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HYPERELLIPTIC MODULAR CURVES

By

N. ISHII and F. MOMOSE

Let $N \geqq 1$ be an integer, and Δ be a subgroup of $(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$. Let $X_{\Delta}=X_{\Delta}(N)$ be the modular curve defined over $\boldsymbol{Q}$ associating to the modular group $\Gamma_{\Delta}=\Gamma_{\Delta}(N)$:

$$\Gamma_{\Delta}(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid c \equiv 0 \bmod N,\ (a \bmod N) \in \Delta\right\}.$$

Since $X_{\Delta}=X_{\langle \pm 1, \Delta \rangle}$ [2], we always assume that -1 belongs to Δ. For $\Delta=\{\pm 1\}$ (resp. $\Delta=(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$), we denote $X_{\Delta}(N)$ by $X_1(N)$ (resp. $X_0(N)$). Ogg [18] determined all the hyperelliptic modular curves of type $X_0(N)$. This work aids the determination of the rational points on the modular curves $X_{split}(N)$ etc. [15, 16, 17] and that of the automorphism groups of $X_0(N)$ [8], [19]. In this paper, we determine all the hyperelliptic modular curves of type $X_{\Delta}(N)$. There are nineteen hyperelliptic modular curves $X_0(N)$ for $N=22$, 23, 26, 28, 29, 30, 31, 33, 35, 37, 39, 40, 41, 46, 47, 48, 50, 59 and 71 [18]. The modular curves $X_{\Delta}(N)$ are subcoverings of $X_1(N) \rightarrow X_0(N)$. Therefore it suffices to discuss the cases for the above nineteen integers N and for the integers N with genus of $X_0(N)$ are 0 or 1 (i.e. $N=17$, 19, 20, 24, 27, 32, 36, 49; 13, 16, 18 and 25). Our result is as follows.

THEOREM. *The hyperelliptic modular curves of type* $X_{\Delta}(N)$ *are the curves* $X_0(N)$ *for the above nineteen integers* N, *and* $X_1(13)$, $X_1(16)$ *and* $X_1(18)$.

By the above result and [18], we see that the hyperelliptic involutions of $X_{\Delta}(N)$ as above are represented by matrices belonging to $\mathrm{GL}_2^{+}(\boldsymbol{Q})$, except for $X_0(37)$ (see also [12]). Our result is used to determine the torsion points on elliptic curves defined over quadratic fields [17].

The automorphism groups $\mathrm{Aut}\, X_{\Delta}(N)$ are determined for $X_0(N)$, [3], [8], [19], and for all Δ with square free integers N [13]. Except for $N=37$ and 63 the automorphisms of $X_0(N)$ with genera $\geqq 2$ are represented by matrices belonging to $\mathrm{GL}_2^{+}(\boldsymbol{Q})$ loc. cit.. In the final section, we determine the automorphism

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groups of the hyperelliptic modular curves as above.

NOTATION. Let Q_p^{ur} denote the maximal unramified extension of Q_p. For a positive integer n, ζ_n is a primitive n-th root of unity, and μ_n is the group consisting of all the n-th roots of unity.

§ 1. Preliminaries

In this section, we give a review on modular curves and add the list of the hyperelliptic modular curves of type $X_0(N)$ [18]. Let $N \geqq 1$ be an integer, and Δ be a subgroup of $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$ containing -1. Let $X_\Delta = X_\Delta(N)$ be the modular curve defined over $\boldsymbol{Q}$ associating to the modular group $\Gamma_\Delta(N)$:

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\boldsymbol{Z}) \mid c \equiv 0 \bmod N,\ (a \bmod N) \in \Delta\right\}$$

Then $X_\Delta(N)$ is the coarse moduli space (over $\boldsymbol{Q}$) of the isomorphism classes of the generalized elliptic curves E with a point $P \bmod \Delta$. We have the Galois covering

$$X_1(N) \longrightarrow X_\Delta(N) \longrightarrow X_0(N),$$

$$(E, \pm P) \longmapsto (E, \Delta P) \longmapsto (E, \langle P \rangle)$$

where $\langle P \rangle$ is the cyclic subgroup generated by P. Let $g_\Delta(N)$, $g_1(N)$ and $g_0(N)$ denote the genera of $X_\Delta(N)$, $X_1(N)$ and $X_0(N)$, respectively, Let $Y_\Delta(N)$, $Y_1(N)$ and $Y_0(N)$ be the open affine subschemes $X_\Delta(N) \setminus \{\text{cusps}\}$ $X_1(N) \setminus \{\text{cusps}\}$, and $X_0(N) \setminus \{\text{cusps}\}$, respectively [2] VI (6.5). Then the covering $Y_1(N) \to Y_0(N)$ ramifies at the points represented by the pairs $(E, \langle P \rangle)$ with $\mathrm{Aut}(E, \langle P \rangle) \neq \{\pm 1\}$ and $\mathrm{Aut}(E, \pm P) = \{\pm 1\}$. The modular invariants of the remification points on $Y_0(N)$ are 0 or 1728.

