lambda_\nu^j u_\nu=i(P_{1\nu}+[\Lambda_\nu^j,\ P_{1\nu}]\Lambda_\nu^{-j}+\Lambda_\nu^j c\Lambda_\nu^{-j})\Lambda_\nu^j u_\nu+\Lambda_\nu^j f$$

and $P_{1\nu,j}=P_{1\nu}+[\Lambda_\nu^j, P_{1\nu}]\Lambda_\nu^{-j}$. Here, $[\Lambda_\nu^j, P_{1\nu}]\Lambda_\nu^{-j}\in C_t^0([0, T]; S^0_{\lambda_\nu})$ and by Lemma 4.3 $\Lambda_\nu^j c\Lambda_\nu^{-j}\in\mathcal{B}_t^0([0, T]; S^0_{0,0})\cap\mathcal{B}_t^1([0, T]; S^1_{0,0})$ uniformly with respect to ν and $P_{1\nu,j}$ satisfies (4.18). Hence we get (4.14) similarly to (4.13). On the other hand, noting

$$\frac{d}{dt}\Lambda_\nu^j u_\nu = i(\Lambda_\nu^j P_{1\nu}\Lambda_\nu^{-j-2}+\Lambda_\nu^j c\Lambda_\nu^{-j-2})\Lambda_\nu^{j+2}u_\nu+\Lambda_\nu^j f$$

where $\sigma(\Lambda_\nu^j P_{1\nu}\Lambda_\nu^{-j-2})\in C_t^0([0, T]; S_{\lambda_\nu}^0)$ and $\Lambda_\nu^j c\Lambda_\nu^{-j-2}\in \mathcal{B}_t^0([0, T]; S_{0,0}^0)\cap\mathcal{B}_t^1([0, T]; S_{0,0}^1)$ (uniformly on ν), we have

$$\left\|\frac{d}{dt}\Lambda_\nu^j u_\nu(t)\right\|\leqq C_1\|\Lambda_\nu^{j+2}u_\nu\|+\|\Lambda_\nu^j f\|.$$

Then from this and (4.14), we get (4.15). Put $u_\nu(t)-u_\nu(t')=\int_{t'}^t\frac{d}{d\tau}u_\nu(\tau)d\tau$. Then we get (4.16) from (4.15). This completes the proof of Lemma 4.4.

PROOF OF THEOREM 4.1. 1) First we assume that $u_0\in H_4$, $f(t)\in C_t^0([0, T]; H_4)$. Then it follows from Lemma 4.4 that there is a solution $u_\nu(t)$ of (4.6). Then, from (4.9) we have

(4.19) $$\|\Lambda_\nu^j u_0\|\leqq\|\Lambda^j u_0\|,\qquad \|\Lambda_\nu^j f\|\leqq\|\Lambda^j f\|$$

for $j\leqq 4$, where $\Lambda=\langle D_x\rangle$.

Therefore by (4.13), (4.14), (4.16) we have

(4.20) $$\|\Lambda_\nu^j u_\nu(t)\| C_1\{\|\Lambda^j u_0\|+\max_{[0,T]}\|\Lambda^j f\|\},$$

(4.21) $$\|\Lambda_\nu^j(u_\nu(t)-u_\nu(t'))\|\leqq C_2'|t-t'|\{\|\Lambda^{j+2}u_0\|+\max_{[0,T]}\|\Lambda^{j+2}f\|\}$$

for $j\leqq 4$.

Therefore if we fix $t_0\in[0, T]$ arbitrarily, then from (4.20) with $j=0$, $\{u_\nu(t_0)\}_{\nu=1}^\infty$ is a bounded sequence in L_2. Hence there exists $u(t_0)\in L_2$ and a subsequence such that $\{u_{\nu_m}\}_{m=1}^\infty$ of L_2 we have

$$u_{\nu_m}\longrightarrow u(t_0)\text{ (weakly) in }L_2\qquad(m\to\infty).$$

Let $\{t_k\}_{k=1}^\infty$ be a dense set in $[0, T]$. Then by the diagonal method there exists $u(t_k)\in L_2$ and we have

(4.22) $$u_{\nu_m}(t_k)\longrightarrow u(t_k)\text{ (weakly) in }L_2\qquad(m\to\infty).$$

Then from (4.21) with $j=0$, for any $t, t'\in\{t_k\}_{k=1}^\infty$ we have

(4.23) $$\|u(t)-u(t')\|\leqq C_2|t-t'|\{\|\Lambda^2 u_0\|+\max_{[0,T]}\|\Lambda^2 f\|\}.$$

