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AN APPROXIMATE RESOLUTION OF THE PRODUCT WITH A COMPACT FACTOR

By

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Abstract. For any given approximate resolution

$$p=\{p_a \mid a\in A\} : X \longrightarrow \mathcal{X}=(X_a, \mathcal{V}_a, p_{aa'}, A)$$

of a topological space X, where $\mathcal{X}$ is uniform, all X_a are paracompact, all $\mathcal{V}_a$ are locally finite and A is cofinite, and any given compact Hausdorff space Y, the approximate resolution

$$r=p\times 1=\{r_b=p_a\times 1 \mid b=(a, \varphi)\in B\} : X\times Y \longrightarrow \mathcal{X}\times Y$$
$$=(X_a\times Y, \mathcal{V}_a\times\varphi[\mathcal{V}_a], p_{aa'}\times 1, B)$$

of the product space $X\times Y$ is constructed. Here, the indexing set B is obtained by means of the set A and certain subfamilies of

$$\Phi(a)=\{\varphi \mid \varphi : \mathcal{V}_a \longrightarrow Cov(Y)\}, \qquad a\in A,$$

while the mesh $\mathcal{V}_a\times\varphi[\mathcal{V}_a]$ is a stacked covering of $X_a\times Y$ over $\mathcal{V}_a$.

1. Introduction.

The notion of approximate resolution of a space was introduced recently by S. Mardešić and T. Watanabe ([7]). It is a logical synthesis of a suitable restriction of the approximate inverse limit and a generalization of the (classical) resolution ([15] and [6], [2] and [3]). The underlying theory eliminates many previously observed defects of non-compact or compact non-metric inverse limits.

Let us briefly recall the main definitions from [7] that we need in the sequel. A *normal* or *numerable* (open) covering of a (topological) space X is an open covering $\mathcal{V}$ of X which admits a subordinate partition of unity. The set of all normal coverings of X is denoted by $Cov(X)$. For any subset $X'\subseteq X$

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and $\mathcal{V}\in Cov(U)$, the union of all $U\in\mathcal{V}$ with $U\cap X'\neq\emptyset$ is eenoted by $st(X', \mathcal{V})$ and called the *star* of X' (with respect to $\mathcal{V}$). If $\mathcal{V}, \mathcal{W}\in Cov(X)$ and $\mathcal{W}$ refines $\mathcal{V}$, we write $\mathcal{W}<\mathcal{V}$. For any two maps $f, g: Y\to X$ that are $\mathcal{V}$-near, i.e. for every $y\in Y$ there is $U\in\mathcal{V}$ with $f(y), g(y)\in U$, we write $(f, g)<\mathcal{V}$.

An *aproximate inverse system* $\mathcal{X}$ is a collection $(X_a, \mathcal{V}_a, p_{aa'}, A)$ consisting of

(i) a preordered indexing set $A=(A, <)$ (it may be not antisymmetric) which is directed and unbounded (no maximal element);

(ii) for each $a\in A$, a space X_a and a normal covering $\mathcal{V}_a$ of X_a (called the *mesh* of X_a);

(iii) for every two related indices $a<a'$, a (continuous) map $p_{aa'}: X_{a'}\to X_a$ ($p_{aa}=1_{X_a}$ is the identity map on X_a). Furthermore, the following three conditions must be satisfied:

(A1) For any three related indices $a<a'<a''$,

$$(p_{aa'}p_{a'a''}, p_{aa''})<\mathcal{V}_a;$$

(A2) For each $a\in A$ and each $\mathcal{V}\in Cov(X_a)$ there exists $a'>a$ such that

$$(p_{aa_1}p_{a_1a_2}, p_{aa_2})<\mathcal{V},$$

whenever $a_2>a_1>a'$;

(A3) For each $a\in A$ and each $\mathcal{V}\in Cov(X_a)$ there exists $a'>a$ such that

$$\mathcal{V}_{a''}<p_{aa''}^{-1}[\mathcal{V}]=\{p_{aa''}^{-1}[U]\,|\,U\in\mathcal{V}\},$$

whenever $a''>a'$.

An approximate inverse system $\mathcal{X}$ is called *uniform*, if it satisfies the additional condition:

(AU) For any two related indices $a<a'$,

$$\mathcal{V}_{a'}<p_{aa'}^{-1}[\mathcal{V}_a].$$

With every approximate inverse system $\mathcal{X}=(X_a, \mathcal{V}_a, p_{aa'}, A)$ we can associate a uniform one $\mathcal{X}^*=(X_a, \mathcal{V}_a, p_{aa'}, A^*)$, $A^*=(A, <^*)$, such that only the ordering $<$ on A is slightly changed according to $a<^*a'\Rightarrow a<a'$ ([7], (1.6) Remark).