(1.1) Let $\mathbf{0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be the $\boldsymbol{Q}$-rational cusps on $X_0(N)$ which are represented by the pairs $(\boldsymbol{G}_m \times \boldsymbol{Z}/N\boldsymbol{Z}, \boldsymbol{Z}/N\boldsymbol{Z})$ and $\{\mathbf{G}_m, \mu_N\}$, respectively [2] II. For a positive divisor d of N and for an integer i prime to d, let $\begin{pmatrix} i \\ d \end{pmatrix}$ denote the cusp on $X_0(N)$ which is represented by $(\mathbf{G}_m \times \boldsymbol{Z}/(N/d)\boldsymbol{Z}, \langle \zeta_N^i, 1 \rangle)$. Then $\begin{pmatrix} i \\ d \end{pmatrix}$ is defined over $\boldsymbol{Q}(\zeta_n)$ for $n = $ G.C.D. of d and N/d, and $\begin{pmatrix} i \\ d \end{pmatrix} = \begin{pmatrix} j \\ d \end{pmatrix}$ if and only if $i \equiv j \bmod n$. The ramification index of the covering $X_1(N) \to X_0(N)$ at the cusp $\begin{pmatrix} i \\ d \end{pmatrix}$ is G.C.D. of d and N/d. Let $\mathbf{O}_i$ $(1 \leqq i \leqq \#((\boldsymbol{Z}/N\boldsymbol{Z})^\times/\Delta))$ be the cusps on $X_\Delta(N)$ lying over the cusp $\mathbf{O}$ on $X_0(N)$. Then $\mathbf{O}_i$ are all $\boldsymbol{Q}$-rational.

We call them **O**-cusps.

Let $C_0=\binom{i}{d}$ be a cusp on $X_0(N)$, and C be a cusp on $X_\Delta(N)$ lying over C_0. We here discuss the field of definition of the cusp C. Put $N=d_1\cdot N_d$ for coprime divisors d_1 and N_d such that d and d_1 have same prime divisors. Put $\Delta'_d=\{a \bmod d_1 \mid a\in\Delta,\ a\equiv 1 \bmod N/d\}$, $\Delta''_d=\{a\in(\boldsymbol{Z}/d_1\boldsymbol{Z})^\times \mid a\equiv 1 \bmod d\}$, and let Δ_d be the subgroup generated by Δ'_d and Δ''_d.

LEMMA 1.2. *With the notation as above, let $k(\Delta, d)$ be the field associating to the subgroup Δ_d of $(\boldsymbol{Z}/d_1\boldsymbol{Z})^\times$. Then $k(\Delta, d)$ is the field of definition of the cusp C. For $C=\infty$, we know $\Delta_d=\Delta$.*

PROOF. The cusp C is represented by the pair

$$(\mathbf{G}_m\times\boldsymbol{Z}/(N/d)\boldsymbol{Z},\ (\zeta, 1) \bmod \Delta)$$

for a primitive d-th root $\zeta=\zeta_d$ of unity (1.1). The subgroup Δ acts by $(\zeta, 1)\mapsto(\zeta^a, a)$ for $a\in\Delta$. Further, as a generalized elliptic curve, $\mathrm{Aut}(\mathbf{G}_m\times\boldsymbol{Z}/(N/d)\boldsymbol{Z})$ is generated by $(x, i)\mapsto(\zeta_{N/d}^i\cdot x, i)$ and $(x, i)\mapsto(x^{-1}, -i)$ (see [2] I). □

