TSUKUBA J. MATH.
Vol. 15 No. 2 (1991), 351-387

A FACTOR OF SINGULAR HOMOLOGY

By

Katsuya EDA and Katsuro SAKAI

0. Introduction

Singular homology is a beautiful theory, in which we can see a clear correspondence between Algebra and Topology. However, it behaves badly on topological spaces which are not locally simply connected in comparison with Čech homology. In the present paper we introduce a canonical factor $H_n^T(X)$ of singular homology $H_n(X)$, which agrees with singular homology on ANR's and behaves well on the indicated spaces. We also introduce a notion "quasi-homotopy" for continuous maps. It turns out that the factor is invariant under quasi-homotopy.

We state definitions and basic facts in Section 1. In Section 2, we prove that $H_n^T(X)=H_n(X)$ for every ANR X. In Section 3, we show that $H_0^T(X)$ is isomorphic to a free abelian group whose rank is equal to the cardinality of equivalence classes with respect to a certain kind of connectedness. There we also introduce a notion "quasi-homotopy" and show that the factor is invariant under quasi-homotopy. In Sections 4, 5 and 6, one can see the advantage of $H_n^T(X)$ to the singular homology groups $H_n(X)$. More precisely, $H_n^T(X)$ are calculated for certain spaces such as the so-called Hawaiian earring and infinite products and so on. Furthermore, certain natural abelian groups are realized as $H_n^T(X)$ by natural topological spaces X. We shall show that any homomorphism from $H_n^T(X)$ to $H_n^T(Y)$ is induced by a continuous map if X and Y are obtained by attaching copies of $\boldsymbol{S}^n$ in certain ways. For example, for the Hawaiian earring $\boldsymbol{H}$, any homomorphism from $H_1^T(\boldsymbol{H})$ to itself can be induced by a continuous map from $\boldsymbol{H}$ to itself, though this fact does not hold for $H_1(\boldsymbol{H})$. We also characterize slenderness of abelian groups by using $H_1^T(\boldsymbol{H})$ and the notion of spatial homomorphism. (See Theorems 6.2, 6.3 and Remark 6.4.) Ralph [22] has defined a factor of singular chain and homology groups HA and HM. In Section 4, one can see that H_n^T has similar effect as HA.

Received July 14, 1989. Revised January 21, 1991.

1. Definitions and basic facts

All (topological) spaces in this paper are Tychonoff spaces and all groups are abelian, unless otherwise stated. For a space X, let $\mathrm{Pm}(X)$ denote the set of all continuous pseudo-metrics on X. ANR's mean absolute neighborhood retracts for metrizable spaces [13].

By Δ_n, we denote the standard n-simplex with vertices $\boldsymbol{e}_0, \cdots, \boldsymbol{e}_n$, where $\boldsymbol{e}_i(i)=1$ and $\boldsymbol{e}_i(j)=0$ for $j\neq i$. We regard $\Delta_n\subset\Delta_{n+1}$. Let $C(\Delta_n, X)$ be the set of all continuous maps from the n-simplex Δ_n to a topological space X, $S_n(X)$ the free abelian group generated by $C(\Delta_n, X)$ and $\partial=\partial_{n+1}: S_{n+1}(X)\to S_n(X)$ be the boundary operator as usual [3, Ch. III 2.1]. Namely $\varepsilon_i: \Delta_n\to\Delta_{n+1}$ is the linear map defined by: $\varepsilon_i(\boldsymbol{e}_j)=\boldsymbol{e}_j$ for $j<i$ and $\varepsilon_i(\boldsymbol{e}_j)=\boldsymbol{e}_{j+1}$ for $j\geqq i$. Then $\partial_{n+1}(u)=\sum_{i=0}^{n+1}(-1)^i u\circ\varepsilon_i$ for $u\in C(\Delta_{n+1}, X)$. The singular homology is defined by: $H_n(X)=Z_n(X)/B_n(X)$, where $Z_n(X)=\mathrm{Ker}\,\partial_n$ and $B_n(X)=\mathrm{Im}\,\partial_{n+1}$.

Now we state the definition of the factor. Regarding $C(\Delta_n, X)$ as the topological space with the compact-open topology, we consider $S_n(X)$ as the free abelian topological group on $C(\Delta_n, X)$ (in the sence of Markov [18]). Then, ∂ becomes a continuous homomorphism, hence the closure $\overline{B_n(X)}$ of $B_n(X)$ is included in $Z_n(X)$. (Refer to [18], [12] and [17] for the definition of free abelian topological groups and [20] for summary.) We define $H_n^T(X)=Z_n(X)/\overline{B_n(X)}$. Though $H_n^T(X)$ naturally becomes a topological group, we ignore this topology. By definition, there exists a natural epimorphism $\sigma_X: H_n(X)\to H_n^T(X)$. Let $f_*: H_n(X)\to H_n(Y)$ be the homomorphism induced by a continuous map $f: X\to Y$.

PROPOSITION 1.1. *Each continuous map $f: X\to Y$ induces a homomorphism $f_*^T: H_n^T(X)\to H_n^T(Y)$ such that $f_*^T\circ\sigma_X=\sigma_Y\circ f_*$.*

In fact, since $S_n(X)$ is a free abelian topological group on $C(\Delta_n, X)$, we can define a continuous homomorphism $f_\#: S_n(X)\to S_n(Y)$ by: $f_\#(u)=f\circ u\in C(\Delta_n, X)$ for $u\in C(\Delta_n, X)$. Then $f_\#(\overline{B_n(X)})\subset\overline{f_\#(B_n(X))}\subset\overline{B_n(Y)}$ by the continuity of $f_\#$ and a basic fact about singular homology. Thus f_*^T can be defined by $f_*^T(u+\overline{B_n(X)})=f_\#(u)+\overline{B_n(Y)}$. We have $f_*^T\circ\sigma_X=\sigma_Y\circ f_*$ by definition.

By the homotopy invariance of singular homology and Proposition 1.1 and by definition, we get the following.

PROPOSITION 1.2. *The groups $H_n^T(X)$ are homotopy invariant.*

PROPOSITION 1.3. *If two continuous maps $f, g: X\to Y$ are homotopic, then*

$f_*^T = g_*^T$.

This will be strengthened in Section 3 (Theorem 3.8). For basic results and notions about algebraic topology, we refer the reader to [3 and 23]. The set of nonzero positive integers is denoted by $\boldsymbol{N}$. As usual we abbreviate the subscript of the boundary operator ∂_n and $Z_n(X)$ or $B_n(X)$ by Z_n or B_n respectively, in case no confusion will occur.

2. $H_n^T(X)=H_n(X)$ for ANR's X

The purpose of this section is to prove the following theorem.

THEOREM 2.1. *Let X be an ANR. Then $B_n(X)$ is closed in the free abelian topological group $S_n(X)$ for each n. Consequently, $H_n^T(X)$ is identical with $H_n(X)$ for each n.*

COROLLAY 2.2. *Let X be a space which has the homotopy type of an ANR. Then $H_n^T(X)=H_n(X)$.*

To show this, Graev's metric on free abelian groups plays a crucial role in spite of the fact that the metric topology is coarser than the free topology except rare cases. It seems impossible to perform the proof only by using the universality of free abelian topological groups [17]. Our proof implies that $B_n(X)$ is not only closed but also open in $Z_n(X)$ and hence $H_n^T(X)$ turns out to be discrete for every ANR X even if we consider its topology.

First we introduce Graev's metric of the free abelian group $A(M)$ generated by a metric space $M=(M, \rho)$ [12]. Fix an element x^* of M and extend ρ to ρ' on the set $M\cup\{0\}\cup -M$ as follows: (1) $\rho'(x^*, 0)=1$; (2) $\rho'(x, 0)=\rho'(-x, 0)=\rho(x, x^*)+1$ for $x\in X$; (3) $\rho'(-x, -y)=\rho(x, y)$ and $\rho'(-x, y)=\rho(x, 0)+\rho(0, y)$ for $x, y\in M$. For $u, v\in A(M)$, define

$$\tilde{\rho}(u, v)=\inf\{\textstyle\sum_{i=0}^m \rho'(x_i, y_i): u=\sum_{i=0}^m x_i,\ v=\sum_{i=0}^m y_i,\ x_i, y_i\in M\cup\{0\}\cup -M,\ m\in \boldsymbol{N}\}.$$

In fact, Graev [12] proved that there exist $x_i, y_i\in M\cup\{0\}\cup -M$ $(0\leqq i\leqq m)$ such that $u=\sum_{i=0}^m x_i$, $v=\sum_{i=0}^m y_i$ and $\tilde{\rho}(u, v)=\sum_{i=0}^m \rho'(x_i, y_i)$ and that $\tilde{\rho}$ is a metric on $A(M)$. Observe that both x_i and y_i in the above belong to either M or $-M$, if $\tilde{\rho}(u, v)<1$. One should remark that the topology induced by $\tilde{\rho}$ is coarser than the topology of $A(M)$ as the free abelian topological group on M. For the simplicity of notation, we also use ρ for $\tilde{\rho}$.

Now we start to prove the theorem. Without loss of generality, we can assume by [19] that X is a closed subset of a normed linear space $Y=(Y, \|\cdot\|)$ and there is a retraction $r: U\to X$ of a uniform neighborhood U of X in Y, i.e.,

$$U=\{y\in Y: \|x-y\|<\delta \text{ for some } x\in X\}$$

for some $\delta>0$. The compact-open topology on $C(\Delta_n, X)$ is induced by the sup-metric

$$\rho(f, g)=\sup\{\|f(\alpha)-g(\alpha)\|: \alpha\in\Delta_n\}.$$

Let ρ $(=\tilde{\rho})$ be the metric on $S_n(X)$ induced by ρ as above. To prove the theorem, it suffices to show that $B_n(X)$ is open in $Z_n(X)$, hence closed under this metric ρ.

To this end, let $u\in Z_n(X)$ and $v\in B_n(X)$ such that $\rho(u, v)<\min\{1, \delta/N\}$, where $N=\max\left\{\binom{n+1}{j}: 0\leqq j\leqq n+1\right\}$. By the remark following the definition of ρ $(=\tilde{\rho})$, we can write $u=\sum_{k=0}^m\lambda_k u_k$ and $v=\sum_{k=0}^m\lambda_k v_k$ where $u_k, v_k\in C(\Delta_n, X)$, $\lambda_k=\pm 1$ and $\rho(u, v)=\sum_{k=0}^m\rho(u_k, v_k)<\delta/N$. We will construct $\sigma\in S_{n+1}(U)$ so that $\partial\sigma=u-v$. Then $r_\#(\sigma)\in S_{n+1}(X)$ and $\partial(r_\#(\sigma))=r_\#(\partial\sigma)=r_\#(u-v)=u-v$ because $u-v\in S_n(X)$. Thus it follows $u=\partial(r_\#(\sigma))+v\in B_n(X)$.