An *approximate mapping* $p=\{p_a\,|\,a\in A\}: X\to\mathcal{X}=(X_a, \mathcal{V}_a, p_{aa'}, A)$ from a space X into an approximate inverse system $\mathcal{X}$ is a family of maps $p_a: X\to X_a$, $a\in A$, such that the following condition holds:

(AS) For any $a\in A$ and any $\mathcal{U}\in Cov(X_a)$ there exists $a'>a$ so that $(p_{aa''}p_{a''}, p_a)<\mathcal{U}$, for every $a''>a'$.

Let (POL) denote the collection of all polyhedra (endowed with the CW-topology).

An *approximate resolution of a space* X is an approximate mapping $p=\{p_a|a\in A\}: X\to\mathcal{X}=(X_a, \mathcal{U}_a, p_{aa'}, A)$ of X into an approximate inverse system $\mathcal{X}$ satisfying the following two conditions:

(R1) For any $P\in(POL)$, $\mathcal{V}\in Cov(P)$ and $f: X\to P$ there is $a\in A$ such that, for every $a'>a$, there exists a map $g: X_{a'}\to P$ with $(gp_{a'}, f)<\mathcal{V}$;

(R2) For any $P\in(POL)$ and $\mathcal{V}\in Cov(P)$ there is a $\mathcal{V}'\in Cov(P)$ such that, for any two maps $g, g': X_a\to P$ with $(gp_a, g'p_a)<\mathcal{V}'$, there exists $a'>a$ so that $(gp_{aa''}, g'p_{aa''})<\mathcal{V}$, for every $a''>a'$.

There are a lot of characterizations of (R1) as well as of (R2). We shall use the following two ([7], §2.):

(B1)* For every $\mathcal{U}\in Cov(X)$ there are $a\in A$ and $\mathcal{V}\in Cov(X_a)$ such that $p_a^{-1}[\mathcal{V}]<\mathcal{U}$;

(B2)** For each $a\in A$ there is $a'\in A$, $a'>a$, such that

$$p_{aa'}[X_{a'}]\subseteq st(p_a[X], \mathcal{U}_{ar}).$$

At the end of this introduction, we should mention that every space X admits a (cofinite and even commutative) approximate resolution with all $X_a\in(POL)$ ([7], (2.19) Theorem).

2. Construction of the indexing set.

The (classical) resolution of a direct product was considered by S. Mardešić ([4], Theorems 4 and 5). In that case any system has been comutative and without meshes, so the desired results followed immediately applying the existence of the stacked covering refinements. In the case of an approximate resolution much more preparations have to be done to obtain an analogous result.

Let an approximate inverse system $\mathcal{X}=(X_a, \mathcal{U}_a, p_{aa'}, A)$ and a space Y be given. First of all, we would like to organize the family $\{X_a\times Y|a\in A\}$ of spaces $X_a\times Y$, $a\in A$, and the family $\{p_{aa'}\times 1|a<a'\in A\}$ of maps $p_{aa'}\times 1: X_{a'}\times Y\to X_a\times Y$, $a<a'\in A$, and $1=1_Y$, to get an approximate inverse system denoted by $\mathcal{X}\times Y$. Secondly, if $p=\{p_a|a\in A\}: X\to\mathcal{X}$ is an approximate mapping (resolution), we would like $p\times 1: X\times Y\to\mathcal{X}\times Y$ to be also an approximate

mapping (resolution).

In order to do it, we will firstly define by induction a new (hypothetic) indexing set B and an orderinng $<$ on it which is refleksive and transitive. After that we shall prove that $(B, <)$ exists and is directed and unbounded, whenever $\mathfrak{X}$ satisfies a few additional conditions.

Denote by $\Phi(a)$ the family of all functions $\varphi: \mathcal{V}_a \to Cov(Y)$, $a \in A$, and define

$$\varphi_1 < \varphi_2 \Longleftrightarrow \varphi_2(U^a(<\varphi_1(U^a) \quad \text{for every } U^a \in \mathcal{V}_a . \tag{1}$$

If $a' \in A$, $a < a'$, and $\varphi \in \Phi(a)$, $\varphi' \in \Phi(a')$, define

$$\varphi <^* \varphi' \Longleftrightarrow \text{If } U^a \in \mathcal{V}_a,\ U^{a'} \in \mathcal{V}_{a'} \text{ and } p_{aa'}[\mathcal{V}^{a'}] \subseteq \mathcal{V}^a, \text{ then } \varphi'(U^{a'}) < \varphi(U^a) . \tag{2}$$

Observe that if $\varphi_1, \varphi_2 \in \Phi(a)$, $\varphi' \in \Phi(a')$ and $\varphi_1 < \varphi_2 <^* \varphi'$ then $\varphi_1 <^* \varphi'$; similarily, $\varphi <^* \varphi_1' < \varphi_2'$ implies $\varphi <^* \varphi_2'$.