(1.3) Let $M\neq 1$ be a positive divisor of N prime to N/M. The matrix $\begin{pmatrix} Ma & b \\ Nc & Md\end{pmatrix}$ for integers a, b, c, d with $adM^2-cdN=M$ defines an automorphism w_M of $X_1(N)$. For a choice of a primitive M-th root ζ_M of unity. w_M is defined by

$$(E, \pm P)\longmapsto(E/\langle P_M\rangle, \pm(P+Q_M) \bmod \langle P_M\rangle),$$

where $P_M=(N/M)P$ and Q_M is a point of order M such that $e_M(P_M, Q_M)=\zeta_M$ and $e_M: E_M\times E_M\to\mu_M$ is the e_M (Weil)-pairing. Then w_M induces the involution of $X_0(N)$ defined by

$$((E, A)\longmapsto(E/A_M, (A+E_M)/A_M),$$

where A_M is the cyclic subgroup of order M of A. For an integer i prime to N, let $[i]$ denote the automorphism of $X_1(N)$ represented by $g\in\Gamma_0(N)$ such that $g\equiv\begin{pmatrix} i & * \\ 0 & *\end{pmatrix} \bmod N$, then $[i]$ acts as $(E, \pm P)\mapsto(E, \pm iP)$. We denote also by w_M and $[i]$ the automorphisms of a subcovering $X_\Delta(N)$ which are induced by w_M and $[i]$, respectively.

(1.4) There are exactly nineteen values of N for which $X_0(N)$ are hyperelliptic curves and they are listed in the table below [18]:

N	*genus*	*hyperelliptic involution*
22	2	w_{11}
23	2	w_{23}
26	2	w_{26}
28	2	w_7
29	2	w_{29}
30	3	w_{15}
31	2	w_{31}
33	3	w_{11}
35	3	w_{35}
37	2	$s \cdots (*)$
39	3	w_{39}
40	3	$\begin{pmatrix} -10 & 1 \\ -120 & 10 \end{pmatrix}$
41	3	w_{41}
46	5	w_{23}
47	4	w_{47}
48	3	$\begin{pmatrix} -6 & 1 \\ -48 & 6 \end{pmatrix}$
50	2	w_{50}
59	5	w_{59}
71	6	w_{71}

(*) s is not represented by any 2×2 matrix [12] §5, [18].

§2. Hyperelliptic modular curves $X_\Delta(N)$

In this section, we determine the hyperelliptic modular curves of type $X_\Delta(N)$. To determine the hyperelliptic modular curve $X_\Delta(N)$ (of genus $g_\Delta(N)\geqq 2$), it suffices to discuss the following three cases (1), (2) and (3):

Case (1) $g_0(N)\geqq 2$ (see (1.4)).
Case (2) $g_0(N)=1$ ($N=17$, 19, 20, 24, 27, 32, 36 and 49)
Case (3) $g_0(N)=0$ ($N=13$, 16, 18 and 25)

THEOREM 2.1. *All the hyperelliptic modular curves* $X_\Delta(N)$ *are the following twenty-two modular curves:*

$$X_0(N) \qquad \textit{for the nineteen integers } N \textit{ in } (1.4),$$

and

	genus	*hyperelliptic involution v*
$X_1(13)$	2	$[5]=[2]^3$
$X_1(16)$	2	$[7]=[5]^2$
$X_1(18)$	2	$w_2\circ[7]$

PROOF. Suppose that $X_\Delta=X_\Delta(N)$ has the hyperelliptic involution w. Then w is defined over $\boldsymbol{Q}$ and belongs to the center of $\mathrm{Aut}\,X_\Delta(N)$. If moreover $g_0(N)\geqq 2$, then w induces the hyperelliptic involution v of $X_0(N)$.

CASE (1) $g_0(N)\geqq 2$: At first, we discuss the case when the hyperelliptic involutions v of $X_0(N)$ are of type w_M (1.4). For $N=23$, 26, 29, 31, 35, 39, 41, 47, 50, 59 and 71, $v(\mathbf{0})=\infty$ and the cusps lying over ∞ are defined over the fields associated with the subgroup Δ of $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$ by lemma 1.2. For $N=22$, 28, 30, 33 and 46, by Lemma 1.2, we see that the cusps on $X_\Delta(N)$ lying over $v(\mathbf{0})$ are not defined over $\boldsymbol{Q}$ for $\Delta\neq(\boldsymbol{Z}/N\boldsymbol{Z})^\times$. Now we discuss the remaining case for $N=40$, 48 and 37.