For any $t_0\in[0, T]$ we choose subsequence $\{t_{k_s}\}_{s=1}^\infty$ of $\{t_k\}$ such that $t_{k_s}\to t_0$. Then by (4.23), $\{u(t_{k_s})\}_{s=1}^\infty$ becomes a Cauchy sequence in $L_2(R^n)$. Thus there exists $u(t_0)\in L_2$ and we have $u(t_{k_s})\to u(t_0)$ in L_2 $(s\to\infty)$. Moreover for all $u(t)$

we obtain (4.23) and

$$u_{\nu_m}(t) \longrightarrow u(t) \text{ (weakly) in } L_2 \qquad (m\to\infty).$$

Thus for $\varphi\in\mathcal{S}$ we have

$$(4.24) \qquad (\Lambda^k_{\nu_m}u_{\nu_m}(t),\ \varphi)=(u_{\nu_m}(t),\ \Lambda^k_{\nu_m}\varphi) \longrightarrow (u(t),\ \Lambda^k\varphi) \qquad (m\to\infty).$$

Therefore noting $|(\Lambda^k_{\nu_m}u_{\nu_m}(t),\ \varphi)|\leqq\|\Lambda^k_{\nu_m}u_{\nu_m}(t)\|\|\varphi\|$, we get from (4.20) and (4.24),

$$(4.25) \qquad |(u(t),\ \Lambda^k\varphi)|\leqq C_1\{\|\Lambda^k u_0\|+\max_{[0,T]}\|\Lambda^k f\|\}\|\varphi\| \qquad (\varphi\in\mathcal{S})$$

for $k\leqq 4$. We define $u_{R,k}(t, x)\in L^2$ for any $t\in[0, T]$ and $R>0$ such that $\hat{u}_{R,k}(t, \xi)=\langle\xi\rangle^k\hat{u}(t, \xi)$ $(|\xi|\leqq R)$, and $\hat{u}_{R,k}(t, \xi)=0$ $(|\xi|>R)$ and take the sequence $\{\varphi_j\}_{j=1}^\infty$ of $\mathcal{S}$ as $\operatorname{supp}\hat{\varphi}\subset\{|\xi|\leqq R\}$, $\hat{\varphi}_j\to\hat{u}_{R,k}(t, \xi)$ in L_2 $(j\to\infty)$. Then we have

$$(u(t),\ \Lambda^k\varphi_j)=\int\hat{u}(t, \xi)\langle\xi\rangle^k\overline{\hat{\varphi}_j(\xi)}d\xi \longrightarrow \int_{|\xi|\leqq R}\langle\xi\rangle^{2k}|\hat{u}(t, \xi)|^2d\xi \quad (j\to\infty).$$

Therefore from (4.25) we get

$$\|u_{R,k}(t)\|\leqq C_1\{\|\Lambda^k u_0\|+\max_{[0,T]}\|\Lambda^k f\|\} \qquad (k\leqq 4).$$

Hence taking $R\to\infty$, we have $\Lambda^k u\in L_2$ and

$$(4.26) \qquad \|\Lambda^k u\|\leqq C_1\{\|\Lambda^k u_0\|+\max_{[0,T]}\|\Lambda^k f\|\} \qquad (k\leqq 4).$$

From (4.21) we have similarly to (4.26)

$$(4.27) \qquad \|\Lambda^k(u(t)-u(t'))\|\leqq C_2'|t-t'|\{\|\Lambda^{k+2}u_0\|+\max_{[0,T]}\|\Lambda^{k+2}f\|\} \qquad (k\geqq 2).$$

Thus we obtain $u(t)$, $\Lambda u(t)$, $\Lambda^2 u(t)\in C_t^0([0, T]\,;\ L_2)$.

We take $\varphi(t, x)=\varphi_1(t)\varphi_2(x)\in C_0^\infty((0, T)\times R^n)$ arbitrarily. Noting

$$L^*_{\nu_m}\varphi \longrightarrow L^*\varphi \qquad \text{in } C_t^0([0, T]\,;\ L_2),$$

we have

$$\iint_{\Omega_T}Lu\cdot\bar{\varphi}\,dx\,dt=\iint_{\Omega_T}u\cdot\overline{L^*\varphi}\,dx\,dt$$

$$=\lim_{m\to\infty}\iint_{\Omega_T}u_{\nu_m}\cdot\overline{L^*_{\nu_m}\varphi}\,dx\,dt=\lim_{m\to\infty}\iint_{\Omega_T}L_{\nu_m}u_{\nu_m}\cdot\bar{\varphi}\,dx\,dt$$

$$=\iint_{\Omega_T}f\cdot\bar{\varphi}\,dx\,dt.$$

Thus $Lu=f$ $(t\in[0, T])$. Therefore u is a solution of (4.2). Also noting $u(t)$, $\Lambda u(t)$, $\Lambda^2 u(t)\in C_t^0([0, T]\,;\ L_2)$ and $\partial_t u=iP(t)u+f$, we get $u(t)\in C_t^1([0, T]\,;\ L_2)\cap C_t^0([0, T]\,;\ H_2)$.

