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AUTOMORPHISMS OF ORDER 4 OF THE SIMPLY CONNECTED COMPACT LIE GROUP E_6

By

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Using the theory of Kac-Moody Lie algebras, for compact simple Lie algebra $\mathfrak{g}$, automorphisms ρ of finite order of $\mathfrak{g}$ can be classified and the type of Lie subalgebras $\mathfrak{g}^{\rho}$ of fixed points are determined [1]. Now for the simply connected compact Lie group E_6, we realize automorphisms ρ of order 4 and determine the subgroups $(E_6)^{\rho}$ of fixed points. Among compact exceptional Lie groups, only E_6 has outer automorphisms, so we consider the case of E_6. As results, the group E_6 has eight inner automorphisms named as $\gamma_1, \gamma_2, \cdots, \gamma_5, \sigma_1, \sigma_2, \sigma_3$ and three outer automorphisms named as $\tau\gamma_2'$, $\tau\gamma_3$, $\tau\sigma_3$, and the subgroups $(E_6)^{\rho}$ of fixed points are given as follows.

ρ	$(\mathfrak{e}_6)^{\rho}$	$(E_6)^{\rho}$
γ_1	$T^1\oplus A_1\oplus A_4$	$(Sp(1)\times S(U(1)\times U(5)))/\boldsymbol{Z}_2$
σ_2	$T^1\oplus A_1\oplus A_1\oplus A_3$	$(Sp(1)\times S(U(2)\times U(4)))/\boldsymbol{Z}_2$
γ_2	$T^1\oplus A_1\oplus A_2\oplus A_2$	$(Sp(1)\times S(U(3)\times U(3)))/\boldsymbol{Z}_2$
γ_3	$T^1\oplus A_5$	$(U(1)\times SU(6))/\boldsymbol{Z}_2$
γ_4	$T^2\oplus A_4$	$(U(1)\times S(U(1)\times U(5)))/\boldsymbol{Z}_2$
γ_5	$T^2\oplus A_1\oplus A_3$	$(U(1)\times S(U(2)\times U(4)))/\boldsymbol{Z}_2$
σ_1	$T^1\oplus D_5$	$(U(1)\times Spin(10))/\boldsymbol{Z}_4$
σ_3	$T^2\oplus D_4$	$(U(1)\times(Spin(2)\times Spin(8)))/(\boldsymbol{Z}_2\times\boldsymbol{Z}_4)$
$\tau\gamma_2'$	$A_1\oplus D_3$	$(Sp(1)\times SO(6))/\boldsymbol{Z}_2$
$\tau\gamma_3$	$T^1\oplus C_3$	$(U(1)\times Sp(3))/\boldsymbol{Z}_2$
$\tau\sigma_3$	$A_1\oplus B_3$	$(SU(2)\times Spin(7))/\boldsymbol{Z}_2$.

1. Preliminaries

Let $\mathfrak{C}=\boldsymbol{H}\oplus\boldsymbol{H}e$ ($\boldsymbol{H}$ is the field of quaternions with the basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$) be the Cayley algebra with the multiplication $(m+ae)(n+be)=(mn-\bar{b}a)+(a\bar{n}+bm)e$, the conjugation $\overline{m+ae}=\bar{m}-ae$, the inner product $(x, y)=(\bar{x}y+\bar{y}x)/2$ and the length $|x|=\sqrt{(x, x)}$, and $\mathfrak{C}^C$ be its complexification. Let $\mathfrak{J}=\{X\in M(3, \mathfrak{C})\,|\,X^*=X\}$

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be the exceptional Jordan algebra with the multiplication $X\circ Y=(XY+YX)/2$ and $\mathfrak{J}^C$ be its complexification. $\mathfrak{J}$ and $\mathfrak{J}^C$ have the inner product $(X, Y)=\mathrm{tr}(X\circ Y)$, the Freudenthal multiplication $X\times Y=(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X, Y))E)/2$ and the determinant $\det X=(X\times X, X)/3$. ($\mathfrak{J}(3, \boldsymbol{H})=\{M\in M(3, \boldsymbol{H})\mid M^*=M\}$ and $\mathfrak{J}(3, \boldsymbol{H})^C$ are also defined). The complex conjugations of $\mathfrak{C}^C$, $\mathfrak{J}^C$ are denoted by τ. In $\mathfrak{J}^C$, the positive definite inner product $\langle X, Y\rangle$ is defined by $(\tau X, Y)$. Now

$$E_6=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C)\mid\det\alpha X=\det X,\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}$$
$$=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C)\mid\alpha X\times\alpha Y=\tau\alpha\tau(X\times Y),\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}$$

is the simply connected compact Lie group of type E_6 [2]. Throughout this paper, we use such notations and theorems in [4] as E, E_i, $F_i(x_i)$, $i=1, 2, 3$ of $\mathfrak{J}$, $\mathfrak{J}^C$ and Lie subgroups $F_4=\{\alpha\in E_6\mid\tau\alpha=\alpha\tau\}=\{\alpha\in E_6\mid\alpha E=E\}$, $Spin(9)=\{\alpha\in F_4\mid\alpha E_1=E_1\}$, $Spin(10)=\{\alpha\in E_6\mid\alpha E_1=E_1\}$ of E_6 etc..

2. Inner automorphisms $\gamma_1, \gamma_2, \cdots, \gamma_5$ of order 4 of E_6

The field $\boldsymbol{H}$ is embedded in $M(2, \boldsymbol{C})$ by $k:\boldsymbol{H}=\boldsymbol{C}\oplus\boldsymbol{C}\boldsymbol{j}\to M(2, \boldsymbol{C})$ (where $\boldsymbol{C}=\{x+y\boldsymbol{i}\mid x, y\in\boldsymbol{R}\}$) by $k(a+b\boldsymbol{j})=\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix}$, $a, b\in\boldsymbol{C}$. This k is naturally extended to $\boldsymbol{R}$-linear mappings $k: M(3, \boldsymbol{H})\to M(6, \boldsymbol{C})$, $k:\boldsymbol{H}^3\to M(2, 6, \boldsymbol{C})$. Moreover these k are extended to C-$\boldsymbol{C}$-linear isomorphisms $k: M(3, \boldsymbol{H})^C\to M(6, \boldsymbol{C})$, $k:(\boldsymbol{H}^3)^C\to M(2, 6, \boldsymbol{C})$,

$$k(M_1+iM_2)=k(M_1)+\boldsymbol{i}k(M_2), \qquad M_i\in M(3, \boldsymbol{H}),$$
$$k(\boldsymbol{a}_1+i\boldsymbol{a}_2)=k(\boldsymbol{a}_1)+\boldsymbol{i}k(\boldsymbol{a}_2) \qquad \boldsymbol{a}_i\in\boldsymbol{H}^3.$$

Finally we define the $\boldsymbol{C}$-vector space $\mathfrak{S}(6, \boldsymbol{C})$ by $\{S\in M(6, \boldsymbol{C})\mid {}^tS=-S\}$ and the C-$\boldsymbol{C}$-linear isomorphism $k_J:\mathfrak{J}(3, \boldsymbol{H})^C\to\mathfrak{S}(6, \boldsymbol{C})$ by

$$k_J(M_1+iM_2)=k(M_1)J+\boldsymbol{i}k(M_2)J, \qquad M_i\in\mathfrak{J}(3, \boldsymbol{H})$$

where $J=\mathrm{diag}(J, J, J)$, $J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$.