If $a \in A$, let $|a|$ denote the cardinal number of the set $\{a' \in A \mid a' < a \neq a'\}$ of all predecessors of a in A. Assume that there is $a \in A$, $|a| = 0$. Define

$$B_0 = \bigcup_{a \in A,\ |a|=0} B^a , \tag{3}$$

where $B^a = \{b = (a, \varphi) \mid \varphi \in \Phi(a)\}$ for each a, $|a| = 0$. For $b, b' \in B_0$ we put

$$b <_0 b' \Longleftrightarrow b = b' . \tag{4}$$

If $a \in A$ with $|a| = 1$ exists and $a_1 < a$ is the unique predecessor of a, $|a_1| = 0$, consider any $b_1 = (a_1, \varphi_1) \in B^{a_1} \subseteq B_0$ and the family $\Phi(a\,; b_1) = \{\varphi \mid \varphi \in \Phi(a),\ \varphi_1 <^* \varphi\}$. Now define $B_1 = \bigcup_{a \in A,\ |a|=1} B^a$, where $B^a = \bigcup_{b_1 \in B^{a_1}} B^a_{b_1}$ and $B^a_{b_1} = \{b = (a, \varphi) \mid \varphi \in \Phi(a, b_1)\}$. For $b, b' \in B_1$ put $b <_1 b'$ if and only if $b = b'$.

Suppose that all sets B_m and relations $<_m$, $1 \leqq m \leqq n-1$, $n \in N$, $n \geqq 2$, are defined such that the following conditions are satisfied:

$$B_m = \bigcup_{a \in A,\ |a|=m} B^a ; \tag{5}$$

$$B^a = \bigcup_{(b_1, \cdots, b_m) \in B^{a_1} \times \cdots \times B^{a_m}} B^a_{(b_1, \cdots, b_m)} , \tag{6}$$

where $a_1, \cdots, a_m < a$ are all the predecessors of a and $b_i = (a_i, \varphi_i) \in B^{a_i}$, $i = 1, \cdots, m$;

$$B^a_{(b_1, \cdots, b_m)} = \{b = (a, \varphi) \mid \varphi \in \Phi(a\,; b_1, \cdots, b_m)\}, \tag{7}$$

where $\Phi(a\,; b_1, \cdots, b_m) = \{\varphi \mid \varphi \in \Phi(a),\ \varphi_i <^* \varphi,\ i = 1, \cdots, m\}$;

$$b <_m b' \Longleftrightarrow b = b' , \tag{8}$$

for any pair $b, b' \in B_m$.

Assume that there is $a \in A' = A \setminus \{a' \mid a' \in A, |a'| \leqq n-1\}$ with $|a|'=0$ in A' with the restricted ordering of A. Let $a_1, \cdots, a_k < a$, $k \geqq a$, be all the predecessors of a in A. Then $|a_i| \leqq n-1$ for every $i=1, \cdots, k$. By the inductive assumption there are all the sets $(B_{|a_i|}, <_{|a_i|})$, for all a_i, $i=1, \cdots, k$. Now for any k-tuple $b_1=(a_1, \varphi_1) \in B^{a_1} \subseteqq B_{|a_1|}, \cdots, b_k=(a_k, \varphi_k) \in B^{a_k} \subseteqq B_{|a_k|}$, let us consider the family $\Phi(a; b_1, \cdots, b_k) = \{\varphi \mid \varphi \in \Phi(a), \varphi_i <^* \varphi, i=1, \cdots, k\}$ and define the sets $B^a_{(b_1, \cdots, b_k)}$, B^a, B_k and the relation $<_k$ according to (7), (6), (5) and (8) respectively. If $k \geqq n+1$, define $B_n = \cdots = B_{k-1} = \emptyset$. Finally, let us define the set

$$B = \bigcup_{n \geqq 0} B_n . \tag{9}$$

Let the ordering $<$ on B be an extension of all relations $<_n$ as follows: Take any two indices $b=(a, \varphi)$, $b'=(a', \varphi') \in B$ such that $b \in B_m$, $b' \in B_n$ and $m \neq n$. Then $\varphi \in \Phi(a; b_1, \cdots, b_m)$, where $b_i=(a_i, \varphi_i) \in B^{a_i}$, while $a_1, \cdots, a_m$ are all the predecessors of a; similarily, $\varphi' \in \Phi(a'; b'_1, \cdots, b'_n)$, $b'_j=(a'_j, \varphi'_j) \in B^{a'_j}$, where $a'_1, \cdots, a'_n$ are all the predecessors of a'. We put

$$b < b' \Longleftrightarrow (a < a'; \varphi_1 < \varphi'_{j_1}, \cdots, \varphi_m < \varphi'_{j_m}, \varphi < \varphi'_{j_{m+1}}) \tag{10}$$

where, because of $a < a'$, $m < n$ and $\{a_1, \cdots, a_m, a\} \subseteqq \{a'_1, \cdots, a'_n\}$ must hold, i.e. for every $i \in \{1, \cdots, m\}$, $a_i = a'_{j_i}$ and $a = a'_{j_{m+1}}$ for some $j_i \in \{1, \cdots, n\}$. It is easy to verify that $<$ is a reflexive and transitive relation on B. Observe that $b=(a, \varphi)<(a', \varphi')=b'$ and $p_{aa'}[U^{a'}] \subseteqq U^a$, $U^a \in \mathcal{V}_a$, $U^{a'} \in \mathcal{V}_{a'}$, imply $\varphi(U^{a'}) < \varphi(U^a)$.

