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INTEGRAL DOMAINS FINITE OVER EACH UNDERRING

By

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Abstract. A characterization is given of integral domains R such that R is a finitely generated S-module for each subring S of R which has the same quotient field as R. Apart from the absolutely algebraic fields of positive characteristic, such R are subrings of the rings of integers of certain global fields.

Throughout, let R denote a (commutative integral) domain with quotient field K. As in [4], an *underring* of R is a subring of R that also has quotient field K. The main purpose of this note is to sharpen the focus of the following main result of [4] from integrality to module-finiteness.

THEOREM A [4, Corollary 3.7]. *R is integral over each underring of R if and only if one of the following three conditions holds:*

(1) *R is isomorphic to a subring of the ring of all algebraic integers;*

(2) *$R=K$ is an algebraic field extension of some $\boldsymbol{F}_p$;*

(3) *$ch(R)>0$ and precisely one valuation ring of K does not contain R.*

More precisely, we shall determine which of the domains R in Theorem A satisfy the stronger condition that R is module-finite over each of its underrings. The answer is stated next. (As usual, by an *algebraic number field,* we mean a finite-dimensional field extension of $\boldsymbol{Q}$.)

THEOREM B. *R is module-finite over each underring of R if and only if one of the following three conditions holds:*

(1) *R is isomorphic to a subring of the ring of integers of some algebraic number field;*

(2) *$R=K$ is an algebraic field extension of some $\boldsymbol{F}_p$;*

(3) *$ch(R)=p>0$, precisely one valuation ring of K does not contain R, and K is a finitely generated field extension of $\boldsymbol{F}_p$.*

We pause to record a number-theoretic insight into one benefit of shifting

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focus from Theorem A to Theorem B.

If R satisfies (1) in Theorem B, then R is inside the ring of integers of a global field; by [4, Theorem 2.3(iii)], the same conclusion holds if R satisfies (3) in Theorem B. By way of a partial converse, note, by the proof of [4, Remark 2.4(a)], that $\boldsymbol{F}_p[X]$ satisfies (3) in Theorem B. (A contrast between the two results is afforded by the example of $F[X]$, where F denotes the algebraic closure of $\boldsymbol{F}_p$; indeed, $F[X]$ satisfies (3) in Theorem A but does not satisfy (3) in Theorem B.) However, it should be noted (in positive characteristic) that not all rings of integers of global fields satisfy (3) in Theorem B. Indeed, [4, Remark 2.4(c)] gives an example of a global field (in any positive characteristic $p\neq 2$) whose ring of integers fails to satisfy the weaker condition (3) in Theorem A.

Before proving Theorem B, we state the following variant. In view of Theorem A, it is straightforward to show that Theorem C is equivalent to Theorem B.

THEOREM C. *R is module-finite over each underring of R if and only if at least one of the following two conditions holds:*

(i) *$R=K$ is an algebraic field extension of some $\boldsymbol{F}_p$;*

(ii) *R is integral over each underring of R, and K is a finitely generated field extension of its prime subfield.*

Following the proof of Theorem B and C, we remark on additional motivation for studying domains which are module-finite over each underring.

PROOF OF THEOREMS B AND C. We prove the "if" assertion first. If $R=K$ is algebraic over $\boldsymbol{F}_p$, then R is its only underring, and so R is module-finite over each underring. Suppose next that $R\neq K$ satisfies (ii); let S be an underring of R. If $\mathrm{ch}(R)=p>0$, [4, Theorem 2.3(iii)] assures that $\mathrm{t.d.}(K/\boldsymbol{F}_p)=1$; thus, we can find a transcendence basis $\{X\}$ inside S (cf. [8]). Let T denote the ring of integers of K; that is, the integral closure of $\boldsymbol{F}_p[X]$ in K. By a fundamental finiteness result (cf. [8, Theorem 9, p. 267]), T is module-finite over $A=\boldsymbol{F}_p[X]$. Since $A\subset S\subset R\subset T$ and A is a Noetherian ring, R is module-finite over A, and so R is module-finite over S. A similar argument, with $\boldsymbol{Z}$ replacing $A=\boldsymbol{F}_p[X]$, takes care of the case in which $R\neq K$ satisfies (ii) and has characteristic zero.

We turn now to the "only if" assertion. Suppose that R is module-finite over each of its underrings. If $R=K$ then, by integrality, R has no proper underrings, and so [4, Proposition 2.1] assures that (i) holds. We may assume henceforth that $R \neq K$. Since R is integral over each underring, it remains only to show that K is a finitely generated field extension of its prime subfield.