In $\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C$, we define the Freudenthal multiplication [2] as

$$(M+\boldsymbol{a})\times(N+\boldsymbol{b})=\Big(M\times N-\frac{1}{2}(\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a})\Big)-\frac{1}{2}(\boldsymbol{a}N+\boldsymbol{b}M).$$

Then $\mathfrak{J}^C$ is isomorphic to $\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C$ by the correspondence

$$\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}\longleftrightarrow\begin{pmatrix}\xi_1 & m_3 & \bar{m}_2\\ \bar{m}_3 & \xi_2 & m_1\\ m_2 & \bar{m}_1 & \xi_3\end{pmatrix}+(a_1, a_2, a_3)$$

(where $x_i=m_i+a_ie$, m_i, $a_i\in \boldsymbol{H}^C$) as Freudenthal algebra [4]. Hereafter we identify $\mathfrak{J}^C$ and $\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C$. We define an involutive C-linear mapping $\gamma: \mathfrak{J}^C\to\mathfrak{J}^C$ by

$$\gamma(M+\boldsymbol{a})=M-\boldsymbol{a}, \qquad M+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C=\mathfrak{J}^C.$$

Then $\gamma\in E_6$ and $\gamma^2=1$.

PROPOSITION 2.1. $(E_6)^\gamma\cong(Sp(1)\times SU(6))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.

PROOF. Let $Sp(1)=\{p\in\boldsymbol{H}\mid \bar{p}p=1\}$ and $SU(6)=\{A\in M(6, \boldsymbol{C})\mid A^*A=E, \det A=1\}$. Now the mapping $\phi: Sp(1)\times SU(6)\to(E_6)^\gamma$,

$$\phi(p, A)(M+\boldsymbol{a})=k_J^{-1}(Ak_J(M)^tA)+pk^{-1}(k(\boldsymbol{a})A^*), \qquad M+\boldsymbol{a}\in\mathfrak{J}^C$$

induces the required isomorphism. The details of proof are in [2] or [4].

REMARK. $\phi: Sp(1)\times SU(6)\to(E_6)^\gamma$ satisfies $\gamma=\phi(-1, E)$ and $\tau\phi(p, A)\tau=\phi(p, -J\bar{A}J)$.

Using $\phi: Sp(1)\times SU(6)\to(E_6)^\gamma$ of Proposition 2.1, we define

$$\gamma_1=\phi(1, \boldsymbol{i}I_1), \qquad I_1=\mathrm{diag}(-1, 1, 1, 1, 1, 1),$$
$$\gamma_2=\phi(1, \boldsymbol{i}I_3), \qquad I_3=\mathrm{diag}(-1, -1, -1, 1, 1, 1),$$
$$\gamma_3=\phi(\boldsymbol{i}, E),$$
$$\gamma_4=\phi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\Gamma_1), \qquad \boldsymbol{\varepsilon}=(1+\boldsymbol{i})/\sqrt{2}, \ \Gamma_1=\mathrm{diag}(\boldsymbol{i}, 1, 1, 1, 1, 1),$$
$$\gamma_5=\phi(\boldsymbol{i}, I_2), \qquad I_2=\mathrm{diag}(-1, -1, 1, 1, 1, 1).$$

Then $\gamma_i\in E_6$ and the order of γ_i is 4, for $i=1, 2, \cdots, 5$.

THEOREM 2.2. (1) $(E_6)^{\gamma_1}\cong(Sp(1)\times S(U(1)\times U(5)))/\boldsymbol{Z}_2$,
(2) $(E_6)^{\gamma_2}\cong(Sp(1)\times S(U(3)\times U(3)))/\boldsymbol{Z}_2$,
(3) $(E_6)^{\gamma_3}\cong(U(1)\times SU(6))/\boldsymbol{Z}_2$,
(4) $(E_6)^{\gamma_5}\cong(U(1)\times S(U(2)\times U(4)))/\boldsymbol{Z}_2$
where $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$ *in any case.*

PROOF. (1) Since $\gamma_1^2=\gamma$, we have $(E_6)^{\gamma_1}\subset(E_6)^\gamma$. Hence, for $\alpha\in(E_6)^{\gamma_1}$ there exist $p\in Sp(1)$, $A\in SU(6)$ such that $\alpha=\phi(p, A)$ (Proposition 2.1). From the condition $\gamma_1\alpha=\alpha\gamma_1$, we have $\phi(p, \boldsymbol{i}I_1A)=\phi(p, \boldsymbol{i}AI_1)$, that is, $I_1A=AI_1$, therefore $A\in S(U(1)\times U(5))$. Thus we have the required isomorphism.

(2), (3), (4) are proved to be similar to (1).

THEOREM 2.3. $(E_6)^{\gamma_4}\cong(U(1)\times S(U(1)\times U(5)))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.