LEMMA 1. *Let $\mathfrak{X}=(X_a, \mathcal{V}_a, p_{aa'}, A)$ be a uniform approximate inverse system with all $\mathcal{V}_a$ locally finite and A cofinite, and let Y be any space. Then the set B with the ordering $<$, defined by* (3)—(10), *can be constructed. Mroeover, B is a directed and unbounded set.*

PROOF. According to the previous considerations, it is sufficient to prove that B is not empty and that for any pair $b, b' \in B$ there exists $b'' \in B$ such that $b < b'' \neq b$ and $b' < b'' \neq b'$. In order to do it, we are proving four claims.

CLAIM 1. For every $a \in A$, the set $B^a \subseteqq B_{|a|}$ is not empty. More precisely, all the families $\Phi(a)$ and $\Phi(a; b_1, \cdots, b_{|a|})$ are not empty.

We prove this claim by induction on $|a| \in N_0$, $a \in A$. The set A is cofinite, so every $a \in A$ has at most finitely many predecessors, and there is $a \in A$ with $|a|=0$. Let us consider any $a \in A$, $|a|=0$. Then $\Phi(a) \neq \emptyset$ (for instance, the function $\varphi: \mathcal{V}_a \to Cov(Y)$, $\varphi(U^a)=\{Y\}$ for all $U^a \in \mathcal{V}_a$, belongs to $\Phi(a)$) and consequently $B^a=\{b=(a, \varphi) | \varphi \in \Phi(a)\} \neq \emptyset$. Suppose that for all $a \in A$, $|a| \leqq n-1$, $n \in N$, the sets B^a and all the families $\Phi(a; b_1, \cdots, b_{|a|})$ are not empty. Let $a \in A$ be any element with $|a|'=0$ in the set $A'=A \setminus \{a' | a' \in A, |a'| \leqq n-1\}$ with restricted ordering of A, and let $a_1, \cdots, a_k < a$, $k \geqq n$, be all the predecessors of a in A. Then $|a_i| \leqq n-1$ for every $i=1, \cdots, k$, so all $B^{a_i} \neq \emptyset$ and all $\Phi(a_i; b_1^i, \cdots ‘b^i_{|a_i|}) \neq \emptyset$ by the inductive assumption. For each i choose one $b_i=(a_i, \varphi_i) \in B^{a_i}$, $i=1, \cdots, k$, and define $\varphi: \mathcal{V}_a \to Cov(Y)$ by the following relation

$$\varphi(U^a)=\bigwedge_i (\bigwedge_{U^{a_i}} \varphi_i(U^{a_i})), \qquad \text{where } U^a \in \mathcal{V}_a \text{ with } p_{a_i a}[U^a] \subseteq U^{a_i} \in \mathcal{V}_{a_i},\ i \in \{1, \cdots, k\};$$

$$\varphi(U^a)=\text{any covering from } Cov(Y), \text{ otherwise.}$$

Here $\bigwedge_{j \in J} \mathcal{V}_j$ denotes the family $\{\bigcap_j V^j | V^j \in \mathcal{V}_j, j \in J\}$. Since $\mathcal{V}_{a_i}$ are locally finite coverings and $1 \leqq i \leqq k \in N$, the function φ is well defined. Moreover $\varphi_i <^* \varphi$, $i=1, \cdots, k$, i.e. $\varphi \in \Phi(a; b_1, \cdots, b_k) \neq \emptyset$ and thus $(a, \varphi) \in B^a \neq \emptyset$.

CLAIM 2. For each $b=(a, \varphi) \in B$ and each $\bar{\varphi} \in \Phi(a)$, if $\varphi < \bar{\varphi}$ then $(a, \bar{\varphi})= \bar{b} \in B$.

This claim is an immediate consequence of the definition of B^a and relations (1) and (3).

CLAIM 3. For each $b=(a, \varphi) \in B$ and each $a' \in A$, $a < a' \neq a$, there exists $\varphi' \in \Phi(a')$ such that $(a', \varphi')=b' \in B$ and $b < b' \neq b$.