There are two cases, depending on $\mathrm{ch}(R)$. Suppose that $\mathrm{ch}(R)=p>0$. As above, R contains a transcendence basis $\{X\}$ of K over $F=\boldsymbol{F}_p(X)$. Put $A=\boldsymbol{F}_p[X]$. Consider any nonzero (proper) ideal I of R. By integrality, we can pick a nonzero element $n \in I \cap A$. Then $S=A+nR$ is an underring of R (since they share the nonzero ideal nR) and so, by hypothesis, R is a finitely generated S-module. It follows that $R^*=R/nR$ is a finitely generated module over $S^*=S/nR$. By a standard homomorphism theorem, $S^* \cong A/(A \cap nR)$. Since A is integrally closed, it follows easily that $A \cap nR=nA$, whence $S^* \cong A/nA$. As (the nonunit) n is a polynomial of positive degree, A/nA is a finite-dimensional extension of $\boldsymbol{F}_p$; thus, $S^* \cong A/nA$ is a finite ring. Consequently, R^* is also finite, and hence so is its factor ring R/I. In particular, R/I is a Noetherian ring. By a standard homomorphism theorem, there is no strictly ascending chain of ideals in R that begins at I. Since I was arbitrary, R is a Noetherian ring. Also, by integrality (cf. [3]), $\dim(R)=\dim(A)=1$. It follows from the Krull-Akizuki Theorem (cf. [3, Proposition 5, p. 500]) that R', the integral closure of R, is a Dedekind domain. It also follows from the Krull-Akizuki Theorem (cf. [7, Theorem 2]) that if $P \in \mathrm{Spec}(R')$ is nonzero, then R'/P is finitely generated as a module over R and, a fortiori, module-finite over $R/(P \cap R)$. We show next that $[K:F]<\infty$.

Consider any nonzero prime ideal P of R'. Since A is a *PID*, $P \cap A=pA$ for some prime element $p \in A$. Now, let L be a finite-dimensional field extension of F contained in K; let B be the integers of L (that is, the integral closure of A in L). By classical ramification theory (cf. [8, Corollary, p. 287]), $\sum e_i f_i=[L:F]$, where $pB=\prod P_i^{e_i}$ is a product of prime powers and $f_i=[B/P_i : A/pA]$. Working in the Dedekind domain R', we can write $P_iR'=\prod P_{ij}^{e_{ij}}$. Then $f_{ij}=[R'/P_{ij} : A/pA]=[R'/P_{ij} : R/(P_{ij} \cap R)] \cdot [R/(P_{ij} \cap R) : A/pA]$. The first factor is bounded above by the size of a minimal generating set of R'/P_{ij} as an R-module, and the second factor is bounded above by the (finite) number of elements in $R/(P_{ij} \cap R)$; thus, $f_{ij}<\infty$. Also, $f_i \leqq f_{ij}$. Consider

$$pR'=(pB)R'=(\prod P_i^{e_i})R'=\prod (P_iR')^{e_i}=\prod P_{ij}^{e_{ij}e_i}.$$

Since R' is a Dedekind domain, we see that the data $(P_{ij}, e_{ij}e_i)$ is independent of L. Moreover, for fixed i,

$$\sum_j (e_{ij}e_i) f_{ij} \geqq \sum_j (e_{ij}e_i) f_i \geqq \sum_j e_i f_i \geqq e_i f_i .$$

Hence, $\sum_{i,j}(e_{ij}e_i)f_{ij} \geqq \sum_i e_i f_i = [L:F]$. If $[K:F]=\infty$, algebraicity would permit us to choose L so that $[L:F] > \sum_{i,j}(e_{ij}e_i)f_{ij}$, a contradiction. Thus, $[K:F] < \infty$, completing the proof in case $\mathrm{ch}(R) \neq 0$.

If $\mathrm{ch}(R)=0$, essentially the same argument as above goes through, by replacing (A, F) with $(\boldsymbol{Z}, \boldsymbol{Q})$. The proof is complete.

REMARK. (a) As noted in the first paragraph of [4], new "underring" studies can often be viewed as "dual" to corresponding known work on overrings. In this regard, the "dual" context that suggested the topic of Theorem B was the work on "module-finite pairs" in [6]. It would be interesting to find a "pair" generalization of the above results, in the same sense that the "Noetherian pair" context of [7] generalizes the context of [5].

(b) Another concept that now seems to deserve attention is the one dual to "conducive". Recall from [1] (cf. [2]) that R is a *conducive domain* in case $(R:T) \neq 0$ for each overring $T \neq K$ of R. Accordingly, we say that R is a *coconducive domain* in case $(S:R) \neq 0$ for each underring S of R. It is clear that if R is module-finite over each of its underrings, then R is coconducive. Hence, if R satisfies any of the conditions (1)-(3) in Theorem B, then R is coconducive. It would be interesting to find other sufficient conditions for the "coconducive" property.

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