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ON A TOPOLOGICAL INVARIANT OF FINITE TOPOLOGICAL SPACES AND ENUMERATIONS

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By

Shōji OCHIAI

1. Introduction and preliminaries.

Let $T(n)$ be the number of topologies on a set with n-element. It is an old and difficult problem to determine this number. Many authors attacked this problem and so far determined this number $T(n)$ for small value n. Recently, M. Erné pronounced that he determined this number $T(n)$ for all $n\leqq 13$, by using a new reduction formula [4].

In this paper, we will define a topological invariant which relates to the determination of $T(n)$, investigate its property and make some computations in connection with its invariant rather than intend to determine $T(n)$.

If X is a finite topological space with n-element, then X is determined by the minimal open set U_x containing each of its point x. If $U_x\subseteq U_y$ holds, then we define the relation $\leq$ by $x\leq y$ on X. This relation $\leqq$ is reflexive and transitive and so $(X, \leq)$ is a quasi ordered set.

Conversely, with a given quasi ordered set $(X, \leq)$, if we define U_x by $U_x=\{y \mid y\leq x\}$, we can associate the finite topological space with the minimal base $\{U_x \mid x\in X\}$. This gives a one to one correspondence between all topologies on X and all quasi orders on X, and also induces a one to one correspondence between all T_0 topologies on X and all partial orders on X [**1**, p, 28], [**2**, p. 14], [**7**, p. 142]. It is also well known that there is a one to one correspondence between all topologies on X and all the labeled transitive digraphs with n-element [**5**]. Furthermore there is a one to one correspondence between all T_0 topologies on a set with n-element, all finite distributive lattices L of rank n, all the labeled transitive acyclic digraphs with n-element [**5**], [**11**]. Let $T_0(n)$ denote the number of all T_0 topologies on a set with n-element. J. W. Evance, F. Harary and M. S. Lynn proved that

$$T(n)=\sum_{m=1}^{n}S(n, m)T_0(m),$$

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where $S(n, m)$ denotes the number of partitions of an n-set into m-disjoint subsets (Stirling number of the second kind) [**5**].

In this paper, we use the $n\times n$, (0, 1) matrix defined as follows. Let $X=\{x_1, x_2, \cdots, x_n\}$ be the finite topological space (X, T) with the minimal base $\{U_1, U_2, \cdots, U_n\}$. We define the $n\times n$, (0, 1) matrix $A=[a_{i,j}]$ by

$$a_{i,j}=\begin{cases} 1 & x_j\in U_i \\ 0 & x_j\notin U_i . \end{cases}$$

We denote this matrix A corresponding to the topology T by $M(T)$. We define two operations $+, \cdot$, on $B=\{0, 1\}$ as follows, $0+0=0$, $1+1=0+1=1+0=1$, $1\cdot 1=1$, $0\cdot 1=1\cdot 0=0\cdot 0=0$. A Boolean vector of dimension n is an n-tuple $[a_1, a_2, \cdots, a_n]$ over B and a Boolean matrix A is the matrix over B. The i-th row (column) of A is denoted by $A_{i*}(A_{*i})$. The matrix multiplication, the matrix addition, the vector addition and the scalar multiplication are defined as usual where those involve Boolean arithmetic. A Boolean matrix A is called reflexive if for all i, $a_{ii}=1$. H. Sharp proves the next result.

THEOREM [**9**]. *A reflexive, $n\times n$, Boolean matrix A corresponds to a topology on X with n-element if and only if $A^2=A$.*

2. Definition and properties of a topological invariant.

We will give a definition of a topological invariant which we will study.

DEFINITION 1. Let A be an n-th matrix corresponding to a topology and $A^{(1)}, A^{(2)}, \cdots, A^{(k)}$ be all of the matrices corresponding to topologies in the form of

$$\begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ b_1 & & & & \\ b_2 & & A & & \\ \vdots & & & & \\ b_n & & & & \end{bmatrix}$$

We put $a=[a_1, a_2, \cdots, a_n]$ and $b={}^t[b_1, b_2, \cdots, b_n]$.

Let $\alpha_0(A)$, $\alpha_1(A)$ and $\alpha_2(A)$ denote the number of the above matrices satisfying the following conditions (i), (ii) and (iii) respectively.

(i) $a=[0, 0, \cdots, 0]$.

(ii) a is equal to some row of A.

(iii) a is not equal to any rows of A and $a\neq[0, 0, \cdots, 0]$.