Since $\partial u=\sum_{k=0}^m\sum_{i=0}^n(-1)^i\lambda_k u_k\varepsilon_i=0$, we have a partition P_u of the set $(m+1)\times(n+1)$ such that each member of P_u has exactly two elements and $\{(k, i), (k', i')\}\in P_u$ implies $(-1)^i\lambda_k u_k\varepsilon_i+(-1)^{i'}\lambda_{k'}u_{k'}\varepsilon_{i'}=0$, i.e., $(-1)^i\lambda_k+(-1)^{i'}\lambda_{k'}=0$ and $u_k\varepsilon_i=u_{k'}\varepsilon_{i'}$. (We identify a natural number n with the set $\{i: 0\leqq i\leqq n-1\}$ to simplify the notation.) We denote $(k, i)\sim_u(k', i')$ if $\{(k, i), (k', i')\}\in P_u$. Similarly, we have a relation $\sim_v$ on $(m+1)\times(n-1)$ such that $(k, i)\sim_v(k', i')$ implies $(-1)^i\lambda_k v_k\varepsilon_i+(-1)^{i'}\lambda_{k'}v_{k'}\varepsilon_{i'}=0$. Next, we extends $\sim_u$ and $\sim_v$ to relations on $(m+1)\times\{F: F\subset n+1 \text{ and } |F|=j\}$ for $0\leqq j\leqq n$ as follows: $(k, F)\sim_u(k', F')$ if there exist $i\in F$ and $i'\in F'$ such that $(k, i)\sim_u(k', i')$ and $g(F\setminus\{i\})=F'\setminus\{i'\}$ for the order preserving (o.p.) bijection $g: n+1\setminus\{i\}\to n+1\setminus\{i'\}$. For $F\subset n+1$, let $\varepsilon_F: \Delta_{n-|F|}\to\Delta_n$ be the linear map such that $\varepsilon_F(e_i)=e_{f(i)}$, where $f: n+1-|F|\to n+1\setminus F$ is the o.p. bijection. Since $(k, i)\sim_u(k', i')$ implies $u_k\varepsilon_i=u_{k'}\varepsilon_{i'}$, $(k, F)\sim_u(k', F')$ implies $u_k\varepsilon_F=u_{k'}\varepsilon_{F'}$. Since for each (k, i) there exists a unique (k', i') such that $(k, i)\sim_u(k', i')$, $|\{(k', F'): (k', F')\sim_u(k, F)\}|<|F|$. Extend $\sim_v$ similarly, which has the same properties as $\sim_u$. A *j-block* B is a nonempty minimal subset of $(m+1)\times\{F: F\subset n+1 \text{ and } |F|=j\}$ which satisfies the following: If $(k, F)\sim_u(k', F')$ or $(k, F)\sim_v(k', F')$ then $(k, F)\in B$ implies $(k', F')\in B$. In other words, B is an equivalence class of the equivalence relation generated by $\sim_u$ and $\sim_v$. For the convenience of notation, we regard $\{(k, \emptyset)\}$ as a 0-block for each k. For each block B, we choose a representative $(k(B), \mathcal{F}(B))\in B$. Note that every (k, F) belongs to only one j-block, where $j=|F|$.

LEMMA 2.3. *Suppose that* (k, F) *and* (k', F') *belong to the same* j*-block* B, $F\subset G\subset n+1$ *and* $G'=F'\cup g(G\setminus F)$ *for the o.p. injection* $g: n+1\setminus F\rightarrow n+1\setminus F'$. *Then* (k, G) *and* (k', G') *belong to the same* $|G|$*-block.*

PROOOF. The case $j=0$ is clear. Let $j>0$. Note that (k, F) and (k', F') are combined by a sequence of members of B such that adjoining members have the relations $\sim_u$ or $\sim_v$. We prove the lemma by induction on the length of this sequence. Suppose that the lemma holds for (k, F) and (k^*, F^*), i.e., (k, G) and (k^*, G^*) belong to the same $|G|$-block, where $G^*=F^*\cup g^*(G\setminus F)$ for the o.p. bijection $g^*: n+1\setminus F\rightarrow n+1\setminus F^*$. Now let $(k^*, F^*)\sim_x(k', F')$, where $x=u$ or v. By definition, there exist $i^*\in F^*$ and $i'\in F'$ such that $(k^*, i^*)\sim_x (k', i')$ and $F'=\{i'\}\cup f(F^*\setminus\{i^*\})$ for the o.p. bijection $f: n+1\setminus\{i^*\}\rightarrow n+1\setminus\{i'\}$. Observe $g=f\circ g^*$. Then it follows

$$\begin{aligned} G' &= F'\cup f\circ g^*(G\setminus F) \\ &= \{i'\}\cup f(F_*\setminus\{i^*\})\cup f(G^*\setminus F^*) \\ &= \{i'\}\cup f(G^*\setminus\{i^*\}). \end{aligned}$$

Therefore $(k^*, G^*)\sim_x(k', G')$. Thus (k, G) and (k', G') belong to the same block.

LEMMA 2.4. *If* (k, F) *and* (k', F') *belong to the same* j*-block* B, *then* $\rho(u_k\varepsilon_F, u_{k'}\varepsilon_{F'})<\delta$, $\rho(u_k\varepsilon_F, v_{k'}\varepsilon_{F'})<\delta$ *and* $\rho(v_k\varepsilon_F, v_{k'}\varepsilon_{F'})<\delta$.

PROOF. Observe the following facts: For each $0\leqq h\leqq m$, $|\{G:(h, G)\in B\}|\leqq\binom{n+1}{j}$; $(h, G)\sim_u(h', G')$ implies $u_h\varepsilon_G=u_{h'}\varepsilon_{G'}$; $(h, G)\sim_v(h', G')$ implies $v_h\varepsilon_G=v_{h'}\varepsilon_{G'}$; $\rho(u_h\varepsilon_G, v_h\varepsilon_G)\leqq\rho(u_h, v_h)$. Considering a shortest sequence combining (k, F) and (k', F') by $\sim_u$ and $\sim_v$, we obtain

$$\begin{aligned} \rho(u_k\varepsilon_F, u_{k'}\varepsilon_{F'}) &\leqq \binom{n+1}{j}\textstyle\sum_{h=0}^{m}\rho(u_h, v_h)\leqq\binom{n+1}{j}\rho(u, v) \\ &< \binom{n+1}{j}\delta/N\leqq\delta \end{aligned}$$

and the others similarly.

Let $\varepsilon_i^0=\varepsilon_i\times\mathrm{id}:\Delta_{m-1}\times\Delta_n\to\Delta_m\times\Delta_n$ $(0\leqq i\leqq m)$ and $\varepsilon_i^1=\mathrm{id}\times\varepsilon_i:\Delta_m\times\Delta_{n-1}\to\Delta_m\times\Delta_n$ $(0\leqq i\leqq n)$. For each $0\leqq j\leqq n$, the set of all injections $s:j\to n+1$ is denoted by $I(j, n+1)$. Here, we admit $\emptyset$ as a map from 0 and identify s with the j-tuple $(s_0, \cdots, s_{j-1})$. For each $s\in I(j, n+1)$ and $0\leqq p\leqq j$, let $f_s^p: n+1\setminus\{s_q: 0\leqq q<p\}\to n+1-p$ be the o.p. bijection.

For each k, we define $\tau_\emptyset^k:\Delta_n\times\Delta_1\to Y$ by:

$$\tau_\emptyset^k(\alpha, (1-\lambda)e_0+\lambda e_1)=(1-\lambda)\cdot v_k(\alpha)+\lambda\cdot u_k(\alpha).$$

For each $s\in I(j, n+1)$ $(j>0)$, we define inductively $\tau_s^k:\Delta_{n-j}\times\Delta_{j+1}\to Y$ as follows: $\tau_s^k(\alpha, \cdot)$ is linear for each $\alpha\in\Delta_{n-j}$ as a map from Δ_{j+1} to Y,

$$\tau_s^k(\alpha, e_0)=v_{k(B)}\circ\varepsilon_{\mathcal{F}(B)}(\alpha) \quad \text{and}$$

$$\tau_s^k(\alpha, e_{p+1})=\tau_{\underline{s}}^k(\varepsilon_{f_s^{j-1}(s_{j-1})}(\alpha), e_p) \quad \text{for } 0\leqq p\leqq j,$$

where $\underline{s}=(s_0, \cdots, s_{j-2})$ and B is the j-block which contains $(k, \mathrm{Im}\, s)$.

We represent τ_s^k a little bit more directly by using the notion of blocks. Let $B_s^{k,p}$ be the $(j-p)$-block which contains $(k, \{s_q: 0\leqq q<j-p\})$ and $F_s^{k,p}=\mathcal{F}(B_s^{k,p})\cup\{g(s_q): j-p\leqq q<j\}$, where $g: n+1\setminus\{s_q: 0\leqq q<j-p\}\to n+1\setminus\mathcal{F}(B_s^{k,p})$ is the o.p. bijection. Then, $B_s^{k,0}=B$ and $B_s^{k,p+1}=B_{\underline{s}}^{k,p}$ for $p\geqq 0$. Moreover, $F_s^{k,p+1}=F_{\underline{s}}^{k,p}\cup\{g(s_{j-1})\}$. Then by induction, we have the following.

LEMMA 2.5. *For each* $s\in I(j, n+1)$ *and* $0\leqq p\leqq j\leqq n$,

$$\tau_s^k(\alpha, e_p)=v_{k(B_s^{k,p})}\circ\varepsilon_{F_s^{k,p}}(\alpha) \ (\textit{especially } \tau_s^k(\alpha, e_j)=v_k\circ\varepsilon_{\mathrm{Im}\, s}(\alpha)) \ \textit{and}$$

$$\tau_s^k(\alpha, e_{j+1})=u_{k(B_s^{k,j})}\circ\varepsilon_{F_s^{k,j}}(\alpha)=u_k\circ\varepsilon_{\mathrm{Im}\, s}(\alpha).$$

By Lemmas 2.3, 2.4 and 2.5 we have

CORLLARY 2.6. *For each* $s\in I(j, n+1)$ $(0\leqq j\leqq n)$, $\mathrm{Im}\,\tau_s^k\subset U$.

Next we explicitly represent triangulations of products of simplexes. The order $\leqq$ on $(m+1)\times(n+1)$ is defined by: $(i, j)\leqq(i', j')$ if $i\leqq i'$ and $j\leqq j'$. We triangulate $\Delta_m\times\Delta_n$ so that vertices are (e_i, e_j) and simplexes are spanned by $(e_{i_0}, e_{j_0}), \cdots, (e_{i_p}, e_{j_p})$ with $(i_0, j_0)<\cdots<(i_p, j_p)$. By $O(m, n)$ we denote the set of all o.p. injections from $m+n+1$ to $(m+1)\times(n+1)$. For $t\in O(m, n)$, let $\mu_t:\Delta_{m+n}\to\Delta_m\times\Delta_n$ be the linear map defined by $\mu_t(e_i)=(e_{t_0(i)}, e_{t_1(i)})$, where $t(i)=(t_0(i), t_1(i))$.

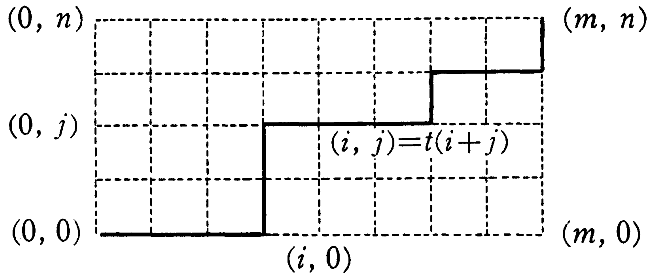
For $\tau:\Delta_m\times\Delta_n\to U$ $(m, n\geqq 0)$, let

$$C_{m,n}(\tau)=\sum_{t\in O(m,n)}(-1)^{c(t)}\tau\circ\mu_t\,,$$

$$\text{where } c(t)=\sum_{j=0}^{m+n-1}t_0(j)(t_1(j+1)-t_1(j))\,,$$

and especially $C_{-1,n}(\tau)=C_{m,-1}(\tau)=0$. The following figure helps us to prove the next lemma. In the figure, $t\in O(m, n)$ is written as a shortest path from $(0, 0)$ to (m, n) on the lattice.

(Figure)



The next lemma can be seen as a generalization of the so-called prism lemma.