To prove it, observe that all the predecessors $a_1, \cdots, a_m$ of a and a are some of all the predecessors $a_1', \cdots, a_n'$ of a', $n \geqq m+1 \geqq 1$. Denote $a_1'=a_1, \cdots, a_m'=a_m$, $a_{m+1}'=a$. Let b belong to $B^a_{(b_1, \cdots, b_m)}$, i.e. $b_i=(a_i, \varphi_i) \in B^{a_i}$ and $\varphi_i <^* \varphi$, $i=1, \cdots, m$. Take $b_{m+1}=(a, \varphi) \in B^a=B^{a'_{m+1}}$. By Claim 1, every $B^{a_j'}$ is not empty, so choose any $b_j=(a_j', \varphi_j) \in B^{a_j'}$, $j=m+2, \cdots, n$. Let us define φ': $\mathcal{V}_{a'} \to Cov(Y)$ as follows:

$$\varphi'(U^{a'})=\bigwedge_j (\bigwedge_{U^{a_j'}} \varphi_j(U^{a_j'})), \qquad \text{where } U^{a'} \in \mathcal{V}^{a'} \text{ and } p_{a_j' a'}[U^{a'}] \subseteq U^{a_j'} \in \mathcal{V}_{a_j'},\ j \in \{1, \cdots, n\};$$

$$\varphi'(U^{a'})=\text{any covering from } Cov(Y), \text{ otherwise.}$$

We conclude, as in the proof of Claim 1, that $\varphi' \in \Phi(a'; b_1, \cdots, b_n)$ and $(a', \varphi') =b' \in B^{a'}_{(b_1, \cdots, b_n)} \subseteq B^{a'} \subseteq B$. From the definition of φ' and (10) follows $b < b' \neq b$.

CLAIM 4. For every pair $b=(a, \varphi)$, $\bar{b}=(a, \bar{\varphi}) \in B^a \subseteq B$ and every $a' > a$ there exists $b'=(a', \varphi')\in B$ such that $b<b'$ and $\bar{b}<b'$.

To see that, take $\varphi': \mathcal{U}_{a'}\to Cov(Y)$,

$$\varphi'(U^{a'})=\bigwedge_j(\bigwedge_{U^{a'_j}}(\varphi_j(U^{a'_j})\wedge\bar{\varphi}_j(U^{a'_j}))),\ U^{a'}\in\mathcal{U}_{a'}$$

$$\text{and } p_{a'_j a'}[U^{a'}]\subseteq U^{a'_j}\in\mathcal{U}_{a'_j},\ j\in\{1, \cdots, n\};$$

$$\varphi'(U^{a'})=\text{any covering from } Cov(Y), \quad \text{otherwise},$$

where $a_1, \cdots, a_m$ are all the predecessors of a; $a'_1=a_1, \cdots, a'_m=a_m$, $a'_{m+1}=a, \cdots, a'_n$ are all the predecessors of a'; $\varphi_1, \cdots, \varphi_m$ and $\bar{\varphi}_1, \cdots, \bar{\varphi}_m$ are determinated by b and $\bar{b}$ respectively; $\varphi_{m+1}=\varphi$ and $\bar{\varphi}_{m+1}=\bar{\varphi}$; $\varphi_j=\bar{\varphi}_j$ is determinated by any $b_j=(a_j, \varphi_j)\in B^{a'_j}$, $j=m+2, \cdots, n$.

Then φ' is well defined and belongs to $\Phi(a'; b_1, \cdots, b_n)$, so $(a', \varphi')=b'\in B$. By the definition of φ' and (10) we infer that $b<b'$ and $\bar{b}<b'$ hold.

To finish the proof of Lemma 1, observe only that it is an immediate consequence of the four previous claims.

REMARK 1. (a) The set $(B, <)$ in Lemma 1 is generally non-cofinite (even Y is compact);

(b) Claim 1 and $B_n=\emptyset$ for some n do not contradict each other.

3. Construction of the approximate product resolution.

Let $\mathcal{U}$ be any open covering of a space X, and let, for every $U\in\mathcal{U}$, an open covering $\mathcal{V}_U$ of a space Y be given. Then the family $\mathcal{S}=\{U\times V \mid U\in\mathcal{U}, V\in\mathcal{V}_V\}$ is an open covering of the product space $X\times Y$, so called a *stacked covering* of $X\times Y$ (over $\mathcal{U}$).

The well known fact is that for every space X, every compact Hausdorff space Y and every normal covering $\mathcal{W}$ of $X\times Y$, there exist a normal covering $\mathcal{U}$ of X and a stacked covering $\mathcal{S}$ of $X\times Y$ (over $\mathcal{U}$) which refines $\mathcal{W}$ ([1], pp. 357, 361). Moreover, $\mathcal{S}$ is normal, i.e. it belongs to $Cov(X\times Y)$. We shall denote this stacked covering $\mathcal{S}$ by $\mathcal{U}\times(\mathcal{V}_U)_{U\in\mathcal{U}}$.