PROOF. Since the operation of γ_4 on $\mathfrak{J}^C=\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C$ is given by

$$\gamma_4\left(\begin{pmatrix}\xi_1 & m_3 & \overline{m}_2\\ \overline{m}_3 & \xi_2 & m_1\\ m_2 & \overline{m}_1 & \xi_3\end{pmatrix}+(a_1, a_2, a_3)\right)$$

$$=\begin{pmatrix}-\xi_1 & i\varepsilon m_3 & i\varepsilon\overline{m}_2\\ i\varepsilon\overline{m}_3\bar{\varepsilon} & i\xi_2 & im_1\\ i\varepsilon m_2\bar{\varepsilon} & i\overline{m}_1 & i\xi_3\end{pmatrix}+(-i\varepsilon a_1\bar{\varepsilon}, -i\varepsilon\varepsilon a_2, -i\varepsilon\varepsilon a_3)$$

(where $\varepsilon=(1+i)/\sqrt{2}$), the eigen C-vector spaces $(\mathfrak{J}^C)_\nu$, $\nu=1, -1, i, -i$ with respect to γ_4 are

$$(\mathfrak{J}^C)_1=\{M+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C \mid \gamma_4(M+\boldsymbol{a})=M+\boldsymbol{a}\}$$
$$=\{(a_1(\boldsymbol{i}-i), (\boldsymbol{i}+i)a_2, (\boldsymbol{i}+i)a_3) \mid a_1\in\boldsymbol{C}\boldsymbol{j},\ a_2, a_3\in\boldsymbol{H}\},$$

$$(\mathfrak{J}^C)_{-1}=\{M+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C \mid \gamma_4(M+\boldsymbol{a})=-M-\boldsymbol{a}\}$$
$$=\left\{\begin{pmatrix}\xi_1 & (\boldsymbol{i}+i)m_3 & \overline{m_2(\boldsymbol{i}-i)}\\ \overline{(\boldsymbol{i}+i)m_3} & 0 & 0\\ m_2(\boldsymbol{i}-i) & 0 & 0\end{pmatrix}+(a_1(\boldsymbol{i}+i), 0, 0)\ \middle|\ \begin{matrix}\xi_1\in C,\\ a_1\in\boldsymbol{C}\boldsymbol{j},\ m_2, m_3\in\boldsymbol{H}\end{matrix}\right\},$$

$$(\mathfrak{J}^C)_i=\{M+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C \mid \gamma_4(M+\boldsymbol{a})=i(M+\boldsymbol{a})\}$$
$$=\left\{\begin{pmatrix}0 & (\boldsymbol{i}-i)m_3 & \overline{m_2(\boldsymbol{i}+i)}\\ \overline{(\boldsymbol{i}-i)m_3} & \xi_2 & m_1\\ m_2(\boldsymbol{i}+i) & \overline{m}_1 & \xi_3\end{pmatrix}\ \middle|\ \begin{matrix}\xi_2, \xi_3\in C,\\ m_1\in\boldsymbol{H}^C,\ m_2, m_3\in\boldsymbol{H}\end{matrix}\right\},$$

$$(\mathfrak{J}^C)_{-i}=\{M+\boldsymbol{a}\in\mathfrak{J}(3, \boldsymbol{H})^C\oplus(\boldsymbol{H}^3)^C \mid \gamma_4(M+\boldsymbol{a})=-i(M+\boldsymbol{a})\}$$
$$=\{(a_1, (\boldsymbol{i}-i)a_2, (\boldsymbol{i}-i)a_3)) \mid a_1\in\boldsymbol{C}^C,\ a_2, a_3\in\boldsymbol{H}\}$$

where $\boldsymbol{C}\boldsymbol{j}=\{s\boldsymbol{j}+t\boldsymbol{k} \mid s, t\in\boldsymbol{R}\}$. These spaces are invariant under the group $(E_6)^{\gamma_4}$. We shall show that $(\boldsymbol{H}^3)^C$ is invariant under $(E_6)^{\gamma_4}$. From the forms of $(\mathfrak{J}^C)_\nu$, it is sufficient to show that $\alpha\boldsymbol{a}\in(\boldsymbol{H}^3)^C$ for $\alpha\in(E_6)^{\gamma_4}$ and $\boldsymbol{a}=(a(\boldsymbol{i}+i), 0, 0)=F_1((a(\boldsymbol{i}+i))e)$ $(a\in\boldsymbol{C}\boldsymbol{j})$. Now, in fact,

$$\begin{aligned}\alpha F_1((a(\boldsymbol{i}+i))e)&=4\alpha((F_3((\boldsymbol{i}-i)\bar{a})\times F_1(1))\times F_3(e))\\&=4(\alpha F_3((\boldsymbol{i}-i)\bar{a})\times\alpha F_1(1))\times\tau\alpha\tau F_3(e)\\&\in 4(\mathfrak{J}(3,\boldsymbol{H})^C\times\mathfrak{J}(3,\boldsymbol{H})^C)\times(\boldsymbol{H}^3)^C\subset\mathfrak{J}(3,\boldsymbol{H})^C\times(\boldsymbol{H}^3)^C\subset(\boldsymbol{H}^3)^C.\end{aligned}$$

Thus we see that $(\boldsymbol{H}^3)^C$ is invariant under $(E_6)^{\gamma_4}$, hence $\mathfrak{J}(3,\boldsymbol{H})^C=((\boldsymbol{H}^3)^C)^{\perp}=\{X\in\mathfrak{J}^C\,|\,\langle X,Y\rangle=0$ for all $Y\in(\boldsymbol{H}^3)^C\}$ is also invariant under $(E_6)^{\gamma_4}$. Consequently, $\alpha\in(E_6)^{\gamma_4}$ commutes with γ, that is, $(E_6)^{\gamma_4}\subset(E_6)^{\gamma}$. Hence, for $\alpha\in(E_6)^{\gamma_4}$, there exist $p\in Sp(1)$, $A\in SU(6)$ such that $\alpha=\psi(p,A)$ (Proposition 2.1). From the condition $\gamma_4\alpha=\alpha\gamma_4$, we have $\psi(\boldsymbol{\varepsilon}p,\boldsymbol{\varepsilon}\Gamma_1A)=\psi(p\boldsymbol{\varepsilon},A\boldsymbol{\varepsilon}\Gamma_1)$, that is, $\boldsymbol{\varepsilon}p=p\boldsymbol{\varepsilon}$, $\Gamma_1A=A\Gamma_1$ (or $\boldsymbol{\varepsilon}p=-p\boldsymbol{\varepsilon}$, $\Gamma_1A=-A\Gamma_1$ (which is impossible)), therefore $p\in U(1)$, $A\in S(U(1)\times U(5))$. Thus we have the required isomorphism.

3. Inner automorphisms σ_1, σ_2, σ_3 of order 4 of E_6

Let $U(1)=\{\theta\in C\,|\,(\tau\theta)\theta=1\}$ (where $C=\boldsymbol{R}^C$) and we define an embedding ϕ: $U(1)\to E_6$ by

$$\phi(\theta)\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}=\begin{pmatrix}\theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2\\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1\\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3\end{pmatrix}$$

and put $\sigma=\phi(-1)\in E_6$.