LEMMA 2.7. *For* $\tau:\Delta_m\times\Delta_n\to U$ $(m, n\geqq 0)$,

$$\partial C_{m,n}(\tau)=\sum_{i=0}^{m}(-1)^{i+n}C_{m-1,n}(\tau\circ\varepsilon_i^0)+\sum_{i=0}^{n}(-1)^{i}C_{m,n-1}(\tau\circ\varepsilon_i^1)\,.$$

PROOF. First observe

$$\partial C_{m,n}(\tau)=\sum_{t\in O(m,n)}\sum_{k=0}^{m+n}(-1)^{c(t)+k}\tau\circ\mu_t\circ\varepsilon_k\,.$$

If $m=0$ or $n=0$, then $C_{m,n}(\tau)=\tau$ and hence we have the formula. Next, observe that $\tau\circ\mu_t\circ\varepsilon_k$ is cancelled in the above sum if $1\leqq k\leqq m+n-1$ and $(t_\nu(k)-t_\nu(k-1))(t_\nu(k+1)-t_\nu(k))=0$ for both $\nu=0, 1$, i.e., t is bent at the point $(t_0(k), t_1(k))$ in the above figure. Therefore,

$$\partial C_{m,n}(\tau)=\sum_{(t,k)\in P_0}(-1)^{c(t)+k}\tau\circ\mu_t\circ\varepsilon_k+\sum_{(t,k)\in P_1}(-1)^{c(t)+k}\tau\circ\mu_t\circ\varepsilon_k\,,$$

where

$$P_\nu=\{(t, k)\in O(m, n)\times(m+n+1):t_\nu(k-1)=t_\nu(k)=t_\nu(k+1) \text{ or}$$
$$k=0 \;\&\; t_\nu(0)=t_\nu(1) \text{ or } k=m+n \;\&\; t_\nu(m+n+1)=t_\nu(m+n)\}$$

for $\nu=0, 1$. For $(t, k)\in P_1$, let $i=t_0(k)$ and $j=t_1(k)$. Then $i+j=k$. Define $t^*\in O(m-1, n)$ by: $t^*(p)=t(p)$ for $p<k$, and $t^*(p)=(t_0(p+1)-1, t_1(p+1))$ for $p\geqq k$. It is a routine to check $c(t)-c(t^*)=n-j=n+i-k$, hence $(-1)^{c(t)+k}=(-1)^{n+i+c(t^*)}$. On the other hand, $\tau\circ\mu_t\circ\varepsilon_k=\tau\circ\varepsilon_i^0\circ\mu_{t^*}$. Therefore,

$$\sum_{(t,k)\in P_1}(-1)^{c(t)+k}\tau\circ\mu_t\circ\varepsilon_k=\sum_{i=0}^{m}\sum_{t*\in O(m-1,n)}(-1)^{i+n+c(t*)}\tau\circ\varepsilon_i^0\circ\mu_{t*}$$
$$=\sum_{i=0}^{m}(-1)^{i+n}C_{m-1,n}(\tau\circ\varepsilon_i^0).$$

For $(t, k)\in P_0$, set i and j as above. Define $t^*\in O(m, n-1)$ by: $t^*(p)=t(p)$ for $p<k$, and $t^*(p)=(t_0(p+1), t_1(p+1)-1)$ for $p\geqq k$. Then, $(-1)^{k+c(t)}=(-1)^{j+c(t^*)}$ and $\tau\circ\mu_t\circ\varepsilon_k=\tau\circ\varepsilon_j^1\circ\mu_{t*}$. Therefore,

$$\sum_{(t,k)\in P_0}(-1)^{c(t)+k}\tau\circ\mu_t\circ\varepsilon_k=\sum_{j=0}^{n}\sum_{t*\in O(m,n-1)}(-1)^{j+c(t*)}\tau\circ\varepsilon_j^1\circ\mu_{t*}$$
$$=\sum_{j=0}^{n}(-1)^{j}C_{m,n-1}(\tau\circ\varepsilon_j^1).$$

Now, we have shown the lemma.

For $s\in I(j, n+1)$, let $S_s=\prod_{p=1}^{j}(-1)^{p+f_s^{p-1}(s_{p-1})}$ and $S_s=1$ if $j=0$. Finally let $\sigma_j=\sum_{k=0}^{m}\lambda_k\sum_{s\in I(j,n+1)}S_s\cdot C_{n-j,j+1}(\tau_s^k)$ for $0\leqq j\leqq n$ and $\sigma=\sum_{j=0}^{n}(-1)^j\cdot\sigma_j$. Then, σ belongs to $S_{n+1}(U)$ by Corollary 2.6. We want to show $\partial\sigma=\sum_{j=0}^{n}(-1)^j\partial\sigma_j=u-v$. By Lemma 2.7,

$$\partial\sigma_j=\sum_{k=0}^{m}\lambda_k\sum_{s\in I(j,n+1)}S_s\cdot\partial C_{n-j,j+1}(\tau_s^k)$$
$$=\sum_{k=0}^{m}\lambda_k\sum_{s\in I(j,n+1)}S_s\cdot\sum_{i=0}^{n-j}(-1)^{i+j+1}C_{n-j-1,j+1}(\tau_s^k\circ\varepsilon_i^0)$$
$$+\sum_{k=0}^{m}\lambda_k\sum_{s\in I(j,n+1)}S_s\cdot C_{n-j,j}(\tau_s^k\circ\varepsilon_0^1)$$
$$+\sum_{k=0}^{m}\lambda_k\sum_{s\in I(j,n+1)}S_s\cdot\sum_{i=1}^{j-1}(-1)^i\cdot C_{n-j,j}(\tau_s^k\circ\varepsilon_i^1)$$
$$+\sum_{k=0}^{m}\lambda_k\sum_{s\in I(j,n+1)}S_s\cdot\sum_{i=j}^{j+1}(-1)^i\cdot C_{n-j,j}(\tau_s^k\circ\varepsilon_i^1),$$

especially,

$$\partial\sigma_0=\sum_{k=0}^{m}\lambda_k\sum_{i=0}^{n}(-1)^{i+1}C_{n-1,1}(\tau_\phi^k\circ\varepsilon_i^0)+\sum_{k=0}^{m}\lambda_k(u_k-v_k).$$

LEMMA 2.8. *For each* $j>0$

$$\sum_{k=0}^{m}\lambda_k\sum_{s\in I(j,n+1)}S_s\cdot(-1)^j\cdot C_{n-j,j}(\tau_s^k\circ\varepsilon_j^1)=0 \quad \textit{and}$$
$$\sum_{k=0}^{m}\lambda_k\sum_{s\in I(j,n+1)}S_s\cdot(-1)^{j+1}\cdot C_{n-j,j}(\tau_s^k\circ\varepsilon_{j+1}^1)=0.$$

PROOF. For each $s\in I(j, n+1)$ and $0\leqq k\leqq m$, $s'\in I(j, n+1)$ and $0\leqq k'\leqq m$ are uniquely determined so that $(k, s_0)\sim_u(k', s_0')$ and $g(s_p)=s_p'$ for $0<p<j$, where $g: n+1\setminus\{s_0\}\to n+1\setminus\{s_0'\}$ is the o.p. bijection. Observe $u_k\circ\varepsilon_{s_0}=u_{k'}\circ\varepsilon_{s_0'}$. It follows $u_k\circ\varepsilon_{\mathrm{Im}\,s}=u_{k'}\circ\varepsilon_{\mathrm{Im}\,s'}$. Since $(k, \{s_q: 0\leqq q<j-p\})\sim_u(k', \{s_q': 0\leqq q<j-p\})$ for $0\leqq p<j$, $B_s^{k,p}=B_{s'}^{k',p}$ and $F_s^{k,p}=F_{s'}^{k',p}$ for $0\leqq p<j$. By Lemma 2.5, $\tau_s^k(\alpha, e_p)=\tau_{s'}^{k'}(\alpha, e_p)$ for $0\leqq p<j$ and $\tau_s^k(\alpha, e_{j+1})=u_k\circ\varepsilon_{\mathrm{Im}\,s}(\alpha)=u_{k'}\circ\varepsilon_{\mathrm{Im}\,s'}(\alpha)=\tau_{s'}^{k'}(\alpha, e_{j+1})$. Thus we have $\tau_s^k\circ\varepsilon_j^1=\tau_{s'}^{k'}\circ\varepsilon_j^1$. By definition of $\sim_u$, $\lambda_k(-1)^{s_0}+\lambda_{k'}(-1)^{s_0'}=0$. Since $f_s^0(s_0)=s_0$, $f_{s'}^0(s_0')=s_0'$ and $f_s^{p-1}(s_{p-1})=f_{s'}^{p-1}(s_{p-1}')$ for $2\leqq p<j$, we have $\lambda_kS_s+\lambda_{k'}S_{s'}=0$. Thus we get the first equation. By replacing $\sim_u$ by $\sim_v$, we have $\tau_s^k\circ\varepsilon_{j+1}^1$

$=\tau_{s'}^{k'}\circ\varepsilon_{j+1}^{1}$ and $\lambda_k S_s+\lambda_{k'}S_{s'}=0$, which imply the second equation.

LEMMA 2.9. *The following equation holds for $j>0$ and $0\leqq k\leqq m$:*

$$\sum_{s\in I(j,\,n+1)}S_s\cdot\sum_{p=1}^{j-1}(-1)^p C_{n-j,\,j}(\tau_s^k\circ\varepsilon_p^1)=0\,.$$

PROOF. Let $0<p<j$. For each $s\in I(j,\,n+1)$, $\bar{s}\in I(j,\,n+1)$ is uniquely determined so that $\bar{s}_q=s_q$ for $q<p-1$ or $q>p$, $\bar{s}_{p-1}=s_p$ and $\bar{s}_p=s_{p-1}$. Then, $B_s^{k,q}=B_{\bar{s}}^{k,q}$ and $F_s^{k,q}=F_{\bar{s}}^{k,q}$ for $q\neq p$, hence $\tau_s^k\circ\varepsilon_p^1=\tau_{\bar{s}}^k\circ\varepsilon_p^1$. Observe $f_s^q(s_q)=f_{\bar{s}}^q(\bar{s}_q)$ for $q<p-1$ or $q>p$. If $s_{p-1}<s_p$, then $f_s^{p-1}(s_{p-1})=f_{\bar{s}}^p(\bar{s}_p)$ and $f_s^p(s_p)+1=f_{\bar{s}}^{\bar{s}}(\bar{s}_p)$. Otherwise, i.e. $s_p<s_{p-1}$, $f_{\bar{s}}^{p-1}(\bar{s}_{p-1})=f_s^p(s_p)$ and $f_{\bar{s}}^p(\bar{s}_p)+1=f_s^p(s_p)$. In any case, $S_s+S_{\bar{s}}=0$, which implies the lemma.

LEMMA 2.10. *For $0\leqq j<n$,*

$$\sum_{s\in I(j,\,n+1)}S_s\cdot\sum_{i=0}^{n-j}(-1)^{i+j+1}C_{n-j-1,\,j+1}(\tau_s^k\circ\varepsilon_i^0)$$
$$=\sum_{s\in I(j+1,\,n+1)}S_s\cdot C_{n-j-1,\,j+1}(\tau_s^k\circ\varepsilon_0^1)\,.$$

PROOF. For each $s\in I(j,\,n+1)$ and $i\leqq n-j$, $s_*\in I(j+1,\,n+1)$ corresponds uniquely to the pair $(s,\,i)$ so that $s_*(p)=s(p)$ for $p<j$ and $s_*(j)=g(i)$, where $g:n+1-j\to n+1\setminus\{s_p:0\leqq p<j\}$ is the o.p. bijection. Then $f_{s*}^j=g^{-1}$ and $s_{*j}=g(i)$. By the definition of τ_{s*}^k, $\tau_{s*}^k\circ\varepsilon_0^1(\alpha,\,e_p)=\tau_s^k\circ\varepsilon^0_{f^i_{s*}(s_{*j})}(\alpha,\,e_p)=\tau_s^k\circ\varepsilon_i^0(\alpha,\,e_p)$ for each $0\leqq p\leqq j$. Since $f_{s*}^{p-1}(s_{*p-1})=f_s^{p-1}(s_{p-1})$ for $1\leqq p\leqq j$ and $f_{s*}^j(s_{*j})=i$, $S_{s*}=(-1)^{i+j+1}S_s$. Therefore,

$$(-1)^{i+j+1}S_s\cdot C_{n-j-1,\,j+1}(\tau_s^k\circ\varepsilon_i^0)=S_{s*}\cdot C_{n-j-1,\,j+1}(\tau_{s*}^k\circ\varepsilon_0^1)\qquad\text{and}$$
$$(-1)^{j+1}\sum_{s\in I(j,\,n+1)}S_s\cdot\sum_{i=0}^{n-j}(-1)^i C_{n-j-1,\,j+1}(\tau_s^k\circ\varepsilon_i^0)$$
$$=\sum_{s\in I(j+1,\,n+1)}S_s\cdot C_{n-j-1,\,j+1}(\tau_s^k\circ\varepsilon_0^1)\,.$$

Recalling the formula before Lemma 2.8, we have

$$\begin{aligned}\partial\sigma&=\sum_{j=0}^{n}(-1)^j\partial\sigma_j\\&=\sum_{k=0}^{m}\lambda_k(u_k-v_k)\\&\quad+(-1)^n\sum_{k=0}^{m}\lambda_k\sum_{s\in I(n,\,n+1)}S_s\cdot\sum_{i=0}^{0}(-1)^{i+n+1}C_{-1,\,n+1}(\tau_s^k\circ\varepsilon_i^0)\\&=\sum_{k=0}^{m}\lambda_k(u_k-v_k)=u-v\,.\end{aligned}$$

by Lemmas 2.8, 2.9 and 2.10.