Let $\mathcal{X}=(X_a, \mathcal{U}_a, p_{aa'}, A)$, Y and $(B, <)$ be the same as in Lemma 1. For each $b=(a, \varphi)\in B$, let us define the space $Z_b=X_a\times Y$ and the mesh $\mathcal{S}_b=$the stacked covering $\mathcal{U}_a\times(\varphi(U^a))_{U^a\in\mathcal{U}_a}$, and, for each pair $b<b'=(a', \varphi')\in B$, the mapping $r_{bb'}=p_{aa'}\times 1: X_{a'}\times Y=Z_{b'}\to Z_b=X_a\times Y$. The collection $\mathcal{Z}=(Z_b, \mathcal{S}_b, r_{bb'}, B)$ we rather write down as $\mathcal{X}\times Y=(X_a\times Y, \mathcal{U}_a\times\varphi[\mathcal{U}_a], p_{aa'}\times 1, B)$. We are ready to prove that it forms a system.

LEMMA 2. *Let $\mathcal{X}=(X_a, \mathcal{U}_a, p_{aa'}, A)$ be a uniform approximate inverse sys-*

tem with all $\mathcal{V}_a$ locally finite and A cofinite. If all X_a are paracompact and Y is a compact Hausdorff space, then $\mathcal{X}\times Y=(X_a\times Y,\ \mathcal{V}_a\times\varphi[\mathcal{V}_a],\ p_{aa'}\times 1,\ B)$, is a uniform approximate inverse system.

PROOF. The condition (A1) one checks trivially by means of (A1) for $\mathcal{X}$. In order to verify (A2), let $b=(a,\varphi)\in B$ and $\mathcal{W}\in Cov(Z_b)=Cov(X_a\times Y)$ be arbitrary elements. Choose a stacked covering $\mathcal{U}\times(\mathcal{V}_U)_{U\in\mathcal{U}}\in Cov(X_a\times Y)$ refining $\mathcal{W}$, where $\mathcal{U}\in Cov(X_a)$. By (A2) for $\mathcal{X}$, there is $\bar{a}\in A$, $\bar{a}>a$, such that, for all $a''>a'>\bar{a}$, $(p_{aa'}p_{a'a''},\ p_{aa''})<\mathcal{U}$ holds. By Claim 1, there is $\bar{b}=(\bar{a},\bar{\varphi})\in B$. Let $b'=(a',\varphi')$, $b''=(a'',\varphi'')\in B$ be chosen arbitrarily, so that $\bar{b}<b'<b''$. Then $a''>a'>\bar{a}$. Hence $(r_{bb'}r_{b'b''},\ r_{bb''})=((p_{aa''}\times 1)\circ(p_{a'a''}\times 1),\ p_{aa''}\times 1)=((p_{aa'}p_{a'a''})\times 1,\ p_{aa''}\times 1)<\mathcal{U}\times(\mathcal{V}_U)_{U\in\mathcal{U}}<\mathcal{W}$, and (A2) for $\mathcal{X}\times Y$ is satisfied. To prove (A3) for $\mathcal{X}\times Y$, let $b=(a,\varphi)\in B$ and $\mathcal{W}\in Cov(Z_b)=Cov(X_a\times Y)$ be chosen arbitrarily. Since X_a is paracompact, we can choose a stacked covering $\mathcal{U}\times(\mathcal{V}_U)_{U\in\mathcal{U}}\in Cov(X_a\times Y)$ which refines $\mathcal{W}$, where $\mathcal{U}\in Cov(X_a)$ is locally finite. Pick up a covering $\mathcal{U}'\in Cov(X_a)$ such that st $\mathcal{U}'$ refines $\mathcal{U}$. By (A2) and (A3) for $\mathcal{X}$, there is $a'\in A$, $a'>a$, so that, for every $a''>a'$, $(p_{aa'}p_{a'a''}p_{aa''})<\mathcal{U}'$ and $\mathcal{V}_{a''}\ll p_{aa''}^{-1}[\mathcal{U}']$ hold. Let $a_1,\cdots,a_m$ be all the predecessors of a and let $a'_1=a_1,\cdots,a'_m=a_m,\ a'_{m+1}=a,\cdots,a'_n$ be all the predecessors of a'. Then $\varphi\in\Phi(a;b_1,\cdots,b_m)$, where $b_i=(a_i,\varphi_i)\in B^{a_i}$, $i=1,\cdots,m$. Choose now $b'_j=(a'_j,\varphi'_j)\in B^{a'_j}$, $j=1,\cdots,n$, where $b'_1=b_1,\cdots,b'_m=b_m$, $b'_{m+1}=b$ and $b'_j=(a'_j,\varphi'_j)$ are taken arbitrarily for $j=m+2,\cdots,n$, Let us define $\varphi':\mathcal{V}_{a'}\to Cov(Y)$ by

$$\varphi'(U^{a'})=\bigwedge_j(\bigwedge_{U^{a'_j}}\varphi'_j(U^{a'_j}))\wedge(\bigwedge_U\mathcal{V}_U),$$

