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ON THE ABSOLUTELY PARACOMPACT SUBSETS OF $\nabla^{\omega}(\omega+1)^{(*)}$

By

Shoulian YANG

Rudin [R] first proved under CH that the box product $\square^{\omega}(\omega+1)$ of countable many copies of $\omega+1$ is paracompact. But since then it is still unknown if this simplest box product is paracompact in ZFC. Kunen [K] showed that the paracompactness of $\square^{\omega}(\omega+1)$ is equivalent to that of the reduced box product $\nabla^{\omega}(\omega+1)$. In this paper, we give out some special subsets of $\nabla^{\omega}(\omega+1)$ which is paracompact in ZFC (see Theorems 5, 8), hoping that our results will become helpful toward the solution of the paracompactness of $\nabla^{\omega}(\omega+1)$ itself. For survey of box products see van Douwan [vD].

Given spaces $X_i (i\in\omega)$, an *open box* in the Cartesian product $\prod_{i\in\omega} X_i$ is a set of the form $\prod_{i\in\omega} U_i$, where U_i is an open subset of X_i. The topology generated by all open boxes is the box topology. $\prod_{i\in\omega} X_i$ with the product is denoted by $\square_{i\in\omega} X_i$ and is called the *box product*. We define the *reduced* (or nabla) product $\nabla_{i\in\omega} X_i$ as the quotient space $\square_{i\in\omega} X_i/=^*$ by the equivalence relation $=^*$ such that $f=^*g$ iff $f(i)=g(i)$ for almost all $i\in\omega$, that is, $\{i\in\omega: f(i)=g(i)\}$ is finite. Let us use q to denote the quotient map

$$q: \square_{i\in\omega} X_i \longrightarrow \nabla_{i\in\omega} X_i .$$

When all factors are the same space X, we denote $\square_{i\in\omega} X_i$, $\nabla_{i\in\omega} X_i$ by $\square^{\omega}X$, $\nabla^{\omega}X$ respectively. In this paper, we simply denote $\square_{i\in\omega}(\omega+1)$, $\nabla_{i\in\omega}(\omega+1)$ by $\square$, ∇ respectively.

We make our convention that members of $\square$ are denoted by $f, g, h, \cdots$, while members of ∇ are denoted by $x, y, z, \cdots$. For each $x\in\nabla$, we choose a fixed member of $q^{-1}(x)$ and denoted it by $x^{\boxtimes}$. To denote an arbitrary member of $q^{-1}(x)$ we use the symbol $x^{\square}$.

For each $x\in\nabla$, we put

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$$F(x)=\{i\in\omega:\ x^{\square}(i)<\omega\} \quad \text{and} \quad I(x)=\{i\in\omega:\ x^{\square}(i)=\omega\}.$$

If E is an infinite subset of ω, all the above definitions are naturally modified to the product $\prod_{i\in E} X_i$. Let

$$q_E:\ \square^E(\omega+1)\longrightarrow \nabla^E(\omega+1)$$

be the quotient mapping. For each $x\in\nabla$, $x|E$ denotes $q_E(x^{\square}|E)$, where $x^{\square}|E$ is the function $x^{\square}\in{}^{\omega}\omega$ restricted on E.

For f, $g\in\square$, we define

$$f\leqq^* g \text{ iff } f(i)\leqq g(i) \qquad \text{for almost all } i<\omega.$$

$<^*$ is defined by $\leqq^*$ and not $=^*$. Note that $\leqq^*$ is a quasi-order in $\square$. $\leqq^*$ induces a partial order $\leqq$ in ∇, that is,

$$x\leqq y \qquad \text{if } x^{\square}\leqq^* y^{\square}.$$

Similarly, $<^*$ induces $<$. For subsets A, $B\subset\omega$, we define

$$A\subset^* B \qquad \text{iff } A\setminus B \text{ is finite};$$

$$A=^* B \qquad \text{iff } A\subset^* B \text{ and } B\subset^* A.$$

Let ${}^{\omega}\omega\subset\square$ be the set of all functions from ω to ω. Then the image of ${}^{\omega}\omega$ by q is $\nabla^{\omega}\omega\subset\nabla$. Let us denote this $\nabla^{\omega}\omega$ by $\nabla\omega$.