The group $Spin(10)$ is defined by $(E_6)_{E_1}=\{\alpha\in E_6\,|\,\alpha E_1=E_1\}$ which is the covering group of $SO(10)=SO(V^{10})$ where $V^{10}=\{X\in\mathfrak{J}^C\,|\,2E_1\times X=-\tau X\}=\{\xi E_2-\tau\xi E_3+F_1(x)\,|\,\xi\in C,\ x\in\mathfrak{C}\}$. Note that $Spin(10)$ leaves invariant $\{X\in\mathfrak{J}^C\,|\,E_1\times X=0\}=\{F_2(x)+F_3(y)\,|\,x,\ y\in\mathfrak{C}^C\}$.

PROPOSITION 3.1. $(E_6)^{\sigma}\cong(U(1)\times Spin(10))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4=\{(1,1),(-1,\sigma),(i,\phi(-i)),(-i,\phi(i))\}$.

PROOF. The mapping $\varphi: U(1)\times Spin(10)\to(E_6)^{\sigma}$,

$$\varphi(\theta,\beta)=\phi(\theta)\beta$$

induces the required isomorphism. The details of proof are in [2] or [4].

Using $\phi: U(1)\to E_6$ or $\psi: Sp(1)\times SU(6)\to E_6$ of Proposition 2.1, we define

$$\sigma_1=\phi(i)=\psi(-1,\Gamma_4),\qquad \Gamma_4=\mathrm{diag}\,(1,1,\boldsymbol{i},\boldsymbol{i},\boldsymbol{i},\boldsymbol{i}),$$

$$\sigma_2=\gamma\sigma_1=\psi(1,\Gamma_4).$$

Then $\sigma_i\in E_6$ and $\sigma_i{}^2=\sigma$, $\sigma_i{}^4=1$ for $i=1,2$.

THEOREM 3.2. $(E_6)^{\sigma_1}\cong(U(1)\times Spin(10))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4=\langle(i,\phi(-i))\rangle$.

PROOF is clear from Proposition 3.1, because $\sigma_1=\phi(i)$ commutes with any elements of $U(1)$ and $Spin(10)$.

THEOREM 3.3. *$(E_6)^{\sigma_2}\cong(Sp(1)\times S(U(2)\times U(4)))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.*

PROOF. Since $\sigma_2{}^2=\sigma$, we have $(E_6)^{\sigma_2}\subset(E_6)^{\sigma}$. Hence, for $\alpha\in(E_6)^{\sigma_2}$, there exist $\theta\in U(1)$, $\beta\in Spin(10)$ such that $\alpha=\phi(\theta)\beta$ (Proposition 3.1). In particular, α commutes with $\sigma_1=\phi(i)$. Therefore, from the condition $\sigma_2\alpha=\alpha\sigma_2$, that is, $\gamma\sigma_1\alpha=\alpha\gamma\sigma_1$, we have $\gamma\alpha=\alpha\gamma$, namely $\alpha\in(E_6)^{\gamma}$. Hence there exist $p\in Sp(1)$, $A\in SU(6)$ such that $\alpha=\psi(p, A)$ (Proposition 2.1). Moreover from the condition $\sigma_1\alpha=\alpha\sigma_1$, we have $\psi(p, \Gamma_4 A)=\psi(p, A\Gamma_4)$, that is, $\Gamma_4 A=A\Gamma_4$, therefore $A\in S(U(2)\times U(4))$. Thus we have the required isomorphism.

REMARK. The group $(E_6)^{\sigma_2}$ has also the following expression

$$(E_6)^{\sigma_2}\cong(U(1)\times Sp(1)\times(SU(2)\times SU(4)))/(\boldsymbol{Z}_2\times\boldsymbol{Z}_4)$$

where $\boldsymbol{Z}_2=\langle(1, -1, -I_2)\rangle$, $\boldsymbol{Z}_4=\langle(-i, 1, -\Gamma_4)\rangle$. In fact, for $\alpha=\psi(p, A)\in(E_6)^{\sigma_2}$, $p\in Sp(1)$, $A=(P, Q)=\begin{pmatrix} P & 0 \\ 0 & Q\end{pmatrix}\in S(U(2)\times U(4))$, the condition that α belongs to the group $((E_6)^{\sigma_2})_{E_1}\subset Spin(10)$, that is, $\psi(p, (P, Q))E_1=E_1$, is $p\in Sp(1)$, $P\in SU(2)$, $Q\in SU(4)$. From this we have easily the required isomorphism.

The field $\boldsymbol{C}$ of complex numbers is embedded in $\mathfrak{C}$ as $\boldsymbol{C}=\{x+ye\,|\,x, y\in\boldsymbol{R}\}$ and put $\boldsymbol{C}^{\perp}=\{t\in\mathfrak{C}\,|\,(t, \boldsymbol{C})=0\}$. Let $Spin(2)=\{a\in\boldsymbol{C}\,|\,\bar{a}a=1\}$ $(\cong U(1))$ and we define an embedding $D: Spin(2)\to E_6$ by

$$D_a\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}=\begin{pmatrix}\xi_1 & x_3\bar{a} & \bar{x}_2 a \\ a\bar{x}_3 & \xi_2 & a x_1 a \\ \bar{a}x_2 & \overline{a x_1 a} & \xi_3\end{pmatrix}.$$

Put $\sigma_3=D_{-e}$. Then $\sigma_3\in E_6$ and $\sigma_3{}^2=\sigma$, $\sigma_3{}^4=1$.

The group $Spin(8)$ is defined by

$$(E_6)_{E_1, F_1(s)}=\{\alpha\in E_6\,|\,\alpha E_1=E_1,\ \alpha F_1(s)=F_1(s) \text{ for all } s\in\boldsymbol{C}^C\}$$
$$=\{\alpha\in Spin(10)\,|\,\alpha F_1(1)=F_1(1),\ \alpha F_1(e)=F_1(e)\}$$

which is the covering group of $SO(8)=SO(V^8)$ where $V^8=(V^{10})_{\sigma_3}=\{\xi E_2-\tau\xi E_3+F_1(t)\,|\,\xi\in C, t\in\boldsymbol{C}^{\perp}\}$.