Now, we have completed the proof of Theorem 2.1.

3. Broken path connectedness and quasi-homotopy

Let $x, y \in X$. We write $x \underset{p}{\sim} y$ if x and y are connected by a path, i.e., there is a continuous map $f: \boldsymbol{I} \to X$ such that $f(0)=x$ and $f(1)=y$. A finite sequence of continuous maps $f_i: \boldsymbol{I} \to X$ $(i=0, \cdots, n)$ are called a *broken path*. And we say that $(f_i)_{i=0}^n$ *connects* x to y if $f_0(0)=x$ and $f_n(1)=y$. For $\rho \in \mathrm{Pm}(X)$, the *ρ-gap* of a broken path $(f_i)_{i=0}^n$ in X is $\sum_{i=0}^{n-1} \rho(f_i(1), f_{i+1}(0))$. In case $n=0$, the ρ-gap is 0. We write $x \underset{b}{\sim} y$ if for any $\rho \in \mathrm{Pm}(X)$ and $\varepsilon > 0$, x and y are connected by a broken path with the ρ-gap less than ε, i.e., there are $x_i, y_i \in X$ $(i=0, \cdots, n)$ such that $x_0=x$, $y_n=y$, $x_i \underset{p}{\sim} y_i$ for each $i=0, \cdots, n$ and $\sum_{i=0}^{n-1} \rho(y_i, x_{i+1}) < \varepsilon$. Then $\underset{b}{\sim}$ is an equivalence relation on X. It is said that X is *broken path connected* if $x \underset{b}{\sim} y$ for any $x, y \in X$.

PROPOSITION 3.1. *Let $x, y \in X$, then $x \underset{b}{\sim} y$ iff $f(x)=f(y)$ for any continuous map $f: X \to \boldsymbol{R}$ which is constant on each path component.*

PROOF. For an arbitrary continuous map $f: X \to \boldsymbol{R}$ which is constant on each path component, define $\rho \in \mathrm{Pm}(X)$ by: $\rho(u, v)=|f(u)-f(v)|$. Since $u \underset{p}{\sim} v$ implies $\rho(u, v)=0$, $x \underset{b}{\sim} y$ implies $\rho(x, y)=0$, that is, $f(x)=f(y)$. To see the converse implication, for any $\rho \in \mathrm{Pm}(X)$ we define $f: X \to \boldsymbol{R}$ by:

$$f(u)=\inf\{\textstyle\sum_{i=0}^{n-1} \rho(u_i, x_{i+1}) : x_i \underset{p}{\sim} u_i \ (0 \leqq i \leqq n),\ x_0=x,\ u_n=u\}.$$

Then, f is continuous and $f(u)=f(v)$ if $u \underset{p}{\sim} v$. From the assumption, $f(y)=f(x)=0$, which implies $x \underset{b}{\sim} y$.

Let $X=\{(x, y) \mid y=0 \text{ or } y=x^{-1}\} \subset \boldsymbol{R}^2$ and ρ be the Euclidean metric on X. Then each pair of points of X are connected by broken paths with arbitrarily small ρ-gaps. However X is not broken path connected because X is homeomorphic to three parallel straight lines in the plane. The following is such an example in case X is compact.

EXAMPLE 3.2. Concerning Cantor's ternary set, the following is well-known. There exist families $\{I_i : i \in \boldsymbol{N}\}$ and $\{J_i : i \in \boldsymbol{N}\}$ of pairwise disjoint open subintervals of / such that $\mu(\bigcup_{i \in N} I_i)=1$, $\mu(\bigcup_{i \in N} J_i)=1/2$ and both $\boldsymbol{I} \setminus \bigcup_{i \in N} I_i$ and $\boldsymbol{I} \setminus \bigcup_{i \in N} J_i$ are nowhere-dense perfect sets, where μ is the Lebesque meaure. Then there exists a homeomorphism $h: \boldsymbol{I} \to \boldsymbol{I}$ such that $h(\bigcup_{i \in N} I_i)=\bigcup_{i \in N} J_i$.

Let

$$W=\{(x,\ y):\ y=\sin(\pi/x),\ 0<x\leqq 1/2\}\cup$$
$$\{(x,\ -y):\ y=\sin(\pi/(1-x)),\ 1/2<x<1\}.$$

For each $i\in\boldsymbol{N}$, let $h_i:(0,\ 1)\to I_i$ be a homeomorphism and $W_i=(h_i\times\mathrm{id})(W)$. We define $X=(\boldsymbol{I}\setminus\bigcup_{i\in N}I_i)\times[-1,\ 1]\cup\bigcup_{i\in N}W_i$. Then, X is a plane compactum. Any path in X is included in $\{\alpha\}\times[-1,\ 1]$ for some $\alpha\in\boldsymbol{I}$ or W_i for some $i\in\boldsymbol{N}$. Since $\mu(\bigcup_{i\in N}I_i)=1$, the infimum of ρ-gaps of broken paths connecting $(0,\ 0)$ to $(1,\ 0)$ in X is equal to 0, where ρ is the Euclidean metric on the plane. On the other hand, the infimum of ρ-gaps of broken paths connecting $(0,\ 1)$ to $(1,\ 0)$ in $(h\times\mathrm{id})(X)$ is $1/2$ by the same reason. Therefore, X is not broken path connected.

Any path connected space is obviously broken path connected but the converse does not hold. The example has been given in Example 3.2, i.e., $W\cup\{0,\ 1\}\times[-1,\ 1]$ is broken path connected but not path connected. Any broken path connected space is connected but the converse does not hold. In fact, the space in Example 3.2 is such an example. The pseudo-arc $\boldsymbol{P}$ is also such a continuum since it has no nontrivial paths.

One should remark that each equivalence class of $\underset{b}{\sim}$ is closed in X which contains a path component but it need not be connected. For example, let

$$X=\bigcup_{n=1}^{\infty}\{1/n\}\times\boldsymbol{I}\cup\{(0,\ 0),\ (0,\ 1)\}.$$

Then $A=\{(0,\ 0),\ (0,\ 1)\}$ is an equivalence class of $\underset{b}{\sim}$. In case X is compact, it is connected as shown in the next proposition. But even if X is compact metric, it need not be broken path connected. For example, let $\boldsymbol{P}$ be a pseudo-arc in the plane and let A_n $(n\in\boldsymbol{N})$ be a sequence of arcs which converges to $\boldsymbol{P}$ in the hyperspace. Define $X=\bigcup_{n=1}^{\infty}\{1/n\}\times A_n\cup\{0\}\times\boldsymbol{P}\subset\boldsymbol{R}^3$. Then, X is compact metrizable and $\{0\}\times\boldsymbol{P}$ is an equivalence class of $\underset{b}{\sim}$.

PROPOSITION 3.3. *If X is compact, then each equivalence class of $\underset{b}{\sim}$ is connected.*

PROOF. For each $\rho\in\mathrm{Pm}(X)$, take a broken path from x to y with the ρ-gap less than 1 and let K_ρ be the union of the images of its paths. For $\rho,\ \rho'\in\mathrm{Pm}(X)$, define $\rho\leqq\rho'$ by: $\rho(x,\ y)\leqq\rho'(x,\ y)$ for all $x,\ y\in X$. Then $(\mathrm{Pm}(X),\ \leqq)$ is a directed set. Since the hyperspace 2^X with the Vietoris topology is compact, the net $(K_\rho:\rho\in\mathrm{Pm}(X))$ has a cluster point $K\in 2^X$. Clearly, K contains both x and y. Suppose that K is a disjoint union of nonempty closed subsets A and B. Take $\rho\in\mathrm{Pm}(X)$ so that $\rho(a, b)>3$ for each $a\in A$ and $b\in B$. Let $U=\{x\in X: \rho(x, A)<1\}$ and $V=\{x\in X: \rho(x, B)<1\}$. Then there exists $\rho'\in\mathrm{Pm}(X)$ such that $\rho\leqq\rho'$, $K_{\rho'}\subset U\cup V$, $K_{\rho'}\cap U\neq\emptyset$ and $K_{\rho'}\cap V\neq\emptyset$. This contradicts to the fact that the ρ'-gap of the broken path is less than 1.

Now, we prove the following.

THEOREM 3.4. *For any space X, $H_0^T(X)$ is canonically isomorphic to the free abelian group whose rank is equal to the cardinality of equivalence classes with respect to* $\underset{b}{\sim}$.

COROLLARY 3.5. *A space X is broken path connected if and only if $H_0^T(X)$ $=\boldsymbol{Z}$.*

To show the theorem we must recall the topology of the free abelian topological group $A(X)$, because $Z_0(X)=S_0(X)$ is identical with $A(X)$. Any $\rho\in\mathrm{Pm}(X)$ can be extended to $\tilde{\rho}\in\mathrm{Pm}(A(X))$ by the same way as metrics [12] (cf. Section 2). We abuse ρ with $\tilde{\rho}$ as before. It is known that the topology of the free abelian topological group $A(X)$ is determined by all such pseudo-metrics ρ [20, p. 379 or 24, Theorem 1]. Here, we outline the proof. By [16], the topology of $A(X)$ is determined by all invariant continuous pseudo-metrics τ on $A(X)$. Let $\rho\in\mathrm{Pm}(A(X))$ be the extension of $\tau|X$ in the above manner. Since $\tau(a+b, c+d)\leqq\tau(a, c)+\tau(b, d)$ by the invariantness of τ,

$$\{a\in A(X): \rho(a, 0)<\varepsilon\}\subset\{a\in A(X): \tau(a, 0)<\varepsilon\} \qquad \text{for } \varepsilon>0\,.$$

Thus we get the conclusion. The theorem is an immediate consequence of the next lemma, where $C(\Delta_0, X)$ is naturally identified with X.

LEMMA 3.6. $\overline{B_0(X)}=\overline{\langle x-y: x\underset{p}{\sim}y\rangle}=\langle x-y: x\underset{b}{\sim}y\rangle$ *in* $A(X)$.

PROOF. Suppose that $x\underset{b}{\sim}y$. For each $\rho\in\mathrm{Pm}(X)$ and $\varepsilon>0$, we have $x_i, y_i\in X$ $(0\leqq i\leqq n)$ such that $x_i\underset{p}{\sim}y_i$, $x_0=x$, $y_n=y$ and $\sum_{i=0}^{n-1}\rho(y_i, x_{i+1})<\varepsilon$. Then, $\sum_{i=0}^{n}(x_i-y_i)\in B_0$ and

$$\begin{aligned}\rho(x-y, \textstyle\sum_{i=0}^{n}(x_i-y_i))&=\rho(0, \textstyle\sum_{i=0}^{n}x_i-\sum_{i=0}^{n}y_i-x+y)\\ &=\rho(0, \textstyle\sum_{i=1}^{n}x_i-\sum_{i=0}^{n-1}y_i)\\ &\leqq\textstyle\sum_{i=0}^{n-1}\rho(y_i, x_{i+1})<\varepsilon\,.\end{aligned}$$

Hence $x-y\in\overline{B_0}$.