Since the togology of $\omega+1$ is the order topology, the basic set in $\square$ is of the form $\prod_{i\in\omega}[a_i,\ b_i]$, where $a_i<\omega$, or more strictly, we can add the condition that $a_i=b_i$ if $b_i<\omega$. Hence, in ∇, we make a convention that a *basic set* in ∇ means an interval

$$[x,\ y]=\{z\in\nabla:\ x\leqq z\leqq y\}$$

such that (1). $x\in\nabla\omega$;

(2). $x=y$ on $F(y)$, that is, $x^{\square}(i)=y^{\square}(i)$ for almost all $i\in F(y)$.

We say a point $y\in\nabla\omega$ is *increasing* or *unbounded* if some $x^{\square}\in{}^{\omega}\omega$ is so.

Let E be an infinite subset of ω. For an unbounded function $f\in{}^{E}\omega$ we define a function $h(f)\in{}^{\omega}\omega$ by

$$h(f)(n)=f(j), \qquad n\in\omega$$

where

$$j=\min\{i\in E:\ i\geqq n \text{ and } f(i)=\max\{f(k):\ k\in E,\ k\leqq i\}\}.$$

Note that the condition $f(i)=\max\{f(k):\ k\in E,\ k\leqq i\}$ is always satisfied if f is increasing.

We call this $h(f)$ the *hat* of f. For an unbounded $x\in\nabla^{E}\omega$ the *hat* of x is defined by

$$h(x)=q(h(x^{\square}))\in\nabla\omega.$$

For $x\in\nabla$ such that $x|F(x)$ is unbounded, we often use

$$h(x|F(x))$$

and abbreviate this to $h(x)$. Note that $h(x)\in\nabla\omega$, and that $h(x)\leqq x$ if $x|F(x)$ is increasing. When we consider $h(x)$, we always assume that $x|F(x)$ is unbounded.

LEMMA 1. *Let $E\subset\omega$ be infinite, and $x\in\nabla^{E}\omega$ be bounded. If $y\in\nabla\omega$ is increasing, then $y|E\leqq x$ implies $y\leqq h(x)$.*

PROOF. The condition $y|E\leqq x$ implies $h(y|E)\leqq h(x)$. Since y is increasing, we know that $y\leqq h(y|E)$. Hence we get $y\leqq h(x)$.

Recall our convention that the basic set $[x, y]$ is chosen so that $x=y$ on $F(y)$. Then the following lemma is easy to see.

LEMMA 2. *Suppose that $x, y\in\nabla$, and $V_x=[\tilde{x}, x]$, $V_y=[\tilde{y}, y]$ are basic sets. Then $V_x\cap V_y\neq\emptyset$ if all the following three conditions hold:*

(1) $x=y$ *on* $F(x)\cap F(y)$;

(2) $\tilde{x}\leqq y$ *on* $F(y)\setminus F(x)$;

(3) $\tilde{y}\leqq x$ *on* $F(x)\setminus F(y)$.

We define a special relation in ∇, denoted $\prec$, as follows. We write $x\prec y$ if the following two conditions are satisfied:

(i) $x=y$ on $F(x)\cap F(y)$;

(ii) $h(x)<y$ on $F(y)\setminus F(x)$.

Note that if $x\prec y$, then $h(x)\leqq h(y)$.

A subset of $\nabla\omega$ is called *dominating* if it is cofinal in $\langle\nabla\omega, \leqq\rangle$, or equivalently, cofinal in $\langle\omega^{\omega}, \leqq^{*}\rangle$. Define the cardinal

$$\underset{\sim}{d}=\min\{|D| : D \text{ is a dominating subset in } \nabla\omega\}.$$