(iv) Finally, we define $\alpha(A)$ by $\alpha(A)=\alpha_0(A)+\alpha_1(A)+\alpha_2(A)=k$.

DEFINITION 2. Let us define the *row space* of a Boolean matrix by the span of the set of all rows of the matrix. We define the *column space* similarly.

DEFINITION 3. Let $A=[a_{ij}]$, $B=[b_{ij}]$ be $m\times n$ Boolean matrices. If $a_{ij}=1$ implies $b_{ij}=1$ for every i and j, we define $A\leq B$.

Note that the row space and the column space of the matrix are the poset, even if the matrix does not correspond to any poset.

Let $R(A)$ $(C(A))$ denote the row (column) space of A. Then $R(A)$ $(C(A))$ is a lattice with the join and the meet defined as follows. The join of two elements is their sum and the meet of two elements is the sum of the elements of $R(A)$ $(C(A))$ which are less than or equal to both elements. If A is the matrix corresponding to a topology, then $A_{i*}(A_{*i})$ is the join irreducible element of $R(A)$ $(C(A))$ for every i.

From the definition of $\alpha(A)$, it is easy to see that the next lemma is true.

LEMMA 1. *Let $T(n+1)$ be the number of topologies on a set with n-element. Then we have*

$$T(n+1)=\Sigma_A\alpha(A)$$

where A runs over all the n-th matrices corresponding to topologies.

DEFINITION 4. The matrices A and B corresponding to topologies respectively are called *equivalent* if there exists a permutation matrix P such that $B={}^tPAP$.

Let (X, T) and (X, T') be finite topological spaces.

THEOREM 1 [**9**], [**10**]. *The finite topological spaces (X, T) and (X, T') are homeomorphic if and only if $M(T)$ and $M(T')$ are equivalent.*

THEOREM 2. *If finite topological spaces (X, T) and (X, T') are homeomorphic, then we have $\alpha(M(T))=\alpha(M(T'))$.*

PROOF. It is easy to see that there exists a one to one correspondence from the set of the matrices $\left\{\begin{bmatrix} 1 & * \\ * & M(T)\end{bmatrix}\right\}$ corresponding to topologies to the set of matrices $\left\{\begin{bmatrix} 1 & * \\ * & M(T')\end{bmatrix}\right\}$ corresponding to topologies.

Now consider an n-th matrix $A=[a_{i,j}]$ corresponding to a topology. By Theorem 2 [**9**], 3 [**9**], we obtain the following result, LEMMA 2. *$A_{i*} \leq A_{j*}$ holds if and only if $A_{*j} \leq A_{*i}$ holds.*

Let $\sim$ be the equivalence relation in $[n]$ defined by $i \sim j$ if $A_{i*} = A_{j*}$, where $[n]$ denotes $[n] = \{1, 2, \cdots, n\}$. Let $k(1), k(2), \cdots, k(l)$ be the system of representatives for such equivalence classes. Then we define the new l-th matrix corresponding to a topology as $\tilde{A} = [a_{k(i), k(j)}]$, $1 \leq i, j \leq l$. We will call $\tilde{A}$ the *reduced matrix* of A. The matrix $\tilde{A}$ corresponds to a T_0 topology.

LEMMA 3. *If $\tilde{A}$ is the reduced matrix of a matrix A corresponding to a topology, then we have*

$$\alpha(A) = \alpha(\tilde{A}) .$$

LEMMA 4. *If A is the matrix corresponding to a topology, then we have*

$$\alpha(A) = \alpha({}^tA) .$$

The proof of these Lemmas is omitted.

THEOREM 3 (Direct sum theorem). *Let $A_1, A_2, \cdots, A_k$ be the matrices corresponding to topologies respectively and A the matrix corresponding to a topology with $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, that is*

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & 0 \\ & & \ddots & \\ 0 & & & A_k \end{bmatrix} .$$

Then we have $\alpha(A) = 2(\prod_{i=1}^k \alpha_0(A_i) - \sum_{i=1}^k \alpha_0(A_i)) + \sum_{i=1}^k \alpha(A_i) + k - 1$.

PROOF. The proof is carried out by the induction on k. For $k=2$, since $\alpha_0(A)$ is nothing but the cardinality of $C(A)$, $(|C(A)|)$, we obtain $\alpha_0(A) = |C(A)| = |C(A_1)|\,|C(A_2)| = \alpha_0(A_1)\alpha_0(A_2)$.