Case $N=40$: The maximal subgroup of $(\boldsymbol{Z}/40\boldsymbol{Z})^\times=(\boldsymbol{Z}/8\boldsymbol{Z})^\times\times(\boldsymbol{Z}/5\boldsymbol{Z})^\times$ containing ± 1 are $\Delta_1=\langle\pm 1, (3, 1), (-1, 1)\rangle$, $\Delta_2=\langle\pm 1, (3, 2)\rangle$ and $\Delta_3=\langle\pm 1, (1, 2)\rangle$. The hyperelliptic involution v of $X_0(40)$ sends the cusp ∞ to $\begin{pmatrix}1\\4\end{pmatrix}$ (1.4). The cusp C on X_{Δ_i} lying over $\begin{pmatrix}1\\4\end{pmatrix}$ are all $\boldsymbol{Q}$-rational, and those lying over ∞ are defined over the fields associated with the subgroups Δ_i of $(\boldsymbol{Z}/40\boldsymbol{Z})^\times$, cf. Lemma 1.2.

Case $N=48$: The maximal subgroups of $(\boldsymbol{Z}/48\boldsymbol{Z})^\times=(\boldsymbol{Z}/16\boldsymbol{Z})^\times\times(\boldsymbol{Z}/3\boldsymbol{Z})^\times$ are $\Delta_1=\langle\pm 1, (3, 1)\rangle$, $\Delta_2=\langle\pm 1, (9, 1), (1, -1)\rangle$ and $\Delta_3=\langle\pm 1, (3, -1)\rangle$. Tne hyperelliptic involution v of $X_0(48)$ sends the cusp ∞ to $\begin{pmatrix}1\\8\end{pmatrix}$ (1.4). Let P_i and Q_i be the cusps on X_{Δ_i} lying over the cusp ∞ and $\begin{pmatrix}1\\8\end{pmatrix}$, respectively. Then P_i are defined over real quadratic fields, cf. Lemma 1.2. But the cusp Q_1 is defined over $\boldsymbol{Q}(\sqrt{-2})$, and the cusp Q_3 is defined over $\boldsymbol{Q}(\sqrt{-1})$. For Δ_2, suppose that X_{Δ_2} has the hyperelliptic involution v, which induces the hyperelliptic involution w of $X_0(48)$ represented by $\begin{pmatrix}-6 & 1\\-48 & 6\end{pmatrix}$ cf. (1.4). The matrix $\begin{pmatrix}1 & 1/2\\0 & 1\end{pmatrix}$ represents an automorphism u of X_{Δ_2}, and u does not commute with v.

Case $N=37$: The hyperelliptic involution s of $X_0(37)$ sends the cusps to non cuspidal $\boldsymbol{Q}$-rational points, [12] § 5, [18] Theorem 2. Further by [13], any automorphism of $X_\Delta(N)$ is represented by a matrix belonging to $\mathrm{GL}_2^+(\boldsymbol{R})$ for

$\Delta\neq(\boldsymbol{Z}/37\boldsymbol{Z})^{\times}$.

CASE (2) $g_0(N)=1$: Let $\Gamma_{\Delta}^{*}(N)/\boldsymbol{Q}^{\times}$ be the normalizer of $\Gamma_{\Delta}(N)/\pm 1$ in $\mathrm{PGL}_2^{+}(\boldsymbol{Q})$, and put $B_{\Delta}=B_{\Delta}(N)=\Gamma_{\Delta}^{*}(N)/\Gamma_{\Delta}(N)\boldsymbol{Q}^{\times}$, which is a subgroup of $\mathrm{Aut}\,X_{\Delta}(N)$. For square free integers N with $g_{\Delta}(N)\geqq 2$, $B_{\Delta}(N)=\mathrm{Aut}\,X_{\Delta}(N)$ except for $X_0(37)$ [13].

Case $N=17$, 19 and 20: For $\Delta\neq\{\pm 1\}$, $g_{\Delta}(N)=1$. For $N=17$ and 19, $X_1(N)(\boldsymbol{Q})$ consist of the $\boldsymbol{O}$-cusps, and $X_1(20)(\boldsymbol{Q})$ consists of the $\boldsymbol{O}$-cusps and ramified cusps C_1 and C_2 lying over the cusp $\binom{1}{2}$ [10], Lemma 1.2. Suppose that $X_1(N)$ has the hyperelliptic involution v. Then v induces an involution w of $X_0(N)$ such that $X_0(N)/\langle w\rangle\simeq\boldsymbol{P}_{\boldsymbol{Q}}^{1}$, and w commutes with the automorphisms of type w_M cf. [1] §4. Then w fixes $\mathbf{O}$, and $\binom{1}{2}$ for $N=20$. For $N=17$ and 19, there are not such involutions. The orbit of $\left\{\mathbf{O},\binom{1}{2}\right\}$ under the subgroup $\langle w_4, w_5\rangle$ is $\left\{\mathbf{O},\infty,\binom{1}{2},\binom{1}{4},\binom{1}{5},\binom{1}{10}\right\}$, which consists of fixed points of w. This is a contradiction.