LEMMA 3.3. *D_a $(a\in Spin(2))$ and $\beta\in Spin(8)$ commute with each other.*

PROOF.

$$\beta D_a F_1(z)=\beta F_1(aza)=\beta F_1(a^2s+t) \quad (z=s+t\in \boldsymbol{C}^C\oplus(\boldsymbol{C}^{\perp})^C=\mathfrak{C}^C)$$
$$=F_1(a^2s)+\beta F_1(t)=F_1(a^2s)+(\xi_2E_2+\xi_3E_3+F_1(t')) \quad (\xi_i\in C,\ t'\in(\boldsymbol{C}^{\perp})^C)$$
$$=D_a(F_1(s)+\xi_2E_2+\xi_3E_3+F_1(t'))=D_a(F_1(s)+\beta F_1(t))$$
$$=D_a\beta F_1(s+t)=D_a\beta F_1(z).$$
$$\beta D_a F_2(z)=\beta F_2(\bar{a}z)=4\beta((F_1(1)\times F_2(z))\times F_1(a))=4(\beta F_1(1)\times\beta F_2(z))\times\tau\beta\tau F_1(a)$$
$$=4(F_1(1)\times(F_2(x)+F_3(y))\times F_1(a) \quad (\text{for some } x,\ y\in\mathfrak{C}^C)$$
$$=F_2(\bar{a}x)+F_3(y\bar{a})=D_a(F_2(x)+F_3(y))=D_a\beta F_2(z).$$

Similarly $\beta D_a F_3(z)=D_a\beta F_3(z)$. Clearly $D_a\beta=\beta D_a$ on E_1.

$$D_a\beta E_2=D_a(\xi_2E_2+\xi_3E_3+F_1(t))=\xi_2E_2+\xi_3E_3+F_1(t)=\beta E_2=\beta D_a E_2$$

(for some $\xi_i\in C$, $t\in(\boldsymbol{C}^{\perp})^C$). Similarly $D_a\beta E_3=\beta D_a E_3$. Thus we have $D_a\beta=\beta D_a$.

LEMMA 3.4. *Let $\beta\in(Spin(10))^{\sigma_3}$. Then we can put $\beta F_1(1)=F_1(s)$, $\beta F_1(e)=F_1(es)$, $s\in\boldsymbol{C}$, $|s|=1$.*

PROOF. Since the group $(Spin(10))^{\sigma_3}$ acts on $\{F_1(s)\,|\,s\in\boldsymbol{C}\}=\{X\in\mathfrak{J}^C\,|\,\sigma_3X=-X,\ 2E_1\times X=-\tau X\}$, we can put

$$\beta F_1(1)=F_1(s),\qquad \beta F_1(e)=F_1(s'),\qquad s,\ s'\in\boldsymbol{C},\ |s|=|s'|=1.$$

Operate $\tau\beta\tau$ on the relation $F_1(1)\times F_1(e)=-(1,\ e)E_1=0$, then $0=\tau\beta\tau(F_1(1)\times F_1(e))=\beta F_1(1)\times\beta F_1(e)=F_1(s)\times F_1(s')=-(s,\ s')E_1$, hence $(s,\ s')=0$. Together with $|s|=|s'|=1$, we have $s'=es$ or $s'=-es$. The latter case is impossible. In fact, choose $s\in\boldsymbol{C}$ such that $a^2=\bar{s}$ and put $\delta=D_a\beta$, then $\delta F_1(1)=F_1(1)$, $\delta F_1(e)=-F_1(e)$. Then

$$\delta F_2(e)=\delta\sigma_3F_2(1)=\sigma_3\delta F_2(1)=\sigma_3(F_2(x)+F_3(y)) \quad (\text{for some } x,\ y\in\mathfrak{C}^C)$$
$$=F_2(ex)+F_3(ye),$$
$$\delta F_3(1)=2\delta(F_1(1)\times F_2(1))=2\tau\delta\tau F_1(1)\times\tau\delta\tau F_2(1)$$
$$=2F_1(1)\times(F_2(\tau x)+F_3(\tau y))=F_3(\tau\bar{x})+F_2(\tau\bar{y}).$$

Therefore we have

$$F_1(e)=\delta F_1(-e)=2\delta(F_2(e)\times F_3(1))=2\tau\delta\tau F_2(e)\times\tau\delta\tau F_3(1)$$
$$=2(F_2(e(\tau x))+F_3((\tau y)e)\times(F_3(\bar{x})+F_2(\bar{y}))$$
$$=F_1(-x((\tau\bar{x})e)-(e(\tau\bar{y}))y)+*E_2+*E_3.$$

Compare the coefficients of e, then we have $1=-|x_1|^2-|x_2|^2-|y_1|^2-|y_2|^2$ (where $x=x_1+ix_2$, $y=y_1+iy_2$, $x_i, y_i\in\mathfrak{C}$), which is a contradiction. Thus Lemma 3.4 is proved.

THEOREM 3.5. $(E_6)^{\sigma_3}\cong(U(1)\times Spin(2)\times Spin(8))/(\boldsymbol{Z}_2\times\boldsymbol{Z}_4)$,

$$\begin{aligned}\boldsymbol{Z}_2\times\boldsymbol{Z}_4&=\langle(1,-1,\sigma)\rangle\times\langle(i, e, \phi(i)D_e)\rangle\\&=\{(1,1,1),(-1,1,\sigma),(i,e,\phi(i)D_e),(-i,e,\phi(-i)D_e),\\&\qquad(1,-1,\sigma),(-1,-1,1),(i,-e,\phi(-i)D_e),(-i,-e,\phi(i)D_e)\}.\end{aligned}$$

PROOF. We define a mapping $\psi: U(1)\times Spin(2)\times Spin(8)\to(E_6)^{\sigma_3}$ by

$$\psi(\theta, a, \delta)=\phi(\theta)D_a\delta\,.$$

Obviously ψ is well-defined. Since $\phi(\theta)$ ($\theta\in U(1)$), D_a ($a\in Spin(2)$) and $\delta\in Spin(8)$ commute with one another (Lemma 3.3), ψ is a homomorphism. We shall show that ψ is onto. (Although it suffices to show $\dim((\mathfrak{e}_6)^{\sigma_3})=30$, we will give a direct proof). Since $\sigma_3{}^2=\sigma$, we have $(E_6)^{\sigma_3}\subset(E_6)^{\sigma}$. Hence, for $\alpha\in(E_6)^{\sigma_3}$, there exist $\theta\in U(1)$, $\beta\in Spin(10)$ such that $\alpha=\phi(\theta)\beta$ (Proposition 3.1). From $\sigma_3\alpha=\alpha\sigma_3$, we have $\beta\in(Spin(10))^{\sigma_3}$. Hence we can put $\beta F_1(1)=F_1(s)$, $\beta F_1(e)=F_1(es)$, $s\in\boldsymbol{C}$, $|s|=1$ (Lemma 3.4). Choose $a\in\boldsymbol{C}$ such that $a^2=s$ and put $\delta=D_a{}^{-1}\beta$, then $\delta F_1(1)=F_1(1)$, $\delta F_1(e)=F_1(e)$, that is, $\delta\in Spin(8)$. Hence we have a presentation such that

$$\alpha=\phi(\theta)D_a\delta,\qquad \theta\in U(1),\ a\in Spin(2),\ \delta\in Spin(8).$$

Then ψ is onto. $\operatorname{Ker}\psi=\boldsymbol{Z}_2\times\boldsymbol{Z}_4$ is easily obtained. Thus we have the required isomorphism.