Note $\omega_1\leqq\underset{\sim}{d}\leqq c=2^{\omega}$. In the sequel, *we fix a dominating family*

$$\mathcal{D}=\{q_{\alpha} : \alpha\leqq\underset{\sim}{d}\},\quad q_{\alpha}=q(f_{\alpha}),\quad f_{\alpha}\in{}^{\omega}\omega$$

in $\nabla\omega\subset\nabla$ such that each f_{α} is increasing. For every $\alpha<\underset{\sim}{d}$ put

$$\Pi_{\alpha}=\{x\in\nabla : x\leqq q_{\alpha} \text{ on } F(x)\},$$

$$\tilde{\Pi}_{\alpha}=\Pi_{\alpha}\setminus\bigcup_{\beta<\alpha}\Pi_{\beta}.$$

Since $\mathscr{D}$ is dominating, we have

$$\nabla=\cup\{\Pi_\alpha : \alpha<\underline{d}\}.$$

Focusing on the partial order $\prec$, we call that a subset $A\subset\nabla$ is *super-bounded* if for each $x\in\nabla$

$$\{\alpha<\underline{d} : h(y\,|\,F(y)\setminus F(x))\in\Pi_\alpha,\ x\prec y\in A\}$$

is bounded in $\underline{d}$. (Note that if there is no y with $x\prec y\in A$ for each $x\in\nabla$, then A is super-bounded.)

More precisely, let call A *super-bounded* by $g : \nabla\to\underline{d}$ if for every $x\in\nabla$

$$g(x)=\sup\{\alpha<\underline{d} : h(y\,|\,F(y)\setminus F(x))\in\Pi_\alpha,\ x\prec y\in A\}.$$

Let A be super-bounded by g and let $x\in\nabla$ be an arbitrary point. Since $\{q_\alpha : \alpha\leqq g(x)\}$ can not dominate $\nabla\omega$, there exists $y_x\in\nabla\omega$ such that $y_x\nleqq q_\alpha$ for all $\alpha\leqq g(x)$. Fix these y_x's. Let $\mathcal{C}$ be an open cover of ∇. For each $x\in A$, we call $V_x=[\tilde{x}, x]$ a *good* basic neighborhood of x relative to A and $\mathcal{C}$ if it satisfies the following:

(i) V_x is a basic set and is contained in some member of $\mathcal{C}$:

(ii) $\tilde{x}>h(x)$;

(iii) $\tilde{x}>q_\beta$, where β is such that $x\in\tilde{\Pi}_\beta$;

(iv) $\tilde{x}>y_x$, where y_x is as above;

(v) $\tilde{x}$ is increasing.

LEMMA 3. *Let A be super-bounded, and $x, y\in A$. If V_x, V_y are good basic neighborhoods, then the conditions*

$$x\notin V_y \quad \textit{and} \quad y\notin V_x$$

imply

$$V_x\cap V_y=\emptyset\,.$$

PROOF. We consider five cases.

(1) $x\neq y$ *on* $F(x)\cap F(y)$. Then $V_x\cap V_y=\emptyset$ by Lemma 2.

(2) $F(x)\cap F(y)=^*\emptyset$. Then, either $y\not> h(x)$ on $F(y)$ or $x\not> h(y)$ on $F(x)$.

Indeed, if $y>h(x)$ on $F(y)$, then $h(y)>h(x)$ since $F(x)\cap F(y)=^*\emptyset$. Hence $x<h(y)$ on $F(x)$. Since $\tilde{x}>h(x)$ and $\tilde{y}>h(y)$, it follows that either $y\ngeqq\tilde{x}$ on $F(y)$ or $x\ngeqq\tilde{y}$ on $F(x)$ happens; which means $V_x\cap V_y=\emptyset$ by Lemma 2.

Now in the following cases, we can assume that $x=y$ on $F(x)\cap F(y)$ and that $F(x)\cap F(y)$ is infinite. Take α, β such that $x\in\tilde{\Pi}_\beta$, $y\in\tilde{\Pi}_\alpha$ and assume that $\alpha\leqq\beta$.

(3) $F(x_\alpha)\setminus F(y)$ *is infinite.* Since $x_\alpha\leqq q_\alpha$ on $F(x_\alpha)$, and $\tilde{y}>q_\alpha$, we have

$$\tilde{y}\leqq x \qquad \text{on } F(x_{\alpha})\setminus F(y).$$

Since $F(x_{\alpha})\subset^{*}F(x)$, we get

$$\tilde{y}\leqq x \qquad \text{on } F(x)\setminus F(y).$$

Hence $V_{x}\cap V_{y}=\emptyset$ by Lemma 2.