Case $N=21$: The maximal subgroups of $(\boldsymbol{Z}/21\boldsymbol{Z})^{\times}=(\boldsymbol{Z}/3\boldsymbol{Z})^{\times}\times(\boldsymbol{Z}/7\boldsymbol{Z})^{\times}$ are $\Delta_1=\langle\pm 1,(1,-1)\rangle$, $\Delta_2=\langle\pm 1,(1,2)\rangle$, and $g_{\Delta_1}(21)=3$, $g_{\Delta_2}(21)=1$. Suppose that X_{Δ} has the hyperelliptic involution v for $\Delta=\Delta_1$. Then v induces the involution $w=w_3$ or w_{21} [1] §4, [24] table 5. Since $w_{21}(\mathbf{O})=\infty$, $w\neq w_{21}$ cf. Lemma 1.2, hence $w=w_3$. But then v dose not commutes with w_7.

Case $N=24$: Since $X_0(24)(\boldsymbol{Q})=\{\text{cusps}\}$ [24] table 1, and $\Gamma_0(24)/\pm 1$ has no elliptic element, any $\boldsymbol{Q}$-rational automorphism of $X_0(24)$ belongs to $B_0(24)$. The maximal subgroups of $(\boldsymbol{Z}/24\boldsymbol{Z})^{\times}=(\boldsymbol{Z}/8\boldsymbol{Z})^{\times}\times(\boldsymbol{Z}/3\boldsymbol{Z})^{\times}$ are $\Delta_1=\langle\pm 1,(-1,1)\rangle$, $\Delta_2=\langle\pm 1,(3,1)\rangle$ and $\Delta_3=\langle\pm 1,(5,1)\rangle$. For $\Delta=\Delta_1$ and Δ_2, $g_{\Delta}(24)=3$ and $g_{\Delta_3}(24)=1$. Suppose X_{Δ} has the hyperelliptic involution v for $\Delta=\Delta_1$ or Δ_2. Since $\begin{pmatrix}1 & 1/2\\ 0 & 1\end{pmatrix}$ mod $\Gamma_{\Delta}(24)$ does not belong to $\mathrm{Aut}\,X_{\Delta}$, v induces the involution $w=w_8$ or w_{24} [1] §4, [24] table 5. But w_8 and w_{24} are defined over $\boldsymbol{Q}(\sqrt{2})$ for $\Delta=\Delta_1$. For $\Delta=\Delta_2$, w_{24} is defined over $\boldsymbol{Q}(\sqrt{-3})$, hence $w=w_8$. Since $X_{\Delta}(\boldsymbol{Q})$ consisits of the $\mathbf{O}$-cusps and ramified cusps C_1, C_2, C_3, C_4, $w=w_8$ must fix the $\mathbf{O}$-cusps. This is a contradiction.

Case $N=27$: For $\Delta\neq\{\pm 1\}$, $g_{\Delta}(27)=1$, and $g_1(27)=3$. Let $\mathscr{X}=\mathscr{X}_1(27)$ be the normalization of the projective j-line in the function field of $X_1(27)$. Then

$\#\mathcal{X}(\boldsymbol{F}_2)\geqq\#\{\mathbf{O}\text{-cusps}\}=9$, so that $X_1(27)$ is not hyperelliptic cf. [18].

Case $N=32$: For $\Delta'=\langle\pm 1, 1+16\rangle$, $g_{\Delta'}(32)=5$, and for $\Delta''=\langle\pm 1, 1+8\rangle$, $g_{\Delta''}(32)=1$. Let J', J'' be the jacobian varieties of $X_{\Delta'}$ and $X_{\Delta''}$ respectively. Then $J'=J''+A$ for an abelian variety $A(/\boldsymbol{Q})$ of dimension 4. The involution [9] acts by $+1$ on J'', and by -1 on A. If $X_{\Delta'}$ has the hyperelliptic involution v, then [9] v acts by -1 on J'', and $+1$ on A. But there is not such an involution. It is easily seen by Riemann-Hurwitz formula.