4. Outer automorphisms $\tau\gamma_2'$, $\tau\gamma_3$ of oder 4 of E_6

Using $\psi: Sp(1)\times SU(6)\to(E_6)^{\gamma}$ of Proposition 2.1, we define $\gamma_2'=\psi(1, J)$ and consider an automorphism $\tau\gamma_2'$ of E_6: $E_6\ni\alpha\to\tau\gamma_2'\alpha\gamma_2'^{-1}\tau\in E_6$. Then $(\tau\gamma_2')^2=\gamma$, $(\tau\gamma_2')^4=1$.

THEOREM 4.1. $(E_6)^{\tau\gamma_2'}\cong(Sp(1)\times SO(6))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.

PROOF. Since $(\tau\gamma_2')^2=\gamma$, we have $(E_6)^{\tau\gamma_2'}\subset(E_6)^{\gamma}$. Hence, for $\alpha\in(E_6)^{\tau\gamma_2'}$, there exist $p\in Sp(1)$, $A\in SU(6)$ such that $\alpha=\psi(p, A)$ (Proposition 2.1). Since $\tau\gamma_2'\alpha\gamma_2'^{-1}\tau=\alpha$, we have $\tau\psi(p, -JAJ)\tau=\psi(p, A)$, that is, $\psi(p, \bar{A})=\psi(p, A)$ (Remark of Proposition 2.1) then $A=\bar{A}$, hence $A\in SO(6)$. Thus we have the required isomorphism.

We use $\gamma_3=\phi(\boldsymbol{i}, E)$ of Section 2 and consider an automorphism $\tau\gamma_3$ of $E_6: E_6 \ni\alpha\to\tau\gamma_3\alpha\gamma_3{}^{-1}\tau\in E_6$. Then $(\tau\gamma_3)^2=\gamma$, $(\tau\gamma_3)^4=1$.

THEOREM 4.2. $(E_6)^{\tau\gamma_3}\cong(U(1)\times Sp(3))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(1, E), (-1, -E)\}$.

PROOF. Since $(\tau\gamma_3)^2=\gamma$, we have $(E_6)^{\tau\gamma_3}\subset(E_6)^{\gamma}$. Hence, for $\alpha\in(E_6)^{\tau\gamma_3}$, there exist $p\in Sp(1)$, $A\in SU(6)$ such that $\alpha=\phi(p, A)$ (Proposition 2.1). Since $\tau\gamma_3\alpha\gamma_3{}^{-1}\tau=\alpha$, we have $\phi(-\boldsymbol{i}p\boldsymbol{i}, -J\bar{A}J)=\phi(p, A)$ (Remark of Proposition 2.1), then

$$p=-\boldsymbol{i}p\boldsymbol{i},\ A=-J\bar{A}J \quad\text{or}\quad p=\boldsymbol{i}p\boldsymbol{i},\ A=J\bar{A}J.$$

In the first case $p\in U(1)=\{p\in\boldsymbol{C}\mid\bar{p}p=1\}$, $A\in Sp(3)=\{A\in M(6, \boldsymbol{C})\mid JA=\bar{A}J,\ A^*A=E,\ (\det A=1)\}$ (where $\boldsymbol{C}\subset\boldsymbol{H}$). The latter case is impossible. In fact, consider AI where $I=\mathrm{diag}(1, -1, 1, -1, 1, -1)$, then $AI\in Sp(3)$, hence $\det(AI)=1$. On the other hand, $\det(AI)=(\det A)(\det I)=1(-1)=-1$, which contradicts $\det(AI)=1$. Thus we have the required isomorphism.

5. Outer automorphism $\tau\sigma_3$ of order 4 of E_6

As in Section 3, we embed th϶ field $\boldsymbol{C}$ of complex numbers in $\mathfrak{C}$ as $\boldsymbol{C}=\{x+ye\mid x, y\in\boldsymbol{R}\}$. Let $SU(2)=\{A\in M(2, \boldsymbol{C})\mid A^*A=E, \det A=1\}$ and we define a mapping $\phi: SU(2)\to E_6$ by

$$\phi(A)X=(\rho_1(A))X(\rho_1(A))^*, \qquad X\in\mathfrak{J}^C$$

where $\rho_1(A)=\Gamma^{-1}A'\Gamma$, $\Gamma=\mathrm{diag}(i, 1, i)$, $A'=\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}$. Explicitly, for $A=\begin{pmatrix}a & -\bar{b}\\ b & \bar{a}\end{pmatrix}\in SU(2)$, $z\in\mathfrak{C}^C$,

$$\begin{cases}\phi(A)E_1=E_1,\\ \phi(A)(E_2+E_3)=(|a|^2-|b|^2)(E_2+E_3)-2iF_1(a\bar{b}),\\ \phi(A)(E_2-E_3)=E_2-E_3,\end{cases}$$

$$\begin{cases}\phi(A)F_1(z)=-2i(z, \overline{ba})(E_2+E_3)+F_1(aza-\overline{bzb}),\\ \phi(A)F_2(z)=F_2(\bar{a}z)-iF_3(\bar{z}b),\\ \phi(A)F_3(z)=-iF_2(b\bar{z})+F_3(z\bar{a}).\end{cases}$$

Note that $D_a=\phi\begin{pmatrix}a & 0\\ 0 & \bar{a}\end{pmatrix}$ for $a\in\boldsymbol{C}$, $|a|=1$, in particular, $\phi\begin{pmatrix}-e & 0\\ 0 & e\end{pmatrix}=D_{-e}=\sigma_3$.

LEMMA 5.1. *ϕ is well-defined, that is, for $A\in SU(2)$ we have $\phi(A)\in E_6$.*

PROOF. Define $\rho\in E_6$ by $\rho X=\Gamma X\Gamma$, $X\in\mathfrak{J}^C$. We consider an embedding

$SU(3)\subset F_4\subset E_6$, in which $SU(3)\subset F_4$ is given by $h: SU(3)\to F_4$,

$$h(A)(X+M)=AXA^*+MA^*, \qquad X+M\in\mathfrak{J}(3, \boldsymbol{C}^C)\oplus M(3, \boldsymbol{C}^C)=\mathfrak{J}^C$$

(as for notations see [5]). The $\phi(A)$ is nothing but $\phi(A)=\rho^{-1}h(A')\rho\in E_6$.