To see the other inclusion, we introduce some notion and notation. For each $u\in X\cup(-X)$, $|u|=u$ if $u\in X$ and $|u|=-u$ if $u\in -X$. We identify $-(-x)$ with x for $x\in X$. A reduced form of a non-zero $u\in A(X)$ is $u=\sum_{i=1}^{n}u_i$, where $u_i\in X\cup(-X)$ $(1\leqq i\leqq m)$ and $u_i\neq -u_j$ for any i, j. Let $u\in\overline{B_0}$ $(u\neq 0)$ and $u=\sum_{i=1}^{m}u_i$ be a reduced form of u. We claim that there exist $u_i\in X$ and $u_j\in -X$ such that $u_i\underset{b}{\sim}-u_j$. Suppose the contrary. Then there exist $\rho\in\mathrm{Pm}(X)$ and $0<\varepsilon<1$ with the following: $\rho(u, 0)>\varepsilon$; $u_i\in X$ and $u_j\in -X$ imply that the ρ-gap of any broken path connecting u_i to $-u_j$ is greater than ε. Since $u\in\overline{B_0}$, there exist x_i, $y_i\in X$ $(0\leqq i\leqq n)$ such that $x_i\underset{p}{\sim}y_i$ $(0\leqq i\leqq n)$ and $\rho(u, \sum_{i=0}^{n}(x_i-y_i))$ $<\varepsilon$. We may assume that $\sum_{i=0}^{n}(x_i-y_i)$ is a reduced form. Since $u\neq 0$ and $\rho(u, \sum_{i=0}^{n}(x_i-y_i))<1$, there exist a_i, $b_i\in X\cup(-X)$ $(1\leqq i\leqq k)$ such that $u=\sum_{i=1}^{k}a_i$, $\sum_{i=0}^{n}(x_i-y_i)=\sum_{i=1}^{k}b_i$ and $\rho(u, \sum_{i=0}^{n}(x_i-y_i))=\sum_{i=1}^{k}\rho(a_i, b_i)$ [12]. Since $\sum_{i=0}^{n}(x_i-y_i)=\sum_{i=1}^{k}b_i$, k is even. Since $\sum_{i=1}^{m}u_i=\sum_{i=1}^{k}a_i$, m is also even. We may assume that $u_1\in X$, $a_i=u_i$ for $i=1, \cdots, m$ and moreover $a_{m+2i-1}=-a_{m+2i}\in X$ for $i=1, \cdots, (k-m)/2$. We can choose a partion P of $\{1, \cdots, k\}$ such that each element of P has exactly two elements and $\{i, j\}\in P$ implies that $b_i\in X$ iff $b_j\in -X$, and $|b_i|\underset{p}{\sim}|b_j|$. (Note that $b_i=-b_j$ implies $|b_i|\underset{p}{\sim}|b_j|$.) We get a sequence

$$a_{i_1},\ b_{i_1},\ b_{i_2},\ a_{i_2},\ a_{i_3},\ b_{i_3},\ \cdots,\ b_{i_{2h}},\ a_{i_{2h}}$$

such that $i_1=1$, $i_{2h}\leqq m$, $\{i_{2j-1}, i_{2j}\}\in P$ for $j=1, \cdots, h$ and $i_{2j+1}=i_{2j}-1>m$ for $j=1, \cdots, h-1$. Then $b_{i_{2j-1}}\in X$, $b_{i_{2i-1}}\underset{p}{\sim}-b_{i_{2j}}$ for $j=1, \cdots, h$ and $a_{i_{2j+1}}=-a_{i_{2j}}$ $\in X$ for $j=1, \cdots, h-1$. Then,

$$\begin{aligned}
&\rho(|a_{i_1}|, |b_{i_1}|)+\textstyle\sum_{j=1}^{h-1}\rho(|b_{i_{2j}}|, |b_{i_{2j+1}}|)+\rho(|b_{i_{2h}}|, |a_{i_{2h}}|)\\
&\leqq\rho(|a_{i_1}|, |b_{i_1}|)+\textstyle\sum_{j=1}^{h-1}(\rho(|b_{i_{2j}}|, |a_{i_{2j}}|)+\rho(|a_{i_{2j+1}}|, |b_{i_{2j+1}}|))\\
&\qquad\qquad+\rho(|b_{i_{2h}}|, |a_{i_{2h}}|)\\
&=\textstyle\sum_{j=1}^{2h}\rho(|a_{i_j}|, |b_{i_j}|)\\
&\leqq\textstyle\sum_{i=1}^{k}\rho(|a_i|, |b_i|)<\varepsilon,
\end{aligned}$$

that is, we have a broken path connecting $u_1=a_{i_1}$ to $u_{i_{2h}}=a_{i_{2h}}$ with the ρ-gap less than ε. This contradicts to the hypothesis. Now, we have shown the claim, from which lemma follows by induction.

Next we define a notion "quasi-homotopy". For f, $g\in C(X, Y)$, we write $f\underset{h}{\sim}g$ if f and g are homotopic. In case X is locally compact, as is well-known

$f \underset{h}{\sim} g$ iff $f \underset{p}{\sim} g$, where $C(X, Y)$ is endowed with compact-open topology. We say f and g are *quasi-homotopic* (denoted by $f \underset{q}{\sim} g$) if the statement obtained by replacing $\underset{p}{\sim}$ and X by $\underset{h}{\sim}$ and $C(X, Y)$ in the definition of broken path connectedness. More precisely, $f \underset{q}{\sim} g$ if for any $\rho \in \mathrm{Pm}(C(X, Y))$ and $\varepsilon > 0$ there exist f_i, g_i $(0 \leqq i \leqq n)$ such that $f_0 = f$, $g_n = g$, $f_i \underset{h}{\sim} g_i$ and $\sum_{i=0}^{n-1} \rho(g_i, f_{i+1}) < \varepsilon$. In case X is locally compact, $f \underset{q}{\sim} g$ iff $f \underset{b}{\sim} g$.

Analogously to Proposition 3.1 and Lemma 3.5, we get

PROPOSITION 3.7. *Let $C(X, Y)$ be endowed with compact-open topology and $f, g \in C(X, Y)$. Then, $f \underset{q}{\sim} g$ iff $F(f) = F(g)$ for any continuous map $F: C(X, Y) \to \boldsymbol{R}$ such that $F(u) = F(v)$ when $u \underset{h}{\sim} v$.*

PROPOSITION 3.8. *Let $\mathcal{H}(X, Y) = \langle f - g : f \underset{h}{\sim} g,\ f, g \in C(X, Y) \rangle$ be the subgroup of the free abelian topological group $A(C(X, Y))$ over $C(X, Y)$. Then, $f \underset{q}{\sim} g$ iff $f - g \in \overline{\mathcal{H}(X, Y)}$.*

THEOREM 3.9. *For $f, g \in C(X, Y)$, if f and g are quasi-homotopic, then $f_*^T = g_*^T$ holds.*

PROOF. Let $\rho \in \mathrm{Pm}(C(\Delta_n, Y))$. For $z \in Z_n(X)$, define $\rho' \in \mathrm{Pm}(C(X, Y))$ by: $\rho'(f, g) = \rho(f_\# z, g_\# z)$. For each $\varepsilon > 0$, there exist $f_i, g_i \in C(X, Y)$ $(0 \leqq i \leqq m)$ such that $f_0 = f$, $g_m = g$, $f_i \underset{h}{\sim} g_i$ and $\sum_{i=0}^{n-1} \rho'(f_{i+1}, g_i) < \varepsilon$, which implies

$$\rho(f_\# z - g_\# z, \textstyle\sum_{i=0}^{n}(f_{i\#} z - g_{i\#} z)) \leqq \sum_{i=0}^{n-1} \rho(f_{i+1\#} z, g_{i\#} z)$$
$$= \textstyle\sum_{i=1}^{n-1} \rho'(f_{i+1}, g_i) < \varepsilon\,.$$

Hence $f_\# z - g_\# z \in \overline{B_n(Y)}$.

COROLLARY 3.10. *If there exist continuous maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \underset{q}{\sim} \mathrm{id}_X$ and $f \circ g \underset{q}{\sim} \mathrm{id}_Y$, then $H_n^T(X) \cong H_n^T(Y)$.*

Here we give some maps which are quasi-homotopic.

EXAMPLE 3.11. (1) Let

$$X = \{(x, y, 0) : (x-1)^2 + y^2 = 1\} \cup \{(x, y, 1/m) : (x-1)^2 + y^2 \leqq 1,\ n \in \boldsymbol{N}\}$$
$$\cup \{(0, 0, z) : 0 \leqq z \leqq 1\}$$

and define $f_m: X \to X$ by: $$f_m(x, y, z)=\begin{cases}(x, y, z) & \text{if } z \geqq 1/m, \\ (x, y, 1/m) & \text{otherwise.}\end{cases}$$

Then, $\mathrm{id} \underset{q}{\sim} f_1$, since any neighborhood of id_X contains some f_m and $f_m \underset{h}{\sim} f_1$. Consequently $H_n^T(X)=0$.

(2) Let

$$X=\{(x, y, z):(x-\cos 2\pi\theta)^2+y^2+(z-\sin 2\pi\theta)^2 \leqq 1: \theta \in \boldsymbol{Q}\}$$
$$\cup\{(x, y, z):(x-\cos 2\pi\theta)^2+y^2+(z-\sin 2\pi\theta)^2=1: \theta+\sqrt{2} \in \boldsymbol{Q}\}.$$

Then, similarly as the previous example, id_X is quasi-homotopic to a constant map and consequently $H_n^T(X)=0$.

4. $H_n^T(X)$ for products and sums

Let X_i $(i \in I)$ be spaces with base points a_i. For an element u of the direct product $\prod_{i \in I} X_i$, the support of u is the set $\operatorname{supp} u=\{i \in I: u(i) \neq a_i\}$. The *$\Sigma$-product* $\tilde{\prod}_{i \in I} X_i$ denotes the subspaces of $\prod_{i \in I} X_i$ consisting of all u with countable supports. Let $\tilde{\bigvee}_{i \in I} X_i=\{u \in \prod_{i \in I} X_i: \operatorname{supp} u$ is at most one$\}$ denote the subspace of $\prod_{i \in I} X_i$ with base point $a=(a_i)_{i \in I}$. By $\bigvee_{i \in I} X_i$, we denote the quotient space of the discrete sum of X_i by identifying all a_i's, where the identified point a is the base point. These spaces $\tilde{\bigvee}_{i \in I} X_i$ and $\bigvee_{i \in I} X_i$ can be regarded as spaces with the same underlying set. In case each X_i includes a copy A_i of A, $\bigvee_{i \in I}(X_i, A_i)$ is the quotient space of the discrete sum of X_i by identifying all A_i's, which generalizes the one point case.

Corresponding to the above, for groups C_i $(I \in I)$, $\tilde{\prod}_{i \in I} C_i$ denotes the subgroup of the direct product $\prod_{i \in I} C_i$ consisting of all u with countable supports, i.e., $\operatorname{supp} u=\{i \in I: u(i) \neq 0\}$ is countable. The *direct sum* of C_i is denoted by $\bigoplus_{i \in I} C_i$, i.e., $\bigoplus_{i \in I} C_i=\{u \in \prod_{i \in I} C_i: \operatorname{supp} u$ is finite$\}$. In case $C_i \cong C$ for all $i \in I$, $\prod_{i \in I} C_i$, $\tilde{\prod}_{i \in I} C_i$ and $\bigoplus_{i \in I} C_i$ are abbreviated by C^I, $\tilde{\prod}_I C$ and $\bigoplus_I C$ respectively.