Case $N=36$: The maximal subgroups of $(\boldsymbol{Z}/36\boldsymbol{Z})^{\times}=(\boldsymbol{Z}/4\boldsymbol{Z})^{\times}\times(\boldsymbol{Z}/9\boldsymbol{Z})^{\times}$ are $\Delta_1=\langle\pm 1, (1, 4)\rangle$, $\Delta_2=\langle\pm 1, (1, -1)\rangle$, and $g_{\Delta_1}=3$, $g_{\Delta_2}=7$. Snppose X_{Δ} has the hyperelliptic involution v. Then v induces an involution w of $X_0(36)$. At first, we discuss for $\Delta=\Delta_1$. The set $X_{\Delta_1}(\boldsymbol{Q})$ consists of the $\mathbf{O}$-cusps and ramified cusps C_1, C_2 cf. [24] table 1, Lemma 1.2. Then w fixes the set of $\mathbf{O}$-cusps. The matrix $\begin{pmatrix}1 & 1/3\\ 0 & 1\end{pmatrix}$ represents an automorphism g of X_{Δ_1}, and the orbit of $\mathbf{O}$ under the subgroup $\langle g, w_4, w_9\rangle$ is $S=\left\{\mathbf{O}, \infty, \begin{pmatrix}\pm 1\\ 3\end{pmatrix}, \begin{pmatrix}1\\ 9\end{pmatrix}, \begin{pmatrix}1\\ 4\end{pmatrix}, \begin{pmatrix}\pm 1\\ 12\end{pmatrix}\right\}$. Then w must have more than $\#S=8$ fixed points, which is a contradiction. Now consider the case for $\Delta=\Delta_2$. The set $X_{\Delta_2}(\boldsymbol{Q})$ consists of the $\mathbf{O}$-cusps and the cusps lying over the cusps $\begin{pmatrix}1\\ 2\end{pmatrix}, \begin{pmatrix}1\\ 4\end{pmatrix}$, cf. Lemma 1.2. Then v fixes a rational points on X_{Δ_2}, since $\#X_{\Delta_2}(\boldsymbol{Q})=9$. The matrix $\begin{pmatrix}1 & 1/2\\ 0 & 1\end{pmatrix}$ represents an automorphism g of X_{Δ_2}, and the subgroup $\langle g, w_4, \gamma\rangle$ acts transitively on $X_{\Delta_2}(\boldsymbol{Q})$, where γ is a generator of the covering group of $X_{\Delta_2}\to X_0(36)$. Thus v fixes all the points belonging to $X_{\Delta_2}(\boldsymbol{Q})$ and $w_9(X_{\Delta_2}(\boldsymbol{Q}))$. This contradicts to $g_{\Delta}(36)=7$.

Case $N=49$: Let Δ_n be the maximal subgroups of $(\boldsymbol{Z}/49\boldsymbol{Z})^{\times}$ of indices $n=3, 7$. Let $\mathcal{X}_{\Delta}$ be the normalization of the projective j-line $\mathcal{X}_0(1)\cong\boldsymbol{P}^1_{\boldsymbol{Q}}$ in the function field of X_{Δ}. For $\Delta=\Delta_3$, the cusps on X_{Δ} are all defined over $\boldsymbol{Q}(\zeta_7)$, so that $\#\mathcal{X}_{\Delta}(\boldsymbol{F}_8)\geqq 24$. For $\Delta=\Delta_7$, $\#\mathcal{X}_{\Delta}(\boldsymbol{F}_2)\geqq 7$. Therefore X_{Δ_n} are not hyperelliptic cf. [18].

CASE (3) $g_0(N)=0$: For $\Delta\neq\{\pm 1\}$, $X_{\Delta}=\boldsymbol{P}^1_{\boldsymbol{Q}}$. For $N=13$, 16 and 18, [5], [7] and w_2[7] are the hyperelliptic involutions of $X_1(N)$, respectively. There remains the case for $N=25$. Let Δ_n be the maximal subgroups of $(\boldsymbol{Z}/25\boldsymbol{Z})^{\times}$ of index $n=2, 5$. Then $g_{\Delta_2}(25)=0$ and $g_{\Delta_5}(25)=4$. We know that $X_{\Delta_5}(\boldsymbol{Q})$ consists of the $\mathbf{O}$-cusps [6]. Suppose that $X=X_{\Delta_5}$ has the hyperelliptic involution v. Then v fixes a $\mathbf{O}$-cusp, hence v fixes all the $\mathbf{O}$-cusps. Then the divisor class $cl((\mathbf{O}')-(\mathbf{O}''))$ are of order 2 for the $\mathbf{O}$-cusps $\mathbf{O}'$ and $\mathbf{O}''$, $\mathbf{O}'\neq\mathbf{O}''$. But we know that the Mordell-Weil group of the jacobian variety of X is isomorphic to