It is easy to see $\alpha_1(A) = \alpha_1(A_1) + \alpha_1(A_2)$, $\alpha_2(A) = (|R(A_1)| - 1)\,(|R(A_2)| - 1) + \alpha_2(A_1) + \alpha_2(A_2) = \alpha_0(A_1)\alpha_0(A_2) - \alpha_0(A_1) - \alpha_0(A_2) + \alpha_2(A_1) + \alpha_2(A_2) + 1$. Therefore, we obtain $\alpha(A) = 2(\alpha_0(A_1)\alpha_0(A_2) - \alpha_0(A_1) - \alpha_0(A_2)) + \alpha(A_1) + \alpha(A_2) + 1$. Assume that theorem is true for $k = i - 1$. Then $\alpha(A_1 \oplus A_2 \oplus \cdots \oplus A_i) = \alpha((A_1 \oplus \cdots \oplus A_{i-1}) \oplus A_i) = 2(\prod_{j=1}^i \alpha_0(A_j) - \sum_{j=1}^i \alpha_0(A_j)) + \sum_{j=1}^i \alpha(A_j) + k - 1$. This concludes the proof.

COROLLARY. *Let A be the matrix corresponding to a finite topological space (X, T) and A_i $(i = 1, 2, \cdots, k)$ the matrices corresponding to the components C_i $(i = 1, 2, \cdots, k)$ of (X, T) respectively. Then we get $\alpha(A) = 2(\prod_{i=1}^k \alpha_0(A_i) - \sum_{i=1}^k \alpha_0(A_i)) + \sum_{i=1}^k \alpha(A_i) + k - 1$.*

PROOF. Since A is equivalent to $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ [**10**], we obtain this corollary from Theorems 1, 2 and 3.

DEFINITION 5. A subset I of a poset P is called an *order ideal* of P if $x \in I$ and $y \leq x$ imply $y \in I$.

The set of order ideals of P is a distributive lattice denoted by $J(P)$. The join and the meet on order ideals are just ordinary union and intersection as a subset of P.

LEMMA 5. *Let A be an n-th matrix corresponding to a topology. If we choose two posets P and Q as $P=\{A_{k*} \mid k \in K\}$ and $Q=\{A_{*k} \mid k \in K\}$ where $K \subseteq [n]$ respectively, then we have $|J(P)|=|J(Q)|$.*

PROOF. By using Lemma 2, if we denote the dual poset of Q by Q^*, we see that two posets P and Q^* are isomorphic. Therefore, we obtain $J(P) \cong J(Q^*)=(J(Q))^*$. This shows $|J(P)|=|J(Q)|$.

We see immediately that above Lemma is not true in the general Boolean matrix, but the next result is known.

THEOREM [**7**, p. 13]. *Let A be an $m \times n$ Boolean matrix. Then we get $|C(A)|=|R(A)|$.*

THEOREM 4. *Let A be an n-th matrix corresponding to a topology. For every $a \in R(A)$, $B=\begin{bmatrix} 1 & a \\ b & A \end{bmatrix}$ is the matrix corresponding to a topology if and only if b is the element of $C(A)$ and the set $\{i \mid A_{*i} \leq b\}$ is a subset of I_a where $I_a=\{i \mid a \leq A_{i*}\}$.*

PROOF. By Theorem [**9**], B is the matrix corresponding to a topology if and only if $B^2=B$ holds. Since the arithmetic is Boolean, we get $aA=a$, $Ab=b$, $ba \leq A$. From the first two equations, we obtain $a \in R(A)$, $b \in C(A)$. By the relation $ba \leqq A$, if one, for example, a is fixed, another, b is restricted as follows. Let us define the subposet Λ_a of $C(A)$ by $\Lambda_a=\{A_{*i} \mid i \in I_a\}$. Then we can see that b is the element of the sublattice of $C(A)$ generated by the elements of Λ_a. Therefore, that above equations hold is equivalent to $b \in C(A)$ and $\{i \mid A_{*i} \leq b\} \subseteq I_a$.

COROLLARY. *Let A be an n-th matrix corresponding to a topology and let the subposet V_a of $R(A)$ be $V_a=\{A_{i*} \mid a \leq A_{i*}\}$ for every $a \in R(A)$. Then we* *have*

$$\alpha(A)=\sum_{a\in R(A)}|J(V_a)|.$$

PROOF. By above Theorem, for every a, we can see that b is the element of the sublattice of $C(A)$ generated by elements of Λ_a if and only if the matrix $\begin{bmatrix}1 & a\\ b & A\end{bmatrix}$ is the one corresponding to a topology. By the fundamental theorem for finite distributive lattices, [12, p. 106], the number of such matrices corresponding to topologies is equal to $|J(\Lambda_a)|$. By Lemma 5, we get $|J(\Lambda_a)|=|J(V_a)|$. Summing on a, we obtain the desired result.