where $U^{a'}\in\mathcal{V}_{a'}$, $p_{aa'}[U^{a'}]\subseteq U\in\mathcal{U}$, $p_{a'_ja'}[U^{a'}]\subseteq U^{a'_j}\in\mathcal{V}_{a'_j}$ and $j\in\{1,\cdots,n\}$. Since all $\mathcal{V}_{a'_j}$ and $\mathcal{U}$ are locally finite and $\mathcal{X}$ is uniform, the function φ' is well defined. Moreover, that definition implies $\varphi'\in\Phi(a';b_1,\cdots,b_m,b,b'_{m+2},\cdots,b'_n)$ and hence $(a',\varphi')=b'\in B$ and $b'>b$. Let $b''=(a'',\varphi'')\in B$ be any index such that $b''>b'$. Then $a''>a'$ and thus for every $U^{a''}\in\mathcal{V}_{a''}$ there is $U'_1\in\mathcal{U}'$ so that $p_{aa''}[U_{a''}]\subseteq U'_1$. Since $\mathcal{X}$ is uniform, there is $U^{a'}\in\mathcal{V}_{a'}$ so that $p_{aa''}[U^{a''}]\subseteq U^{a'}$, therefore $\varphi''(U^{a''})<\varphi'(U^{a'})$. Finally, $p_{aa'}[U^{a'}]\subseteq U'_2$ for some $U'_2\in\mathcal{U}'$ and $U'_1, U'_2\subseteq st(U',\mathcal{U}')\subseteq U$ for some $U'\in\mathcal{U}'$ and $U\in\mathcal{U}$. Hence $\varphi'(U^{a'})<\mathcal{V}_U$. Thus we may conclude that for every $U^{a''}\in\mathcal{V}_{a''}$ there exists $U\in\mathcal{U}$ such that $p_{aa''}[U^{a''}]\subseteq U$ and $\varphi''(U^{a''})<\mathcal{V}_U$, i.e. $\mathcal{S}_{b''}=\mathcal{V}_{a''}\times\varphi''[\mathcal{V}_{a''}]<p_{aa''}^{-1}[\mathcal{U}]\times(\mathcal{V}_U)_{U\in\mathcal{U}}=(p_{aa''}\times 1)^{-1}\cdot[\mathcal{U}\times((\mathcal{V}_U)_{U\in\mathcal{U}}]<(p_{aa''}\times 1)^{-1}[\mathcal{W}]=r_{bb''}^{-1}[\mathcal{W}]$ and (A3) is verified.

To prove the uniformity condition for $\mathcal{X}\times Y$, let $b=(a,\varphi)$, $b'=(a',\varphi')\in B$ be any two indices with $b<b'$. Since $\mathcal{X}$ is uniform, $\mathcal{V}_{a'}<p_{aa'}^{-1}[\mathcal{V}_a]$ holds. Furthemore, if $p_{aa'}[U^{a'}]\subseteq U^a$ then $\varphi'(U^{a'})<\varphi(U^a)$. Therefore $\mathcal{V}_{a'}\times\varphi'[\mathcal{V}_{a'}]<$

$p_{aa'}^{-1}[\mathcal{U}_a]\times\varphi[\mathcal{U}_a]=(p_{aa'}\times 1)^{-1}[\mathcal{U}_a\times\varphi[\mathcal{U}_a]]$ and Lemma 2 is completely proved.

REMARK 2. The assumption in Lemma 2 that all $\mathcal{U}_a$ are locally finite, beside all X_a are paracompact, may seem superfluous, but stricktly speaking it is not the case. Indeed, if all X_a are paracompact, and some $\mathcal{U}_a$ are not locally finite, we can choose locally finite coverings $\mathcal{U}_a^*\in Cov(X_a)$, $\mathcal{U}_a^*<\mathcal{U}_a$, $a\in A$, to be the new meshes. Of course, one loses now (A1) and (AU) for $\mathcal{X}$, while (A2) and (A3) remain valid. Reordering $(A, <)$ in an obvious way (using (A2) and (A3)) one obtains $A^*=(A, <^*)$, which is also cofinite, and the uniform approximate inverse system $\mathcal{X}^*=(X_a, \mathcal{U}_a^*, p_{aa'}, A^*)$ closely related to $\mathcal{X}$ (compare [7], (1.6) Remark). But now $\mathcal{X}^*$ induces a new set B^* and a new ordering on it. Therefore Lemma 2 can be only restated in these new terms.

We now state our main theorem:

THEOREM. *Let $p=\{p_a|a\in A\}: X\to\mathcal{X}=(X_a, \mathcal{U}_a, p_{aa'}, A)$ be any approximate resolution of a space X, such that $\mathcal{X}$ is uniform, all X_a are paracompact, all $\mathcal{U}_a$ are locally finite and A is confinite. If Y is a compact Hausdorff space and $\mathcal{X}\times Y=(X_a\times Y, \mathcal{U}_a\times\varphi[\mathcal{U}_a], p_{aa'}\times 1, B)$ the system as in Lemma 2, then*

$$r=p\times 1=\{r_b=p_a\times 1|b=(a, \varphi)\in B\}: X\times Y \longrightarrow \mathcal{X}\times Y$$

is a uniform approximate resolution of the product space $X\times Y$.