(4) *$F(y)\setminus F(x_{\alpha})$ is infinite and $F(x_{\alpha})\subset^{*}F(y)$.* If $x\not\prec y$, then $h(x)\not\prec y$ on $F(y)\setminus F(x)$ because x and y satisfy the first condition of $x\prec y$. From $\tilde{x}>h(x)$ it follows that

$$\tilde{x}\not\leqq y \qquad \text{on } F(y)\setminus F(x).$$

Hence $V_{x}\cap V_{y}=\emptyset$ by Lemma 2. Note thrt $F(y)\setminus F(x)=^{*}F(y)\setminus F(x_{\alpha})$ because $F(y)\cap(F(x)\setminus F(x_{\alpha}))=^{*}\emptyset$. So $F(y)\setminus F(x)$ is infinite.

If $x\prec y$, then

$$h(y\,|\,F(y)\setminus F(x))\in\Pi_{\xi}$$

for some $\xi\leqq g(x)$ because $y\in A$ and A is super-bounded by g. This means

$$h(y\,|\,F(y)\setminus F(x))\leqq q_{\xi}\,.$$

On the otheer hand, by the definition of y_{x}, we have

$$y_{x}\not\leqq q_{\xi}\,.$$

From $\tilde{x}>y_{x}$ it follows that

$$\tilde{x}\not\leqq q_{\xi}\,.$$

Hence

$$\tilde{x}\not\leqq h(y\,|\,F(y)\setminus F(x))\,.$$

By Lemma 1 we get

$$\tilde{x}\not\leqq y \qquad \text{on } F(y)\setminus F(x).$$

which shows $V_{x}\cap V_{y}=\emptyset$ by Lemma 2.

(5) *$F(y)=^{*}F(x_{\alpha})$.* Since $x=y$ on $F(y)$ and $x\notin V_{y}$, there exists an infinite subset $G\subset F(x)\setminus F(y)$ such that

$$x<y \qquad \text{on } G\,.$$

Hence $V_{x}\cap V_{y}=\emptyset$.

This completes the proof of Lemma 3.

$x\in\nabla$ is called a *bounded* point if $x^{\square}|F(x)$ is bounded. The points in the previous lemma are unbounded points. For every bounded point x, we simply choose an increasing $\tilde{x}$ so that $V_{x}[\tilde{x}, x]$ is contained in some member of $\mathcal{C}$. Such V_{x} is also called a good neighborhood. The next lemma is easy.

LEMMA 4. *Suppose that x, y are bounded and $x\neq y$, or that x is bounded* *but y is unbounded. Then*

$$V_x \cap V_y = \emptyset$$

for good neighborhoods V_x, V_y.

Now we come to the main theorem.

THEOREM 5. *Every super-bounded subset of ∇ is paracompact. Precisely, every open cover of a super-bounded subset of ∇ has a refinement consisting of pairwise disjoint basic sets.*

PROOF. Suppose that A is the super-bounded subset of ∇. By induction we will define the families $K(\alpha)$ for $\alpha < \underline{d}$ so that the following hold:

(1) $K(\alpha)$ is a disjoint collection consisting of good basic neighborhoods of some points in $A \cap \Pi_\alpha$;

(2) $K(\alpha)$ refines $\mathcal{C}$;

(3) $K(\alpha)$ covers $A \cap \Pi_\alpha$;

(4) $K(\alpha) \subset K(\beta)$ if $\alpha < \beta$.

For a stage $\beta < \underline{d}$, let $B = (A \cap \Pi_\beta) \setminus \cup \{K(\alpha) : \alpha < \beta\}$, and define

$$K'(\beta) = \{V_x : V_x \text{ is good neighborhood of } x \in B\},$$

$$K(\beta) = K'(\beta) \cup \cup \{K(\alpha) : \alpha < \beta\}.$$

By Lemma 3 and 4 we can conclude that $K(\beta)$ satisfies (1). Then, it is easy to check that $K(\alpha)$, $\alpha \leqq \beta$, satisfy (1)-(4). By (3). $\{K(\alpha) : \alpha \leqq \underline{d}\}$ covers $A = \cup \{A \cap \Pi_\alpha : \alpha < \underline{d}\}$. By (1) and (2), $\{K(\alpha) : \alpha < \underline{d}\}$ is a disjoint collection refining $\mathcal{C}$. Thus we can conclude that A is paracompact.