REMARK. It is easy to see that the function α from the poset of the matrices corresponding to topologies with before mentioned order to the set of natural numbers with its usual order is neither monotone increasing nor monotone decreasing.

3. Computation of $\alpha(A)$.

In this section, we will compute $\alpha(A)$ of several matrices A for later use.

LEMMA 6. *Let A be the following n-th triangular matrix corresponding to topology and E_n the n-th identity matrix*

$$A=\begin{bmatrix}1 & 1 & \cdots & 1\\ & 1 & \cdots & 1\\ & & \cdots & \cdot\\ 0 & & \cdot\cdot & \cdot\\ & & \cdot & 1\end{bmatrix}.$$

Then we have $\alpha(A)=n(n+5)/2+1$, $\alpha(E_n)=2^{n+1}+n-1$.

PROOF. The proof comes from facts that $\alpha_0(A)=n+1$, $\alpha_1(A)=n(n+1)/2+n$, $\alpha_2(A)=0$, $\alpha_0(E_n)=2^n$, $\alpha_1(E_n)=2n$, $\alpha_2(E_n)=2^n-n-1$.

LEMMA 7. *Let A be an n-th matrix corresponding to a T_0 topology and B the following matrix corresponding to a topology*

$$B=\begin{bmatrix}1 & 1\cdots 1\\ * & A\end{bmatrix}.$$

If the first row of B equals some other row of B, then we obtain $\alpha(B)=\alpha(A)$. *Otherwise, we obtain* $\alpha(B)=\alpha(A)+\alpha_0(A)+2$.

PROOF. The first case comes from Lemma 3. The second case comes from the following computations. $\alpha_0(B)=\alpha_0(A)+1$, $\alpha_1(B)=\alpha_1(A)+n+2$, $\alpha_2(B)=\alpha_2(A)+\alpha_0(A)-n-1$.

4. On the determination of a type of two matrices.

Firstly, we will define a type of the matrix corresponding to a topology.

DEFINITION 6. Let A be an n-th matrix corresponding to a topology and $A^{(1)}, A^{(2)}, \cdots, A^{(k)}$ be all of the $n+1$-st matrices corresponding to topologies in the form of

$$A'=\begin{bmatrix} 1 & a \\ b & A \end{bmatrix}.$$

Then we define a type of A by the multiset $\{\alpha(A^{(1)}), \cdots, \alpha(A^{(k)})\}$.

If the multiset has k_i elements each of which is equal to i, then we write $1^{k_1}, 2^{k_2}, \cdots$, where terms with $k_i=0$ and the superscript $k_i=1$ are omitted. As a matter of convenience, the notation, for example, $2^3 3^2 2^2=3^2 2^5=32^5 3=\cdots$ is permitted also.

DEFINITION 7. The *weight* of a Boolean vector v, denoted by $w(v)$, is the number of non-zero elements of v.

(i) Type of the n-th identity matrix E_n.

We will classify the set of the $n+1$-st matrices corresponding to topologies in the form of

$$A'=\begin{bmatrix} 1 & a \\ b & E_n \end{bmatrix}$$

into the following three classes.

(1) $a=[0 \cdots 0]$ (2) transpose matrix of (1)

(3) $a=[0 \cdots 0 \overset{i}{1} 0 \cdots 0]$, $b={}^t[0 \cdots 0 \overset{i}{1} 0 \cdots 0]$

where E_{n+1} belongs to the class (1). If the $n+1$-st matrix A' belongs to the class (k), we denote this matrix by $A^{(k)}$.

LEMMA 8. *If the weight of b of $A^{(1)}$ is i $(0\leq i\leq n)$, then we get* $\alpha(A^{(1)})=2^{n+1}+2^{n-i+1}+2^i+n-1$.

PROOF. By computation, we obtain $\alpha_0(A^{(1)})=2^n+2^{n-i}$, $\alpha_1(A^{(1)})=2^i+2n+1$, $\alpha_2(A^{(1)})=2^n+2^{n-i}-n-2$. From these results, we get the conclusion.

LEMMA 9. *If the weight of a of* $A^{(2)}$ *is* i $(1\leq i\leq n)$, *then we obtain* $\alpha(A^{(2)})=2^{n-i+1}+2^{n+1}+2^{i}+n-1$ *and for the matrix* $A^{(3)}$, *we get* $\alpha(A^{(3)})=2^{n+1}+n-1$.