In general H_1 does not commute with direct products for path connected spaces, but H_1^T does, that is,

PROPOSITION 4.1. *Let X_i $(i \in I)$ be $(n-1)$-connected spaces. Then $H_n^T(\prod_{i \in I} X_i) \cong \prod_{i \in I} H_n^T(X_i)$ and $H_n^T(\tilde{\prod}_{i \in I} X_i) \cong \tilde{\prod}_{i \in I} H_n^T(X_i)$ canonically for $n \geqq 1$.*

We need the following lemma.

LEMMA 4.2. *Let X and Y be $(n-1)$-connected spaces, $z \in Z_n(X \times Y)$ and $p_X: X \times Y \to X$, $p_Y: X \times Y \to Y$ be the projections. If $p_{X\#} z \in \overline{B_n(X)}$ and $p_{Y\#} z \in$*

$\overline{B_n(Y)}$, *then* $z\in\overline{B_n(X\times Y)}$.

PROOF. Note that $X\times Y$ is $(n-1)$-connected. By Hurewicz's theorem [23, Ch. 7, §5], we may assume the following: In case n is odd, $z\in C(\Delta_n, X\times Y)$ and $\operatorname{Im} z\circ\varepsilon_i=\{(x_0, y_0)\}$ for $0\leqq i\leqq n$; In case n is even, $z=u-v$ for $u, v\in C(\Delta_n, X\times Y)$ and $\operatorname{Im} u\circ\varepsilon_i=\operatorname{Im} v=\{(x_0, y_0)\}$ for $0\leqq i\leqq n$.

Since the proofs are not so different, we prove only in case n is odd. Define $z_X(\alpha)=(p_X\circ z(\alpha), y_0)$, $z_Y(\alpha)=(x_0, p_Y\circ z(\alpha))$ for $\alpha\in\Delta_n$. Then, $z-(z_X+z_Y)\in B_n(X\times Y)$. Hence, it suffices to show $z_X+z_Y\in\overline{B_n(X\times Y)}$. By the assumption, $z_X\in\overline{B_n(X\times\{y_0\})}$ and $z_Y\in\overline{B_n(\{x_0\}\times Y)}$, where the closures are taken in $S_n(X\times\{y_0\})$ and $S_n(\{x_0\}\times Y)$ respectively. Since $C(\Delta_n, X\times\{y_0\})$ and $C(\Delta_n, \{x_0\}\times Y)$ are retracts of $C(\Delta_n, X\times Y)$, the closures can be regarded as the ones in $S_n(X\times Y)$. Hence $z_X+z_Y\in\overline{B_n(X\times Y)}$.

PROOF OF PROPOSITION 4.1. Let $p_i:\prod_{i\in I}X_i\to X_i$ be the projections and define $\varphi: H_n^T(\prod_{i\in I}X_i)\to\prod_{i\in I}H_n^T(X_i)$ by $\varphi(a)(i)=p_{i*}^T(a)$. By using Hurewicz's theorem, it is easy to see that φ is an epimorphism. It suffices to show $\operatorname{Ker}\varphi=\bigcap_{i\in I}\operatorname{Ker} p_{i*}^T=0$.

Let $z=\sum_{k=0}^m\lambda_k u_k\in Z_n$ ($\lambda_k=\pm1$, $u_k\in C(\Delta_n, \prod_{i\in I}X_i)$) such that $z+\overline{B_n}\in\operatorname{Ker}\varphi$, i.e., $p_{i\#}(z)\in\overline{B_n(X_i)}$ for each $i\in I$. For any open neighborhood U of z in $S_n(\prod_{i\in I}X_i)$, there exist compact subsets K_k^j of Δ_n and basic open sets U_k^j of $\prod_{i\in I}X_i$ $(0\leqq j\leqq r)$ such that $u_k\in\bigcap_{j=0}^r O(K_k^j, U_k^j)$ and $\sum_{k=0}^m\lambda_k(\bigcap_{j=0}^r O(K_k^j, U_k^j))\subset U$, where $O(K, U)=\{u\in C(\Delta_n, \prod_{i\in I}X_i): u(K)\subset U\}$. There exists a finite subset F of I such that U_k^j's only depend on F. Define $u_k'\in C(\Delta_n, \prod_{i\in F}X_i)$ by: $p_{i\#}(u_k')=p_{i\#}(u_k)$ for $i\in F$. Since each X_i is $(n-1)$-connected, $\sum_{k=0}^m\lambda_k u_k'\in\overline{B_n(\prod_{i\in F}X_i)}$ by Lemma 4.2. Pick an element $x_i\in X_i$ for each $i\in I$ and define $u_k''\in C(\Delta_n, \prod_{i\in I}X_i)$ as follows: $u_k''(i)=u_k'(i)$ for $i\in F$ and $u_k''(i)=x_i$ for $i\notin F$. Then, $u_k''\in\bigcap_{j=0}^r O(K_k^j, U_k^j)$ hence $\sum_{k=0}^m\lambda_k u_k''\in U\cap B_n$. Therefore $z\in\overline{B_n}$ and $\operatorname{Ker}\varphi=0$. By separability of Δ_n $\varphi(H_n^T(\tilde{\prod}_{i\in I}X_i))\subset\tilde{\prod}_{i\in I}H_n^T(X_i)$. Therefore, the proof for Σ-products can be done similarly.

To calculate $H_n^T(X)$ for attaching spaces X, we introduce properties $(*_n)$ and $(**)$ for (X, A). These properties are necessary only for $n\geqq2$.

$(*_n)$ If $\sum_{k=0}^m\lambda_k u_k\in Z_n$ ($\lambda_k\in\mathbf{Z}$, $u_k\in C(\Delta_n, X)$) and U_k is an open neighborhood of u_k, then there exist $v_k\in U_k$ $(0\leqq k\leqq m)$ and $Y\subset X$ such that $A\subset Y$, $\operatorname{Im} v_k\subset Y$, $\sum_{k=0}^m\lambda_k v_k\in Z_n$ and some neighborhood of A in Y deforms into A in Y.

(**) For any open neighborhood O of $\mathrm{id}\in C(X, X)$ with the compact-open topology, there exist $Y\subset X$ and $f\in O$, such that $A\subset Y$, $f(X)\subset Y$, $f|A=\mathrm{id}$ and some neighborhood of A in Y deforms into A in Y.

It can be clearly seen that (**) implies $(*_n)$ for every n.

LEMMA 4.3. *If (X_i, A) satisfies the property (**) and A is closed in X_i for each $i\in I$, then $(\bigvee_{i\in I}(X_i, A), A)$ satisfies (**). In case $A=\{a\}$, $(\tilde{\bigvee}_{i\in I}X_i, a)$ also satisfies (**).*

PROOF. Let $K_j\subset\bigvee_{i\in I}(X_i, A)$ be compact and U_j open neighborhoods of K_j $(0\leqq j\leqq m)$. Then there exists a finite subset F of I such that $K_j\subset\bigvee_{i\in F}(X_i, A)$ for every $0\leqq j\leqq m$. There exist $f_i: X_i\to Y_i$ $(i\in F)$ so that $f_i(K_j\cap X_i)\subset U_j\cap X_i$ with other properties in (**). Since A is closed, we have a continuous map $f: \bigvee_{i\in F}X_i\to\bigvee_{i\in F}Y_i$ with $f|X_i=f_i$ for each $i\in I$. Then f satisfies the desired properties.

In case of $\bigvee_{i\in I}X_i$, we may assume that each U_j is a basic open subset for each j. Therefore, there exists a finite subset F of I such that every U_j only depends on $\bigvee_{i\in F}X_i$. Thus the proof is same as the above.

THEOREM 4.4. *Let X and Y be spaces such that $X\cap Y$ $(=A)$ is an acyclic retract of both X and Y.*

(Case $n=1$) $H_1^T(X\bigcup_A Y)\cong H_1^T(X)\oplus H_1^T(Y)$ canonically.

(Case $n\geqq 2$) If X and Y are normal and $(X\bigcup_A Y, A)$ satisfies $(*_n)$, then $H_n^T(X\bigcup_A Y)\cong H_n^T(X)\oplus H_n^T(Y)$ canonically. (In case A consists of one point, the normality is not necessary.)

PROOF. We have retractions $r_X: X\bigcup_A Y\to X$ and $r_Y: X\bigcup_A Y\to Y$ such that $r_X(Y)=r_Y(X)=A$. Let $i_X: X\to X\bigcup_A Y$ and $i_Y: Y\to X\bigcup_A Y$ be the inclusions. Let $\varphi: Z_n(X\bigcup_A Y)\to H_n^T(X)\oplus H_n^T(Y)$ be the homomorphism defined by:

$$\varphi(u)=(r_{X\#}(u)+\overline{B_n(X)})+(r_{Y\#}(u)+\overline{B_n(Y)}).$$

Since $r_Y(X)\subset A$ and A is acyclic, $r_{Y*}^T\circ i_{X*}^T=0$. On the other hand, $r_{X*}^T\circ i_{X*}^T=\mathrm{id}$. Then $\varphi(i_{X\#}(u))=u+\overline{B_n(X)}$ for each $u\in Z_n(X)$. A similar statement holds for Y. Therefore, φ is surjective. We shall show that $\mathrm{Ker}\,\varphi=\overline{B_n(X\cup Y)}$. Then, φ induces the desired isomorphism. First we have

$$\overline{B_n(X\cup Y)}\cap r_{X\#}^{-1}(\overline{B_n(X)})\cap r_{Y\#}^{-1}(\overline{B_n(Y)})=\mathrm{Ker}\,\varphi\,.$$

To see $\operatorname{Ker}\varphi\subset\overline{B_n(X\underset{A}{\cup}Y)}$, let $z=\sum_{k=0}^{m}\lambda_k u_k\in Z_n(X\underset{A}{\cup}Y)$ $(u_k\in C(\Delta_n, X\underset{A}{\cup}Y),\ \lambda_k=\pm 1)$ such that $\varphi(z)=0$. Since $r_{X\#}(z)\in\overline{B_n(X)}$ and $r_{Y\#}(z)\in\overline{B_n(Y)}$, it suffices to show $z-i_{X\#}\circ r_{X\#}(z)-i_{Y\#}\circ r_{Y\#}(z)\in\overline{B_n(X\underset{A}{\cup}Y)}$. Let V be an open neighborhood of $z-i_{X\#}\circ r_{X\#}(z)-i_{Y\#}\circ r_{Y\#}(z)$. Choose an open neighborhood U of z so that $U-i_{X\#}\circ r_{X\#}(U)-i_{Y\#}\circ r_{Y\#}(U)\subset V$ and then open neighborhoods U_k $(\subset C(\Delta_n, X\underset{A}{\cup}Y))$ of u_k so that $\sum_{k=0}^{m}\lambda_k U_k\subset U$. We want to find $v_k\in U_k$ $(0\leqq k\leqq m)$ so that $v-i_{X\#}\circ r_{X\#}(v)-i_{Y\#}\circ r_{Y\#}(v)\in B_n(X\underset{A}{\cup}Y)$, where $v=\sum_{i=0}^{m}\lambda_k v_k$. Then $V\cap B_n(X\underset{A}{\cup}Y)\neq\emptyset$.