We use σ_3 of Section 3 and consider an automorphism $\tau\sigma_3$ of $E_6: E_6\ni\alpha\to\tau\sigma_3\alpha\sigma_3{}^{-1}\tau\in E_6$. Then $(\tau\sigma_3)^2=\sigma$, $(\tau\sigma_3)^4=1$.

LEMMA 5.2. *For $\alpha\in(E_6)^{\tau\sigma_3}$, we have $\alpha E_1=E_1$. In particular, $(E_6)^{\tau\sigma_3}\subset(E_6)_{E_1}=Spin(10)$.*

PROOF. Since $(\tau\sigma_3)^2=\sigma$, we have $(E_6)^{\tau\sigma_3}\subset(E_6)^\sigma$. Hence, for $\alpha\in(E_6)^{\tau\sigma_3}$, there exists $\xi\in C$, $(\tau\xi)\xi=1$ such that $\alpha E_1=\xi E_1$ (Proposition 3.1). From $\xi E_1=\alpha E_1=\alpha\tau\sigma_3E_1=\tau\sigma_3\alpha E_1=\tau\sigma_3(\xi E_1)=\tau\xi E_1$, we have $\tau\xi=\xi$, that is, $\xi\in\boldsymbol{R}$, hence $\xi=\pm1$. The case of $\xi=-1$ is impossible. (Although it follows from the connectedness of $(E_6)^{\tau\sigma_3}$, we will give an elementary proof). Suppose $\xi=-1$. Let $\alpha=\phi(\theta)\beta$, $\theta\in U(1)$, $\beta\in Spin(10)$. Then $-E_1=\alpha E_1=\phi(\theta)\beta E_1=\theta^4E_1$, hence $\theta^4=-1$. Now, for $t\in\boldsymbol{C}^\perp$, we can put $\beta F_1(t)=\eta E_2-\tau\eta E_3+F_1(x)$, $\eta\in C$, $x\in\mathfrak{C}$. Then $\phi(\phi)\beta F_1(t)=\pm i\eta E_2\mp i\tau\eta E_3\pm iF_1(x)$. Since $\tau\sigma_3\alpha=\alpha\tau\sigma_3$, we have $\eta\in i\boldsymbol{R}$, $x\in\boldsymbol{C}$. This shows that for $V=\{F_1(t)\,|\,t\in\boldsymbol{C}^\perp\}$, $\dim V=6$, $\dim(\alpha V)\leqq3$, which contradicts the regularity of α. Thus Lemma 5.2 is proved.

The group $Spin(7)$ is defined by

$$\begin{aligned}(E_6)_{E_1,E,F_1(s)}&=\{\alpha\in E_6\,|\,\alpha E_1=E_1,\ \alpha E=E,\ \alpha F_1(s)=F_1(s)\ \text{for all}\ s\in\boldsymbol{C}\}\\&=\{\alpha\in F_4\,|\,\alpha E_1=E_1,\ \alpha F_1(s)=F_1(s)\ \text{for all}\ s\in\boldsymbol{C}\}\\&=\{\alpha\in Spin(9)\,|\,\alpha F_1(1)=F_1(1),\ \alpha F_1(e)=F_1(e)\}\end{aligned}$$

which is the covering group of $SO(7)=SO(V^7)$ where $V^7=\{\xi(E_2-E_3)+F_1(t)\,|\,\xi\in\boldsymbol{R},\ t\in\boldsymbol{C}^\perp\}$.

LEMMA 5.3. *$\phi(A)\in\phi(SU(2))$ and $\beta\in Spin(7)$ commute with each other.*

PROOF. For $A=\begin{pmatrix}a & -\bar{b}\\ b & \bar{a}\end{pmatrix}\in SU(2)$, $\beta\in Spin(7)$, we shall show

$$\beta\phi(A)X=\phi(A)\beta X, \qquad X\in\mathfrak{J}^C. \tag{i}$$

$$\begin{aligned}\beta\phi(A)(E_2-E_3)&=\beta(E_2-E_3)=\xi(E_2-E_3)+F_1(t)\quad(\text{for some}\ \xi\in C,\ t\in(\boldsymbol{C}^\perp)^C)\\&=\phi(A)(\xi(E_2-E_3)+F_1(t))=\phi(A)\beta(E_2-E_3),\end{aligned}$$

$$\beta\phi(A)(E_2+E_3)=\beta((|a|^2-|b|^2)(E_2+E_3)-2iF_1(\overline{ab}))=\phi(A)(E_2+E_3)$$
$$=\phi(A)\beta(E_2+E_3)\,.$$

Thus (i) is true for $X=E_2$, E_3. For $X=E_1$, (i) is trivial.

$$\beta\phi(A)F_1(z)=\beta\phi(A)F_1(s+t)\quad (z=s+t\in \boldsymbol{C}^C\oplus(\boldsymbol{C}^\perp)^C=\mathfrak{C}^C)$$
$$=\beta(-2i(s,\ \overline{ba})(E_2+E_3)+F_1(a^2s-\bar{b}^2\bar{s})+F_1(t))$$
$$=-2i(s,\ \overline{ba})(E_2+E_3)+F_1(a^2s-\bar{b}^2\bar{s})+(\xi(E_2-E_3)+F_1(t'))$$

(for some $\xi\in C$, $t'\in(\boldsymbol{C}^\perp)^C$)

$$=\phi(A)(F_1(s)+\xi(E_2-E_3)+F_1(t'))=\phi(A)\beta F_1(s+t)=\phi(A)\beta F_1(z)\,.$$

$$\beta\phi(A)F_2(z)=\beta(F_2(\bar{a}z)-iF_3(\bar{z}b))$$
$$=4\beta((F_1(1)\times F_2(z))\times F_1(a))-4i\beta((F_1(1)\times F_3(\bar{z}))\times F_1(\bar{b}))$$
$$=4(\beta F_1(1)\times\beta F_2(z))\times\beta F_1(a)-4i(\beta F_1(1)\times\beta F_3(\bar{z}))\times\beta F_1(\bar{b})$$