The proof is omitted.

By those Lemmas and considering the number of the matrices belonging to each class, we can determine the type of E_n as follows.

THEOREM 5. *The type of* E_n *is given by the next formula.*

$$\Pi_{i=1}^{n}(2^{n+1}+2^{n+1-i}+2^{i}+n-1)^{2\binom{n}{i}}(2^{n+2}+n)(2^{n+1}+n-1)^{n}.$$

Let $A(A')$ be the n $(n+1)-st$ matrix corresponding to a topology in the form of

$$A=\begin{bmatrix}1 & & & \\ & \ddots & 0 & \\ & & \ddots & \\ 0 & & 1 & \\ & & 1 & 1\end{bmatrix},\qquad A'=\begin{bmatrix}1 & a\\ b & A\end{bmatrix}.$$

(ii) Type of A.

Let us classify the $n+1$-st matrices A' into the following 13 classes.

(11) $a=[0\cdots 0]$, $b={}^{t}[b_1\cdots b_{n-2}0\ 0]$

(12) $a=[0\cdots 0]$, $b={}^{t}[b_1\cdots b_{n-2}0\ 1]$

(13) $a=[0\cdots 0]$, $b={}^{t}[b_1\cdots b_{n-2}1\ 1]$

(21) $a=[0\cdots 0\ 1\ 0]$, $b={}^{t}[0\cdots 0\ 0\ 0]$

(22) $a=[0\cdots 0\ 1\ 0]$, $b={}^{t}[0\cdots 0\ 0\ 1]$

(23) $a=[0\cdots 0\ 1\ 0]$, $b={}^{t}[0\cdots 0\ 1\ 1]$

(31) $a=[0\cdots 0\ 1\ 1]$, $b={}^{t}[0\cdots 0\ 0\ 0]$

(32) $a=[0\cdots 0\ 1\ 1]$, $b={}^{t}[0\cdots 0\ 0\ 1]$

(41) $a=[0\cdots 0\ \overset{i}{1}\ 0\cdots 0]$, $b={}^{t}[0\cdots 0]$, $1\leq i\leq n-2$

(42) $a=[0\cdots 0\ \overset{i}{1}\ 0\cdots 0]$, $b={}^{t}[0\cdots 0\ \overset{i}{1}\ 0\cdots 0]$ $1\leq i\leq n-2$

(51) $a=[a_1\cdots a_{n-2}\ 1\ 1]$, $3\leq w(a)$, $b={}^{t}[0\cdots 0]$

(52) $a=[a_1\cdots a_{n-2}\ 0\ 0]$, $2\leq w(a)$, $b={}^{t}[0\cdots 0]$

(53) $a=[a_1\cdots a_{n-2}\ 1\ 0]$, $2\leq w(a)$, $b={}^{t}[0\cdots 0]$.

We denote the matrix belonging to the class (kl) by $A^{(kl)}$ also.

LEMMA 10. *If the weight of* ${}^t[b_1\ b_2 \cdots b_{n-2}]$ *is* i, *then we obtain*

$$\alpha(A^{(11)})=2^n+2^{n-1}+2^{n-i}+2^{n-i-1}+2^i+n$$

$$\alpha(A^{(12)})=2^n+2^{n-1}+2^{n-i}+2^{i+1}+n+1$$

$$\alpha(A^{(13)})=2^n+2^{n-1}+2^{n-i-1}+2^{i+1}+2^i+n$$

PROOF. Firstly, we will compute $\alpha(A^{(11)})$. By an appropriate permutation matrix P, we obtain $n+1$-st matrix as follows

$${}^tPA^{(11)}P=\left[\begin{array}{c|c|cc} E_{n-i-2} & 0 & 0 & \\ \hline 0 & B & 0 & \\ \hline \multirow{2}{*}{0} & \multirow{2}{*}{0} & 1 & 0 \\ & & 1 & 1 \end{array}\right] \quad \text{where } B=\begin{bmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 0 & & 1 \end{bmatrix}.$$

We can compute easily $\alpha_0(E_{n-i-2})=2^{n-i-2}$, $\alpha_0(B)=2^i+1$,

$$\alpha_0\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right)=3.$$

By Lemmas 6 and 7, we obtain $\alpha(E_{n-i-2})=2^{n-i-1}+n-i-3$,

$$\alpha(B)=2^{i+1}+2^i+i+1, \qquad \alpha\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right)=8.$$

By using Theorems 1, 2 and 3,

$$\alpha(A^{(11)})=\alpha({}^tPA^{(11)}P)=\alpha(E_{n-i-2}\oplus B\oplus\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix})=2^n+2^{n-1}+2^{n-i}+2^{n-i-1}+2^i+n.$$

The other cases are computed similarly.