$x, y \in \nabla$ are said be *compatible* if $x = y$ on $F(x) \cap F(y)$. Then, $x \cup y \in \nabla$ is a point such that $F(x \cup y) = F(x) \cup F(y)$, $(x \cup y) | F(x) = x | F(x)$ and $(x \cup y) | F(y) = y | F(y)$.

Let A, B are super-bounded, and $x \in A \subset B$. B is called on *expansion* of A *by* x if we have $x \cup y \in B$ whenever y is a point in A such that: (i) x, y are compatible; $x > h(y)$ on $F(x) \setminus F(y)$; (iii) $y > h(x)$ on $F(y) \setminus F(x)$. Let $x \notin \cup \mathcal{A}$, where $\mathcal{A}$ is a family of super-bounded sets and B is a super-bounded set. Then B is called an *expansion* of $\mathcal{A}$ *by* x if $x \cup y \in B$ whenever y is a point in $\cup \mathcal{A}$ such that (i), (ii), (iii) as above.

LEMMA 6. *Suppose A is a super-bounded set, and $x \notin A$. Then the least expansion of A by x exists.*

PROOF. Let

$$B=A\cup\bigcup\{x\cup y: y\in A \text{ satisfies the above (i), (ii) and (iii)}\}.$$

To show B is the desired expansion, it suffices to show B is super-bounded. Note first that for each $z\in\nabla$

$$\{\alpha: h(x\cup y\,|\,F(x\cup y)\setminus F(z))\in\tilde{\Pi}_{\alpha},\ x\cup yB,\ z\prec x\cup y\} \qquad (*)$$

is bounded in $\underset{\sim}{d}$.

Indeed,

$$\{\alpha: h(x\cup y\,|\,F(x\cup y)\setminus F(z))\in\tilde{\Pi}_{\alpha},\ y\in B,\ z\prec y\}$$

is bounded in $\underset{\sim}{d}$. So let β be the supremum of this set; then

$$h(x\cup y\,|\,F(x\cup y)\setminus F(z))\leqq q_{\beta}\vee h(x\,|\,F(x)\setminus F(z))$$

where $\vee$ is an operation on ∇ such that

$$w\vee v=q(w^{\square}\vee v^{\square}), \qquad (w^{\square}\vee v^{\square})(i)=\max\{w^{\square}(i),\ v^{\square}(i)\}.$$

From the fact $(*)$ it follows that for each $z\in\nabla$,

$$\{\alpha: h(y\,|\,F(y)\setminus F(z))\in\Pi_{\alpha},\ y\in B,\ z\prec y\}$$

is bounded in $\underset{\sim}{d}$. Hence B is super-bounded.

Fix $\beta<\mathrm{cof}(\underset{\sim}{d})$. Let A_{α}, $\alpha<\beta$, be super-bounded subsets in ∇. Let $\mathscr{B}$ be a refinement of C covering $\cup\{A_{\alpha}:\alpha<\beta\}$. $\mathscr{B}$ is called a *good refinement* if every $V_{x}=[\tilde{x},\ x]\in\mathscr{B}$ is a good basic neighborhood of x relative to A_{γ}, where

$$\gamma=\min\{\alpha<\beta: x\in A_{\alpha} \text{ and } V_{x}\in\mathscr{B}\}.$$

LEMMA 7. *If $\beta<\mathrm{cof}(\underset{\sim}{d})$, and $\mathscr{B}$ is a good refinement covering $\cup\{A_{\alpha}:\alpha<\beta\}$, then $\cup\mathscr{B}$ is closed in ∇.*

PROOF. Let A_{α} be super-bounded by g_{α}. Let $g:\nabla\to\underset{\sim}{d}$ be a function with the property that $g(x)\geqq\sup\{g_{\alpha}(x):\alpha<\beta\}$. (Such g exists because $\beta<\mathrm{cof}(\underset{\sim}{d})$) Fix a set B_{0} which is super-bounded by g; then it is clear that $\cup\{A_{\alpha}:\alpha<\beta\}\subset B_{0}$.

Assume $x\notin\cup\mathscr{B}$. Let B be the expansion of B_{0} by x, the existence of which is assured by Lemma 6. Define an $\tilde{x}\in\nabla\omega$ so that:

(i) $V_{x}=[\tilde{x},\ x]$ is a good basic neighborhood of x relative to B:

(ii) $\tilde{x}>q_{\xi}$, where

$$\xi=\sup\{\alpha: h(y)\,|\,F(y)\setminus F(x))\in\tilde{\Pi}_{\alpha},\ x\prec y\in B\}.$$

To sho $\cup\mathscr{B}$ is closed, we will claim that $V_{x}\cap V_{y}=\emptyset$ for every $V_{y}\in\mathscr{B}$.