$Z/71Z$ [6]. □

§ 3. Automorphism groups of hyperelliptic curves $X_\Delta(N)$

In this section, we determined the automorphism groups of hyperelliptic modular curves of type $X_\Delta(N)$. For square free integers N, Aut $X_\Delta(N)$ are determined [13], [19]. Hence it suffices to discuss for $X_1(16)$ and $X_1(18)$ cf. Theorem 2.1.

THEOREM 3.1. *The automorphisms of $X_1(16)$ and $X_1(18)$ are represented by 2×2 matricies.*

PROOF.

Case $N=18$: Let $\mathfrak{X}$ be the minimal model of $X_1(18)$ $(/\boldsymbol{Z})$. The special fibre $\mathfrak{X}\otimes \boldsymbol{F}_2$ has two irreducible components Z, Z' which are isomorphic to $\boldsymbol{P}^1$ and intersect transversally at three supersingular points S_1, S_2 and S_3 [2]. Let $v=w_2[7]$ be the hyperelliptic involution of $X_1(18)$. Since the jacobian variety $J_1(18)$ of $X_1(18)$ has stable reduction at the rational prime 2 [2], any endomorphism of $J_1(18)$ is defined over $\boldsymbol{Q}_2^{ur}$ [22] Lemma 1. Let G be the subgroup of Aut $X_1(18)$ consisting of automorphisms g which fix the irreducible component Z. Then we see that the representation of G into the permutation group $\mathcal{S}_3$ of the set $\{S_1, S_2, S_3\}$ is faithfull. Thus we see that $G=\langle w_3, [7]\rangle$. Further w_2 exchanges Z by Z'. Thus Aut $X_1(18)$ is generated by w_2, w_9 and [7].

Case $N=16$: The hyperelliptic involution $v=\gamma^2$ for $\gamma=[3]$. Put $X=X_1(16)$ and $Y=X/\langle v\rangle$. Let C_1, C_2 (resp. C_3, C_4) be the cusps on X lying over the cusp $\binom{1}{2}$ $\left(\text{resp. } \binom{1}{8}\right)$. Then C_i are the ramification points of the covering $X\to Y$. Let P_1, P_2 be the totally ramified cusps lying over $\binom{1}{4}$ and $\binom{-1}{4}$, respectively. Let S_v be the set of the Weierstrass points of $X: S_v=\{P_1, P_2, C_1, C_2, C_3, C_4\}$, and let $\mathcal{S}_6$ be the permutation group of the elements of S_v. Then $(\mathrm{Aut}\, X)/\langle v\rangle$ becomes a subgroup of $\mathcal{S}_6$.

LEMMA 3.2. $\{g\in \mathrm{Aut}\, X \mid g\gamma g^{-1}=\gamma^{\pm 1}\}=\langle \gamma, w_{16}\rangle$.

PROOF. We can take a local parameter x along the cusp ∞ of $X_0(16)$ such that the modular invariant $j=F(x)/G(x)$ for $F(x)=(x^8+2^4x^7+7\cdot 2^4x^6+7\cdot 2^6x^5+69\cdot 2^4x^4+13\cdot 2^7x^3+11\cdot 2^7x^2+2^{10}x+2^{13})^3$ and $G(x)=x(x+4)(x^2+4x+8)(x+2)^4$ [3] kapitel IV. Further the values $x=0, -2, -2+2\sqrt{-1}, -2-2\sqrt{-1}$ and -4