LEMMA 11. *The numbers* $\alpha(A^{(21)})$, $\alpha(A^{(22)})$ *are determined as follows.*

$$\alpha(A^{(21)})=2^{n+1}+2^{n-1}+n+3$$

$$\alpha(A^{(22)})=2^{n+1}+n+3.$$

The proof of this Lemma is omitted

LEMMA 12. *The numbers* $\alpha(A^{(23)})$, $\alpha(A^{(32)})$, $\alpha(A^{(42)})$ *are determined as follows.*

$$\alpha(A^{(23)})=\alpha(A^{(32)})=\alpha(A^{(42)})=2^n+2^{n-1}+n.$$

PROOF. Since each of the reduced matrices of $A^{(23)}$, $A^{(32)}$, $A^{(42)}$ is A, we get $\alpha(A^{(23)})=\alpha(A^{(32)})=\alpha(A^{(42)})=2^n+2^{n-1}+n$ from Lemma 3.

LEMMA 13. *Let i be the weight of $[a_1 \cdots a_{n-2}]$. The results of computations are*

$$\alpha(A^{(31)})=2^{n+1}+n+3$$

$$\alpha(A^{(41)})=2^{n+1}+2^{n-2}+n+2$$

$$\alpha(A^{(51)})=2^n+2^{n-1}+2^{n-i-1}+2^{i+1}+2^i+n$$

$$\alpha(A^{(52)})=2^n+2^{n-1}+2^{n-i}+2^{n-i-1}+2^i+n$$

$$\alpha(A^{(53)})=2^n+2^{n-1}+2^{n-i}+2^{i+1}+n+1$$

The proof is obtained by using the results of $\alpha(A^{(13)})$, $\alpha(A^{(11)})$, $\alpha(A^{(13)})$, $\alpha(A^{(11)})$, $\alpha(A^{(12)})$ respectively.

From above results, we can obtain the next Theorem.

THEOREM 6. *The type of A is given as follows.*

$$\Pi_{i=0}^{n-2}(2^n+2^{n-1}+2^{n-i}+2^{n-i-1}+2^i+n)^{k(i)}(2^n+2^{n-1}+2^{n-i}+2^{i+1}+n+1)^{l(i)}$$
$$(2^n+2^{n-1}+2^{n-i-1}+2^{i+1}+2^i+n)^{m(i)}(2^n+2^{n-1}+n)^n$$

where $k(i)=2\binom{n-2}{i}-\delta_{i,0}$, $l(i)=2\binom{n-2}{i}$, $m(i)=2\binom{n-2}{i}+\delta_{i,0}$, $\delta_{i,0}$ *is the kronecker delta.*

REMARK. Finally, we will make several remarks. As before mentioned, if finite topological spaces (X, T) and (X, T') are homeomorphic, then we get $\alpha(M(T))=\alpha(M(T'))$, i.e., topological invariant. But, as the following example shows, the converse does not hold. Let (X, T) and (X, T') be non-homeomor phic finite topological spaces with

$$M(T)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad M(T')=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

In this case, $\alpha(M(T))=\alpha(M(T'))=15$ holds, but it is not homotopy type invariant. In spite of this fact the next combinatorial fact seems to hold.

If $\alpha(A)=\alpha(B)$ holds, then the type of A and the type of B are same, that is, the type of A is determined by $\alpha(A)$.

If $\alpha(A)$ is small value, this conjecture is checked and the type of each $\alpha(A)$ is as follows.

$\alpha(A)$	type of A
4	4 8^2 9
8	8^2 13^3 15^3
9	9^2 15^6 18
13	13^3 19^4 22^4 23^2
15	15^3 22^3 23^4 24^2 26^2 28
18	18^3 26^6 28^8 35
19	19^4 26^5 30^5 32^5
22	22^4 30^4 32^4 34^4 35^3 38^2 39
23	23^4 32^6 34^6 37^2 38^4 42
24	24^4 34^{10} 37^5 41^5
26	26^4 35^3 37^4 38^8 41^5 47^2
28	28^4 38^6 39^3 41^6 42^6 47^2 53
35	35^4 47^{20} 53^{10} 68

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Department of Mathematics
Utsunomiya University
Utsunomiya 321
Japan