(*Case* $n=1$) We may assume $u_k\in Z_1(X\underset{A}{\cup}Y)$ for each k. Now fix k, We may also assume $U_k=\bigcap_{j=1}^{l}O(K_j, W_j)$. Let $\bigcup_i O_i=u_k^{-1}(X\underset{A}{\cup}Y\setminus A)$, where O_i are open subintervals of Δ_1 and $O_i\cap O_{i'}=\emptyset$ for $i\neq i'$. If there are only finitely many O_i's, then we can easily see that $u_k-i_{X\#}\circ r_{X\#}(u_k)-i_{Y\#}\circ r_{Y\#}(u_k)\in B_1(X\underset{A}{\cup}Y)$ since A is acyclic. In this case we let $v_k=u_k$. Otherwise we assume that the index i ranges over N. Define $w_n\in C(\Delta_1, X\underset{A}{\cup}Y)$ by: $w_n|O_i=u_k|O_i$ for $i\leqq n$ and $w_n(\alpha)=r\circ u_k(\alpha)$ for $\alpha\notin\bigcup_{i=1}^{n}O_i$, where $r: X\underset{A}{\cup}Y\to A$ is the retraction defined by: $r|X=r_X|X$ and $r|Y=r_Y|Y$. We claim the existence of $w_n\in U_k$. Otherwise, there exists $0\leqq j\leqq l$ such that $w_n\notin O(K_j, W_j)$ for infinitely many w_n's. Therefore, for infinitely many O_i's there exist $\alpha_i\in O_i\cap K_j$ such that $r\circ u_k(\alpha_i)\notin W_j$. Let α^* be an accumulation point of α_i's. Then $r\circ u_k(\alpha^*)\notin W_j$, $u_k(\alpha^*)\in A$ and $\alpha^*\in K_j$. Hence $r\circ u_k(\alpha^*)=u_k(\alpha^*)$, which contradicts to $u_k\in O(K_j, W_j)$. Therefore we get $w_n\in U_k$. Now as in case there are only finitely many O_i's, we can conclude $w_n-i_{X\#}\circ r_{X\#}(w_n)-i_{Y\#}\circ r_{Y\#}(w_n)\in B_1(X\underset{A}{\cup}Y)$. We let $v_k=w_n$.

(*Case* $n\geqq 2$) There exist subsets $\underline{X}\subset X$ and $\underline{Y}\subset Y$ and $v_k\in U_k$ with the properties in $(*_n)$. Let $v=\sum_{k=0}^{m}\lambda_k v_k$. There is an open neighborhood W of A in $\underline{X}\underset{A}{\cup}\underline{Y}$ which is deformable into A in $\underline{X}\underset{A}{\cup}\underline{Y}$. By taking barycentric subdivisions, we can take $v_k'\in C(\Delta_n, X\underset{A}{\cup}Y)$ and λ_k' $(0\leqq k\leqq m')$ so that $v_k'(\Delta_n)\subset W$, or $v_k'(\Delta_n)\cap A=\emptyset$ and $\sum_{k=0}^{m'}\lambda_k'v_k'$ is homologous to v (see [3, Ch. III, §7]). By using the deformation of W into A and an Urysohn map with respect to A and $(\underline{X}\underset{A}{\cup}\underline{Y})\setminus W$, we can construct $w=\sum_{k=0}^{m'}\lambda_k'w_k\in Z_n(\underline{X}\underset{A}{\cup}\underline{Y})$ so that $w_k(\Delta_n)\subset\underline{X}$ or $w_k(\Delta_n)\subset\underline{Y}$ and w is homologous to v. Then we can write $w=w_X+w_Y$, where $w_X\in S_n(\underline{X})$ and $w_Y\in S_n(\underline{Y})$. Since $\partial w_X=-\partial w_Y\in Z_{n-1}(A)$ and A is acyclic, there is a $w^*\in S_n(A)$ such that $\partial w^*=\partial w_X$. Let $w'=w_X-w^*\in Z_n(\underline{X})$ and $w''=w_Y+w^*\in Z_n(\underline{Y})$. Then $w=w'+w''$. Note that $i_{X\#}\circ r_{X\#}(w')=w'$ and $i_{Y\#}\circ r_{Y\#}(w'')=w''$.

Since A is acyclic, $i_{Y\#}\circ r_{Y\#}(w')$, $i_{X\#}\circ r_{X\#}(w'')\in B_n(\underline{X}\bigcup_A\underline{Y})$. Then $i_{X\#}\circ r_{X\#}(w)-w'\in B_n(\underline{X}\bigcup_A\underline{Y})$ and $i_{Y\#}\circ r_{Y\#}(w)-w''\in B_n(\underline{X}\bigcup_A\underline{Y})$. Thus $w-i_{X\#}\circ r_{X\#}(w)-i_{Y\#}\circ r_{Y\#}(w)\in B_n(\underline{X}\bigcup_A\underline{Y})$. Since v and w are homologous, $v-i_{X\#}\circ r_{X\#}(v)-i_{Y\#}\circ r_{Y\#}(v)\in B_n(X\bigcup_A Y)$.

Next we are concerned with spaces obtained by attaching infinitely many spaces.

DEFINITION 4.5. For a space X with $A\subset X$, (X, A) is *primarily n-realizable*, provided

$$H_n^T(X)=\{u-v+\bar{B}_n : u,\ v\in C(\Delta_n, X),\ u(\dot{\Delta}_n),\ v(\dot{\Delta}_n)\subset A,\ u-v\in Z_n\}.$$

If X is $(n-1)$-connected, then (X, A) is primarily n-realizable for any $A\subset X$ by Hurewicz's theorem. We only deal with the case that A is acyclic. In this case, the condition $v(\dot{\Delta}_n)\subset A$ can be replaced by $v(\Delta_n)\subset A$. The notion "primarily n-realizable" is a little bit ad hoc, but it works well later on.

THEOREM 4.6. *Let X_i $(i\in I)$ be spaces with retractions $r_i: X_i\to A_i$ onto copies of a contractible space A, or with base points a_i. Then, the following hold.*

(1) *If (X_i, A) is primarily n-realizable for each $i\in I$, then both $(\bigvee_{i\in I}(X_i, A_i), A)$ and $(\tilde{\bigvee}_{i\in I}X_i, a)$ are primary n-realizable.*

(2) *(Case $n=1$) $H_1^T(\bigvee_{i\in I}(X_i, A_i))\cong\bigoplus_{i\in I}H_1^T(X_i)$ canonically. Suppose that (X_i, a_i) is primarily 1-realizable, then $H_1^T(\tilde{\bigvee}_{i\in I}X_i)\cong\tilde{\prod}_{i\in I}H_1^T(X_i)$ canonically.*
(Case $n\geqq 2$) If each X_i is normal and (X_i, A_i) satisfies (**), then $H_n^T(\bigvee_{i\in I}(X_i, A_i))\cong\bigoplus_{i\in I}H_n^T(X_i)$. If each (X_i, a_i) satisfies (**) and is primarily n-realizable, then $H_n^T(\tilde{\bigvee}_{i\in I}X_i)\cong\tilde{\prod}_{i\in I}H_n^T(X_i)$ canonically. (In case $A=\{a\}$, the normality of spaces is not necessary.)

PROOF. First we prove the statements for $\bigvee_{i\in I}(X_i, A_i)$. Let $p_i:\bigvee_{i\in I}(X_i, A_i)\to X_i$ $(i\in I)$ be the retractions induced by the given retractions r_i $(i\in I)$ and $\varphi: Z_n(\bigvee_{i\in I}(X_i, A_i))\to\prod_{i\in I}H_n^T(X_i)$ be the homomorphism defined by: $\varphi(z)(i)=p_{i\#}(z)+\overline{B_n(X_i)}$. Since the image of $u\in C(\Delta_n, \bigvee_{i\in I}(X_i, A))$ is contained in the union of finite components, we have $\operatorname{Im}\varphi\subset\bigoplus_{i\in I}H_n^T(X_i)$. By Theorem 4.4 and Lemma 4.3, we can see $H_n^T(\bigvee_{i\in I}(X_i, A_i))\cong\bigoplus_{i\in I}H_n^T(X_i)$ canonically. Suppose that (X_i, A_i) $(i\in I)$ are primarily n-realizable. Let $c\in\bigoplus_{i\in I}H_n^T(X_i)$. Then, for each $i\in\operatorname{supp} c$ there exist $u_i,\ v_i\in C(\Delta_n, X_i)$ such that $\varphi(u_i-v_i)=c(i)$. Since A is contractible, taking small simplexes in Δ_n of the same number as $\operatorname{supp} c$ and using $u_i,\ v_i$, we can define $u,\ v\in C(\Delta_n, \bigvee_{i\in I}(X_i, A_i))$ so that $\varphi(u-v)=c$. Therefore, $(\bigvee_{i\in I}(X_i, A_i), A)$ becomes primarily n-realizable.

Next we prove the statements for $\tilde{\bigvee}_{i\in I}X_i$. Let p_i and φ as above. Then $\varphi(\overline{B_n})=0$. Since the image of any $u\in C(\Delta_n, \tilde{\bigvee}_{i\in I}X_i)$ is contained in a countable union of X_i's by the separability of Δ_n, we have $\mathrm{Im}\,\varphi\subset\tilde{\prod}_{i\in I}H_n^T(X_i)$. To see $H_n^T(\tilde{\bigvee}_{i\in I}X_i)\cong\tilde{\prod}_{i\in I}H_n^T(X_i)$, if suffices to show that $\mathrm{Ker}\,\varphi\in\overline{B_n}$ and $\tilde{\prod}_{i\in I}H_n^T(X_i)$ $\subset\mathrm{Im}\,\varphi$.

Suppose that $\varphi(z)=0$ for $z=\sum_{k=0}^m\lambda_k u_k\in Z_n$ ($\lambda_k\in\boldsymbol{Z}$, $u_k\in C(\Delta_n, \tilde{\bigvee}_{i\in I}X_i)$). Then, $p_{i\#}(z)\in\overline{B_n(X_i)}$ for every i. For any neighborhood U of z, u_k $(0\leqq k\leqq m)$ have neighborhoods U_k $(\subset C(\Delta_n, \tilde{\bigvee}_{i\in I}X_i))$ such that $\sum_{k=0}^m\lambda_k U_k\subset U$. We may assume there is a finite subset F of I such that U_k's depend on $\bigvee_{i\in F}X_i$. Let $p_F:\tilde{\bigvee}_{i\in I}X_i\to\bigvee_{i\in F}X_i$ be the retraction and $i_F:\bigvee_{i\in F}X_i\to\tilde{\bigvee}_{i\in I}X_i$ the inclusion map. Then $i_{F\#}\circ p_{F\#}(u_k)\in U_k$ and $\sum_{k=0}^m\lambda_k p_{F\#}(u_k)\in\overline{B_n(\bigvee_{i\in F\#}X_i)}$ by Theorem 4.4. Now $i_{F\#}\circ p_{F\#}(\sum_{k=0}^m\lambda_k u_k)\in\overline{B_n(\tilde{\bigvee}_{i\in I}X_i)}$ and $i_{F\#}\circ p_{F\#}(\sum_{k=0}^m\lambda_k u_k)=\sum_{k=0}^m\lambda_k i_{F\#}\circ p_{F\#}(u_k)\in\sum_{k=0}^m\lambda_k U_k\subset U$. Thus, $U\cap\overline{B}_n\neq\emptyset$ hence $z\in\overline{B_n}$.

Now, let $c\in\tilde{\prod}_{i\in I}H_n^T(X_i)$. Then, $I_0=\mathrm{supp}\,c$ is countable. Since each (X_i, a_i) is primarily n-realizable, there exist $u_i, v_i\in C(\Delta_n, X_i)$ $(i\in I_0)$ such that $c(i)=$ $=u_i-v_i+\overline{B_n(X_i)}$ and $u_i(\dot{\Delta}_n)=v_i(\dot{\Delta}_n)=a_i$. We take small simplexes $E_i\subset\Delta_n$ $(i\in I_0)$ so that each E_i is similar to Δ_n with orientation preserving similar maps $f_i:E_i\to\Delta_n$ and $E_i\cap E_j=\emptyset$ for $i\neq j$. Define $u, v\in C(\Delta_n, \tilde{\bigvee}_{i\in I}X_i)$ by: $u|E_i=u_i\circ f_i$, $v|E_i=v_i\circ f_i$ $(i\in I_0)$ and $u(\alpha)=v(\alpha)=a$ for $\alpha\notin\bigcup_{i\in I_0}E_i$. Since the dia-diameters of E_i's converge to 0, u and v are continuous. For each i, $p_{i\#}(u)|E_i$ $=u_i\circ f_i$, $p_{i\#}(v)|E_i=v_i\circ f_i$ and $p_{i\#}(u)(\Delta_n\setminus E_i)=p_{i\#}(v)(\Delta_n\setminus E_i)=a$. Since $p_{i\#}(u)$ and $p_{i\#}(v)$ are homotopic to u_i and v_i respectively, $p_{i\#}(u)-u_i$, $p_{i\#}(v)-v_i\in B_n(X_i)$. Thus we have

$$c(i)=p_{i\#}(u)-p_{i\#}(v)+\overline{B_n(X_i)}=p_{i\#}(u-v)+\overline{B_n(X_i)}$$

and consequently $\varphi(u-v)=c$. We have not only shown $\tilde{\prod}_{i\in I}H_n^T(X_i)\subset\mathrm{Im}\,\varphi$ but also $(\tilde{\bigvee}_{i\in I}X_i, a)$ is primarily n-realizable.