In the cases that (1) $x\neq y$ on $F(x)\cap F(y)$, or (2) $F(x)\cap F(y)=^{*}\emptyset$, it is easy to prove $V_x \cap V_y = \emptyset$ by the same argument as in the proof of Lemma 3. So, in the next cases (3) and (4), we assume that $x = y$ on the infinite $F(x) \cap F(y)$.

CASE (3): $x \in \tilde{\Pi}_\beta$, $y \in \tilde{\Pi}_\alpha$ and $\beta \geqq \alpha$.

Suppose that $y \in A_\gamma$ and V_γ is the good neighborhood of y in A_γ. Since $g(x) \geqq g_\gamma(x)$ fir all $x \in \nabla$ and we many assume that B is super-bounded by g, we get that $V_x \cap V_y = \emptyset$ by the same way as in Lemma 3.

CASE (4): $x \in \tilde{\Pi}_\beta$, $y \in \tilde{\Pi}_\alpha$ and $\beta < \alpha$.

If $x \not> h(y)$ on $F(x) \setminus F(y)$ or $y \not> h(x)$ on $F(y) \setminus F(x)$, then the conditions $\tilde{x} > h(x)$ and $\tilde{y} > h(y)$ imply that

$$x \ngeqq \tilde{y} \text{ on } F(x) \setminus F(y) \quad \text{or} \quad y \ngeqq \tilde{x} \text{ on } F(y) \setminus F(x).$$

Hence $V_x \cap V_y = \emptyset$ by Lemma 2. If $x > h(y)$ on $F(x) \setminus F(y)$ and $y > h(x)$ on $F(y)/F(x)$, then $x \cup y \in B$. Then

$$h(x \cup y \mid F(x \cup y) \setminus F(x)) \leqq q_\gamma < \tilde{x} .$$

On the other hand,

$$h(y \mid F(y) \setminus F(x)) = h(x \cup y \mid F(x \cup y) \setminus F(x))$$

since

$$y \mid F(y) \setminus F(x) = x \cup y \mid F(x \cup y) \setminus F(x) .$$

So we have

$$h(y \mid F(y) \setminus F(x)) < \tilde{x} .$$

Hence, by Lemma 1, $y \ngeqq \tilde{x}$ on $F(y) \setminus F(x)$, which shows $V_x \cap V_y = \emptyset$ by Lemma 2.

THEOREM 8. *The union of* $\mathrm{cof}(\underline{d})$ *many super-bounded sets is paracompact.*

PROOF. Let A_α, $\alpha < \underline{d}$, be super-bounded subsets. Applying Theorem 5 and Lemma 7 we can show that $\cup\{A_\alpha : \alpha < \mathrm{cof}(\underline{d})\}$ is paracompact. Indeed, let $\beta < \mathrm{cof}(\underline{d})$ and $\mathcal{B}$ be a disjoint good refinement covering $\cup\{A_\alpha : \alpha < \beta\}$. Then, by Lemma 7, $\cup \mathcal{B}$ is closed. For the set $(\nabla \setminus \mathcal{B}) \cap A_\beta$, as a super-bounded set, there is a good refinement covering it by Theorem 5. Since $\nabla \setminus \cup \mathcal{B}$ is open, by suitable contraction we can make $\mathcal{A}$ satisfy that $\mathcal{A} \cup \mathcal{B}$ is a disjoint collection. Thus, by induction, we can get a refinement covering $\cup\{A_\alpha : \alpha < \mathrm{cof}(\underline{d})\}$ consistinng of disjoint basic sets. This completes the proof.

Now remains an open question: Is ∇ a union of $\mathrm{cof}(\underline{d})$ many super-bounded sets in *ZFC*? I conjecture NO. To answer this question, it may be useful to answer first the question whether Lemma 7 remains true if one replaces β by $\mathrm{cof}(\underline{d})$.

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Mathematics Department
Beijing Teacher's Collage
Beijing, China