corresponds to the cusps ∞, $\binom{1}{2}$, $\binom{1}{4}$, $\binom{-1}{4}$ and $\binom{1}{8}$, respectively. If $g\gamma g^{-1}=\gamma^{\pm 1}$, then g induces an automorphism of h of $X_0(16)=\boldsymbol{P}^1(x)$, and h^* sends the set $\{-4, -2\}$ and $\{-2\pm 2\sqrt{-1}\}$ to themselves. If $h^*(-4)=-2$, then $w_{16}{}^*h^*$ fixes both -4 and -2. Changing g by gw_{16}, if necessary, we may assume that h^* fixes both -4 and -2. Let δ be the automorphism of $\boldsymbol{P}^1(x)$ defined by $\delta^*(x)=x+4/x+2$, then $\delta^*(-2+2\sqrt{-1})=1-\sqrt{-1}$, $\delta^*(-2-2\sqrt{-1})=1+\sqrt{-1}$, and $(\delta h\delta^{-1})^*(x)=\alpha x$ for some $\alpha\in\boldsymbol{C}^\times$. If $\alpha\neq 1$, then $\alpha(1+\sqrt{-1})=1-\sqrt{-1}$, so that $\alpha=-\sqrt{-1}$. But then $1+\sqrt{-1}=(\delta h\delta^{-1})^*(1-\sqrt{-1})\neq(-\sqrt{-1})(1-\sqrt{-1})$. Therefore $\alpha=1$, i.e., $h=id$ and g belongs to $\langle\gamma\rangle$. $\square$

At first, we show that any 2-sylow subgroup H of $G=\operatorname{Aut} X$ containg γ and w_{16} is equal to the subgroup $\langle w_{16}, \gamma\rangle$, which is a dihedral group with relation $w_{16}\gamma w_{16}^{-1}=\gamma^{-1}$. If $\#H\neq 8$, then G has a subgroup K of order 16 containing $\langle w_{16}, \gamma\rangle$. Then $\langle\gamma\rangle$ is a normal subgroup of K, since $\langle\gamma\rangle$ is the unique cyclic subgroup of order 4 of $\langle w_{16}, \gamma\rangle$. Then by Lemma 3.2, any $g\in K$ belongs to $\langle w_{16}, \gamma\rangle$. It is a contradiction. Now we show that G is a 2-group. The prime divisors of $\#G$ are 2, 3 or 5. If $g\in G$ is of order 5, then g fixes a Weierstrass point C, which is defined over $\boldsymbol{Q}(\zeta_{16})$. Let t be a local parameter along C. Then $g^*(t)=\zeta_5 t+a_2t^2+\cdots$ for a primitive 5-th root ζ_5 of unity, so that g is not defined over $\boldsymbol{Q}_5^{ur}$. But we know that any endomorphism of the jacobian variety of X is defined over $\boldsymbol{Q}_p^{ur}$ for any prime number $p\neq 2$ [2], [22] Lemma 1. Suppose that an automorphism $g\in G$ is of order 3. By the same way as above, we see that g does not fix any Weierstrass point. Changing the induces of $\{P_i\}$, $\{C_1, C_2\}$ and $\{C_3, C_4\}$, if necessary, we may assume that (1) $g(P_1)=P_2$ or (2) $g(P_1)=C_1$.

CLAIM. $g(P_1)\neq P_2$.

We know that $\gamma=(C_1, C_2)(C_3, C_4) \bmod \langle v\rangle$. If $g(P_1)=P_2$, then $g\gamma g \bmod \langle v\rangle$ is of order 5, so that $g(P_1)\neq P_2$.

Put $h=g\gamma g^{-1}$, which fixes the $\boldsymbol{Q}$-rational cusp C_1. Let t be a local parameter along C_1. Then $h^*(t)=\pm\sqrt{-1}t+\cdots\in\boldsymbol{Q}(\sqrt{-1})[[t]]$, and h is defined over $\boldsymbol{Q}(\sqrt{-1})$. For any $\sigma\in\operatorname{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$, $h^\sigma=h^{\pm 1}$, so that $g^\sigma g^{-1}$ belongs to $\langle w_{16}, \gamma\rangle$ by Lemma 3.2. Since $g^\sigma g^{-1}$ fixes the $\boldsymbol{Q}$-rational cusp C_1, $g^\sigma g^{-1}=1$ or v. Then $(g^\sigma)^2=g^2$. Since g is of order 3, $g^\sigma=g$, so that g is defined over $\boldsymbol{Q}$. But we know that $\operatorname{End}_{\boldsymbol{Q}} J_1(16)\otimes\boldsymbol{Q}\cong\boldsymbol{Q}(\sqrt{-1})$ [14], [20, 21], where $\operatorname{End}_{\boldsymbol{Q}}\cdots$ is the subring consisting of the endomorphisms defined over $\boldsymbol{Q}$. Thus $\operatorname{Aut} X$ is a 2-group. $\square$

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N. Ishii
Dokkyo Secondary High School
1-8, Sekiguchi, Bunkyo-ku
Tokyo 112, Japan

F. Momose
Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112, Japan