II) If $u(t)\in C_t^1([0, T]\,;\ L_2)\cap C_t^0([0, T]\,;\ H_2)$ is a solution of (4.2), we obtain the energy inequality (4.3) similarly to the proof of Lemma 4.4.

Assume $u_0 \in H_2$ and $f(t) \in C_t^0([0, T]; H_2)$. Put $u_{0,\varepsilon} = \chi_\varepsilon(D_x)u_0$ and $f_\varepsilon(t) = \chi_\varepsilon(D_x)f(t)$, where $\chi(\xi) \in \mathcal{S}_\xi$ satisfies $\chi(0)=1$ and $\chi_\varepsilon(\xi)=\chi(\varepsilon\xi)$. Then $u_{0,\varepsilon} \in H_4$, $f_\varepsilon(t) \in C_t^0([0, T]; H_4)$. Therefore from I), we get the solution $u_\varepsilon(t)$ of (4.2) for $u_{0,\varepsilon}$ and f_ε. Then by (4.3) we have

$$\|u_\varepsilon(t) - u_{\varepsilon'}(t)\|_2 \leqq e^{\gamma_1 T}\{\|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_2 + T \max_{[0,T]} \|f_\varepsilon(\tau) - f_{\varepsilon'}(\tau)\|\}.$$

On the other hand when ε, $\varepsilon' \to 0$ we have

$$\begin{cases} \|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_2 = \|(\chi_\varepsilon(D_x) - \chi_{\varepsilon'}(D_x))u_0\|_2 \longrightarrow 0, \\ \max_{[0,T]} \|f_\varepsilon(\tau) - f_{\varepsilon'}(\tau)\|_2 = \max_{[0,T]} \|(\chi_\varepsilon(D_x) - \chi_{\varepsilon'}(D_x))f(\tau)\|_2 \longrightarrow 0. \end{cases}$$

Therefore, noting $\partial_t u_\varepsilon = iPu_\varepsilon + f_\varepsilon$, we can see $\{u_\varepsilon(t)\}_{0<\varepsilon<1}$ becomes a Cauchy sequence in $C_t^1([0, T]; L_2) \cap C_t^0([0, T]; H_2)$ and the limit of this series $u(t)$ is a solution of (4.2). This completes the proof of Theorem 4.1.

PROOF OF THEOREM. By Theorem 4.1 we can see that there exists a unique solution $v(x, t) \in C_t^0([0, T]; H_2) \cap C_t^1([0, T]; L_2)$ of the Cauchy problem (**) with initial data $u_0(x)$ and we get the energy inequality

$$\|v(\cdot, t)\|_{H_k} \leqq C(T)\Big(\|u_0\|_{H_k} + \int_0^t \|K^{-1}f(\cdot, \tau)\|_{H_k} d\tau\Big)$$

for $t \in [0, T]$ and $k=0, 1, 2$. Hence, we obtain the unique solution $u(x, t) = Kv(x, t)$ of the equation (*) with initial data $u_0(x)$ and by using the Calderón-Vaillancourt theorem we get the energy inequality

$$\|u(\cdot, t)\|_{H_k} \leqq C'(T)\Big(\|u_0\|_{H_k} + \int_0^t \|f(\cdot, \tau)\|_{H_k} d\tau\Big)$$

for $t \in [0, T]$ and $k=0, 1, 2$. Thus we complete the proof of our Theorem.

Acknowledgement

The author wishes to express his appreciation to his advisor, Prof. K. Kajitani, for his helpful suggestions.

References

[1] H. Kumano-go, Partial differential operators, Kyôritsu shuppan, Tokyo, 1978, (in Japanese).
[2] H. Kumano-go, Pseudo-Differential Operators, MIT Press, Cambridge, 1982.
[3] S. Mizohata, On the Cauchy Problem, Notes and Reports in Math., 3, Academic Press, 1985.

[4] S. Mizohata, Sur quelques équations du type Schrödinger, Séminaire J. Vaillant, 1980/1981, Univ. de Paris-VI.

[5] J. Takeuchi, Le problème de Cauchy pour quelques équations aux dérivées partielles du type de Schrödinger, C.R. Acad. Sci. Paris, t. 310, Série I, p. 823-826, 1990.

Institute of Mathematics
University of Tsukuba
Ibaraki 305 Japan