(put $\beta F_2(z)=F_2(x)+F_3(y)$, $x,\ y\in\mathfrak{C}^C$, then $\beta F_3(\bar{z})=\beta(2F_1(1)\times F_2(z))=2F_1(1)\times\beta F_2(z)$ $=2F_1(1)\times(F_2(x)+F_3(y))=F_2(\bar{x})+F_3(\bar{y})$)

$$=4(F_1(1)\times(F_2(x)+F_3(y)))\times F_1(a)-4i(F_1(1)\times(F_3(\bar{x})+F_2(\bar{y})))\times F_1(\bar{b})$$
$$=F_2(\bar{a}x)+F_3(y\bar{a})-i(F_3(\bar{x}b)+F_2(b\bar{y}))$$
$$=F_2(\bar{a}x)-iF_3(\bar{x}b)-iF_2(b\bar{y})+F_3(y\bar{a})$$
$$=\phi(A)F_2(x)+\phi(A)F_3(y)=\phi(A)(F_2(x)+F_3(y))=\phi(A)\beta F_2(z)\,.$$

Similarly, $\beta\phi(A)F_3(z)=\phi(A)\beta F_3(z)$. Thus (i) is proved.

LEMMA 5.4. *$\phi(SU(2))$ and $Spin(7)$ are contained in $(E_6)^{\tau\sigma_3}$.*

PROOF. $\phi(SU(2))\subset(E_6)^{\tau\sigma_3}$ is clear, noting that $\begin{pmatrix}-e & 0\\ 0 & e\end{pmatrix}\begin{pmatrix}a & -\bar{b}\\ b & \bar{a}\end{pmatrix}\begin{pmatrix}-e & 0\\ 0 & e\end{pmatrix}^{-1}$ $=\begin{pmatrix}a & \bar{b}\\ -b & \bar{a}\end{pmatrix}$, $\tau i=-i$ and b, i appear simultaniously in $\phi(A)X$. Next, $\beta\in Spin(7)$ $\subset F_4$ implies $\tau\beta=\beta\tau$ and $\sigma_3\beta=\beta\sigma_3$ (Lemma 5.3). Hence $Spin(7)\subset(E_6)^{\tau\sigma_3}$.

THEOREM 5.5. $(E_6)^{\tau\sigma_3}\cong(SU(2)\times Spin(7))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(E,\ 1),\ (-E,\ \sigma)\}$.

PROOF. We define a mapping $\psi: SU(2)\times Spin(7)\to(E_6)^{\tau\sigma_3}$ by

$$\psi(A,\ \beta)=\phi(A)\beta\,.$$

Then ψ is well-defined (Lemma 5.4) and is a homomorphism (Lemma 5.3). We shall show that ϕ is onto. (Although it suffices to show $\dim((\mathfrak{e}_6)^{\tau\sigma_3})=24$, we will give a direct proof). Let $\alpha\in(E_6)^{\tau\sigma_3}$. Then we can put

$$\alpha(i(E_2+E_3))=i\eta(E_2+E_3)+F_1(s), \qquad \eta^2+|s|^2=1,\ \eta\in\boldsymbol{R},\ s\in\boldsymbol{C}.$$

In fact, since $\alpha\in(E_6)^{\tau\sigma_3}\subset(E_6)_{E_1}=Spin(10)$ (Lemma 5.2), we can put $\alpha(i(E_2+E_3))$ $=\xi E_2-\tau\xi E_3+F_1(s+t)$, $\xi\in C$, $s+t\in\boldsymbol{C}\oplus\boldsymbol{C}^{\perp}=\mathfrak{C}$. From $\tau\sigma_3\alpha=\alpha\tau\sigma_3$, we have $\xi\in i\boldsymbol{R}$, $t=0$. And $2(\eta^2+|s|^2)=\langle\alpha(i(E_2+E_3)),\ \alpha(i(E_2+E_3))\rangle=\langle i(E_2+E_3),\ i(E_2+E_3)\rangle$ $=2$. Now put

$$P=\frac{1}{\sqrt{2(1-\eta)}}\begin{pmatrix}\bar{s} & 1-\eta\\ -1+\eta & s\end{pmatrix} \qquad (\text{if } \eta=1, \text{ put } P=E).$$

Then $P\in SU(2)$ and $\phi(P)(i\eta(E_2+E_3)+F_1(s))=i(E_2+E_3)$. Hence $\phi(P)\alpha(i(E_2+E_3))$ $=i(E_2+E_3)$, that is, $\delta=\phi(P)\alpha\in(F_4)_{E_1}=Spin(9)$, moreover $((Spin(9))^{\tau\sigma_3}=(Spin(9))^{\sigma_3}\subset$ $(Spin(10))^{\sigma_3}$. Hence we can put

$$\delta F_1(1)=F_1(s_0), \qquad \delta F_1(e)=F_1(es_0), \qquad s_0\in\boldsymbol{C},\ |s_0|=1$$

(Lemma 3.4). Choose $a\in\boldsymbol{C}$ such that $a^2=\bar{s}_0$ and put $\beta=D_a\delta$, then $\beta(i(E_2+E_3))$ $=i(E_2+E_3)$, $\beta F_1(1)=F_1(1)$, $\beta F_1(e)=F_1(e)$, that is, $\beta=D_a\phi(P)\alpha\in Spin(7)$. Therefore we have a presentation such that

$$\alpha=\phi(P^{-1}A)\beta, \qquad P^{-1}A\in SU(2)\ \left(\text{where } A=\begin{pmatrix}\bar{a} & 0\\ 0 & a\end{pmatrix}\right),\ \beta\in Spin(7).$$

Thus ϕ is onto. $\operatorname{Ker}\phi=\boldsymbol{Z}_2=\{(E,1),(-E,\sigma)\}$. In fact, let $\phi(A)\beta=1$, $A=$ $\begin{pmatrix}a & -\bar{b}\\ b & \bar{a}\end{pmatrix}\in SU(2)$, $\beta\in Spin(7)$. Then $E=\phi(A)\beta E=\phi(A)E=E_1+(|a|^2-|b|^2)(E_2+E_3)$. Hence $|a|^2-|b|^2=1$. Together with $|a|^2+|b|^2=1$, we have $b=0$, so $A=\begin{pmatrix}a & 0\\ 0 & \bar{a}\end{pmatrix}$. From $F_1(1)=\phi(A)\beta F_1(1)=\phi(A)F_1(1)=F_1(a^2)$, we have $a^2=1$, hence $a=\pm1$, so $A=\pm E$. Then $\beta=\phi(E)=1$ or $\beta=\phi(-E)=\sigma$. Thus we have the required isomorphism.

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