PROOF. Because of Lemma 1 and Lemma 2 it is sufficient to verify the conditions (AS), (R1) and (R2).

Let $b=(a, \varphi)\in B$ and $\mathcal{W}\in Cov(X_a\times Y)$ be given. Choose a stacked covering $\mathcal{U}\times(\mathcal{V}_U)_{U\in\mathcal{U}}\in Cov(X_a\times Y)$ refining $\mathcal{W}$, where $\mathcal{U}\in Cov(X_a)$. Applying (AS) for p on a and $\mathcal{U}$, take $a'\in A$, $a'>a$, such that, for every $a''>a'$, $(p_{aa''}p_{a''}, p_a)<\mathcal{U}$ holds. Take any $b'=(a', \varphi')\in B$ with $b'>b$, which exists by Claim 1. Let $b''=(a'', \varphi'')\in B$ be given such that $b''>b'$. Then $a''>a'$ and $(r_{bb''}r_{b''}, r_b)=((p_{aa''}\times 1)\circ(p_{a''}\times 1), p_a\times 1)=((p_{aa''}p_{a''})\times 1, p_a\times 1)<\mathcal{U}\times(\mathcal{V}_U)_{U\in\mathcal{U}}<\mathcal{W}$. Therefore $r=p\times 1: X\times Y\to\mathcal{X}\times Y$ is an approximate mapping.

In order to verify (R1) for $r=p\times 1$, recall that (R1) is equivalent to (B1)* ([7], §2):

For every $\mathcal{W}\in Cov(X_a\times Y)$ there exist $b=(a, \varphi)\in B$ and $\mathcal{S}\in Cov(X_a\times Y)$ such that $r_b^{-1}[\mathcal{S}]=(p_a\times 1)^{-1}[\mathcal{S}]<\mathcal{W}$.

Let $\mathcal{W}\in Cov(X\times Y)$ be given. Choose a stacked covering $\mathcal{U}\times(\mathcal{V})_U)_{U\in\mathcal{U}}\in Cov(X\times Y)$ refining $\mathcal{W}$, where $\mathcal{U}\in Cov(X)$. Applying (B1)* for p on $\mathcal{U}$, we get $a\in A$ and $\mathcal{U}'\in Cov(X_a)$ such that $p_a^{-1}[\mathcal{U}']<\mathcal{U}$. Take any $b=(a, \varphi)\in B$. Let

$\mathcal{S}=\mathcal{V}'\times(\mathcal{V}'_{U'})_{U'\in\mathcal{V}'}\in Cov(X_a\times Y)$ be the stacked covering obtained by choosing $\mathcal{V}'_{U'}=\mathcal{V}_U$ for some $U\in\mathcal{V}$ with $U'\subseteq p_a[U]$. Then $(p_a\times 1)^{-1}[\mathcal{S}]=(p_a\times 1)^{-1}[\mathcal{V}'\times(\mathcal{V}'_{U'})_{U'\in\mathcal{V}'}]=p_a^{-1}[\mathcal{V}']\times(\mathcal{V}'_{U'})_{U'\in\mathcal{V}'}<\mathcal{V}\times(\mathcal{V}_U)_{U\in\mathcal{V}}<\mathcal{W}$, and (R1) for $p\times 1$ holds.

It remains to verify (R2) for $r=p\times 1$. We will check the equivalent condition (B2)** ([7], §2):

For every $b=(a,\varphi)\in B$ there exists $b'=(a',\varphi')\in B$, $b'>b$, such that $r_{bb'}[Z_{b'}]\subseteq st(r_b[Z],\mathcal{S}_b)$, i.e. $(p_{aa'}\times 1)[X_{a'}\times Y]\subseteq st((p_a\times 1)[X\times Y],\mathcal{V}_a\times\varphi[\mathcal{V}_a])$.

Let $b=(a,\varphi)\in B$ be given. The condition (B2)** for $\mathcal{X}$ provides $a'\in A$, $a'>a$, such that $p_{aa'}[X_{a'}]\subseteq st(p_a[X],\mathcal{V}_a)$. Take any $b'=(a',\varphi')\in B$, $b'>b$ (Claim 3). Then

$$(p_{aa'}\times 1)[X_{a'}\times Y]=p_{aa'}[X_{a'}]\times Y\subseteq st(p_a[X],\mathcal{V}_a)\times Y$$
$$=st((p_a\times 1)[X\times Y],\mathcal{V}_a\times\varphi[\mathcal{V}_a]),$$

which completes the proof of the theorem.

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