DEFINITION 4.7. For a space X with base point a, the *X-piled spaces* are spaces with base point inductively defined as follows:

(1) X is an X-piled space with base point a;

(2) If X_i $(i\in I)$ are X-piled spaces with base points a_i, then both $\bigvee_{i\in I}X_i$ and $\bigvee_{i\in I}X_i$ are X-pilled spaces.

Along the definition of X-piled spaces, we define their types and orders. A rigorous reader should think that X-piled spaces are not just pointed spaces, but pointed spaces with their construction. However, in some cases we confuse them for short expression. Types are pairs $(\mu, \boldsymbol{P})$, $(\mu, \boldsymbol{S})$ and $(\mu, \boldsymbol{M})$ of ordinals

μ and letters $\boldsymbol{P}$, $\boldsymbol{S}$ and $\boldsymbol{M}$. The partial order $<$ for types is defined as follows: $(\mu, \Gamma)<(\nu, \Gamma')$ for $\mu<\nu$ and $\Gamma, \Gamma'=\boldsymbol{P}, \boldsymbol{M}$ or $\boldsymbol{S}$; $(\mu, \boldsymbol{P})<(\mu, \boldsymbol{M})$; $(\mu, \boldsymbol{S})<(\mu, \boldsymbol{M})$. We identify the types $(0, \boldsymbol{P})$ and $(0, \boldsymbol{S})$ as a special case. The supremum $\sup^{+}S$ of a set of ordinals S is the least ordinal which is strictly greater than every ordinal in S.

DEFINITION 4.8. For an X-piled spaces Y, the *type* $\mathrm{ty}(Y)$ is defined as follows and the ordinal of $\mathrm{ty}(Y)$ is called the order of Y and denoted by $\mathrm{o}(Y)$.

(1*) $\mathrm{ty}(X)=(0, \boldsymbol{P})=(0, \boldsymbol{S})$;

(2*) Let X_i $(i\in I)$ be X-piled spaces of type $\mathrm{ty}(X_i)$, where $|I|\geqq 2$.

Case (a): $\sup^{*}\{\mathrm{o}(X_i): i\in I\}=1$.

$$\mathrm{ty}(\vee_{i\in I}X_i)=\begin{cases}(1, \boldsymbol{S}) & \text{if } I \text{ is infinite;}\\ (0, \boldsymbol{M}) & \text{otherwise.}\end{cases}$$

$$\mathrm{ty}(\tilde{\vee}_{i\in I}X_i)=\begin{cases}(1, \boldsymbol{P}) & \text{if } I \text{ is infinite;}\\ (0, \boldsymbol{M}) & \text{otherwise.}\end{cases}$$

Case (b): $\sup^{*}\{\mathrm{o}(X_i): i\in I\}=\mu+1$ for $\mu\geqq 1$.

$$\mathrm{ty}(\vee_{i\in I}X_i)=\begin{cases}(\mu+1, \boldsymbol{S}) & \text{if } I_P \text{ is infinite;}\\ (\mu, \boldsymbol{S}) & \text{if } I_P \text{ is empty;}\\ (\mu, \boldsymbol{M}) & \text{otherwise,}\end{cases}$$

$$\mathrm{ty}(\tilde{\vee}_{i\in I}X_i)=\begin{cases}(\mu+1, \boldsymbol{P}) & \text{if } I_S \text{ is infinite;}\\ (\mu, \boldsymbol{P}) & \text{if } I_S \text{ is empty;}\\ (\mu, \boldsymbol{M}) & \text{otherwise,}\end{cases}$$

where $I_P=\{i: \mathrm{ty}(X_i)=(\mu, \boldsymbol{P}) \text{ or } (\mu, \boldsymbol{M})\}$ and $I_S=\{i: \mathrm{ty}(X_i)=(\mu, \boldsymbol{S}) \text{ or } (\mu, \boldsymbol{M})\}$.

Case (c): $\sup\{\mathrm{o}(X_i): i\in I\}=\mu$ a limit ordinal.

$$\mathrm{ty}(\vee_{i\in I}X_i)=(\mu, \boldsymbol{S}) \text{ and } \mathrm{ty}(\tilde{\vee}_{i\in I}X_i)=(\mu, \boldsymbol{P}).$$

Since types of X-piled spaces are defined along the inductive definition, it is possible that spaces of the same homotopy type have different types. However, under some condition on (X, a), types of X-piled spaces are quasi-homotopy invariant and so homotopy invariant. To state the condition we need a definition about groups, which is a version of a notion in [4, p. 189]. We refer the reader to [11] for undefined notions about groups. Further results we need will be proved or mentioned in the appendix.

DEFINITION 4.9. For a group A, the maximal divisible subgroup of A is denoted by $D(A)$ [10, p. 100]. The properties (P) and (S) are defined as follows:

(P) The group $n!\cdot\prod_N A/D(A)$ is not isomorphic to a summand of $n!\cdot(A/D(A))^m$ for any $m, n\in \boldsymbol{N}$.

(S) The group $n!\cdot\bigoplus_N A/D(A)$ is not isomorphic to a summand of $n!\cdot(A/D(A))^m$ for any $m, n\in \boldsymbol{N}$.

If a group of finite rank is not isomorphic to a direct sum of a divisible group and a bounded group, then it satisfies both (P) and (S). Especially, a reduced torsion-free group of finite rank satisfies both.

THEOREM 4.10. *Suppose that (X, a) is primarily n-realizable. In case $n\geqq 2$, we also assume (X, a) satisfies $(**)$. Let Y and Y' be X-piled spaces. If there exist continuous maps $f: Y\to Y'$ and $g: Y'\to Y$ such that $f\circ g\underset{q}{\sim} id_{Y'}$, then* $\mathrm{ty}(Y)\leqq \mathrm{ty}(Y')$ *under the following conditions:*
$H_n^T(X)$ satisfies (Γ) and

(a) $\mathrm{ty}(Y')=(m, \nabla)$ *for an odd $m\in N$;*

(b) $\mathrm{ty}(Y')=(m, \Gamma)$ *for an even $m\in N$; or*

(c) *$o(Y')$ is infinite,*

where (1) $\Gamma=\boldsymbol{P}$ *and* $\nabla=\boldsymbol{S}$; *or* (2) $\Gamma=\boldsymbol{S}$ *and* $\nabla=\boldsymbol{P}$.

Particularly $\mathrm{ty}(Y)\leqq\mathrm{ty}(Y')$ *in case $H_n^T(X)$ satisfies both* (P) *and* (S) *and* $o(Y')\geqq 1$. *Therefore, in this case the type is quasi-homotopy invariant for X-piled spaces of nonzero order. In addition if $H_n^T(X)$ is of finite rank, the condition $o(Y')\geqq 1$ is not necessary in the above.*

PROOF. By Theorem 3.8, $H_n^T(Y)$ is isomorphic to a summand of $H_n^T(Y')$. Let $A=H_n^T(X)$. By Theorem 4.6 and the definition of type for σ-Reid(A) groups in the appendix, $H_n^T(Y)$ and $H_n^T(Y')$ belong to the σ-Reid class of A and $\mathrm{ty}(Y)=\mathrm{ty}(H_n^T(Y))$ and $\mathrm{ty}(Y')=\mathrm{ty}(H_n^T(Y'))$. Therefore, the theorem follows from Theorem A.7 and Corollary A.8.

Ralph [22] has defined a factor of singular chain and homology groups HA and HM. He showed that HM can be used to detect the anomalous singular homology constructed by Barratt and Milnor [2], which is $H_q(\tilde{\bigvee}_N \boldsymbol{S}^r)$. Though $HM(X)$ and $\mathrm{Ker}\,\sigma_X$, in the present paper, are different in general, $\mathrm{Ker}\,\sigma_X$ can also be used to detect such a phenomenon. To see this, we show the following.

PROPOSITION 4.11. *Let X be the inverse limit $\varprojlim(X_i, r_{ij}, I)$ of subspaces X_i of X such that each bonding map $r_{ij}: X_j\to X_i$ is a retraction. Then $H_n^T(X)$ is naturally isomorphic to a subgroup of $\varprojlim(H_n^T(X_i), r_{ij*}^T, I)$.*

PROOF. Let $r_i: X\to X_i$ $(i\in I)$ be the retractions and define $\varphi: Z_n(X)\to \varprojlim(H_n^T(X_i), r_{ij*}^T, I)$ by: $r_{i*}^T\circ\varphi(z)=r_{i\#}(z)+\overline{B_n(X_i)}$ for $i\in I$. Let $z=\sum_{i=1}^m \varepsilon_k u_k$ $(u_k\in C(\Delta_n, X),\ \varepsilon_k=\pm 1)$ such that $r_{i*}^T\circ\varphi(z)=0$ for all $i\in I$. For any neighborhood U of z, there exist neighborhoods $U_k\subset C(\Delta_n, X)$ of u_k such that $\sum_{k=1}^m \varepsilon_k U_k\subset U$. Observe $C(\Delta_n, X)=\varprojlim(C(\Delta_n, X_i), (r_{ij})_\#, I)$. Then there exist $j\in I$ and neighborhood V_k of $r_{j\#}(u_k)$ $(1\leqq k\leqq m)$ such that $r_{j\#}^{-1}(V_k)\subset U_k$. Since r_j is a retraction, $e_\#\circ r_{j\#}(u_k)\in r_{j\#}^{-1}(V_k)\subset U_k$ where $e: X_j\to X$ is the inclusion, hence $e_\#\circ r_{j\#}(z)\in U$. On the other hand, $e_\#\circ r_{j\#}(z)\in\overline{B_n(X)}$ since $r_{j\#}(z)\in\overline{B_n(X_j)}$. Therefore $z\in\overline{B_n(X)}$.

COROLLARY 4.12. *Let $X=\varprojlim(X_i, r_{ij}, I)$, where each X_i is a subspace of X and $r_{ij}: X_j\to X_i$ is a retraction. In case $H_n^T(X_i)=\{0\}$ for all $i\in I$, $H_n^T(X)=\{0\}$ and consequently* Ker $\sigma_X=H_n(X)$.

By this corollary $H_q^T(\tilde{\bigvee}_N \boldsymbol{S}^r)=\{0\}$ for $q>r$ which implies Ker $\sigma_X=H_q(\tilde{\bigvee}_N \boldsymbol{S}^r)\neq\{0\}$ with $q\equiv 1 \bmod (r-1)$, $q>1$, $r>1$ [2, Theorem 1].

REMARK 4.13. In Section 3, we have shown the difference among the Čech homology group $\check{H}_0(X)$, the singular homology group $H_0(X)$ and $H_0^T(X)$. Here we demonstrate examples which show the difference among $\check{H}_1(X)$, $H_1(X)$ and $H_1^T(X)$.

(1) Let $D=\{(x, y): x^2+y^2\leqq 1\}$ and $\boldsymbol{S}^1=\dot{D}=\{(x, y): x^2+y^2=1\}$. Define $X=\boldsymbol{S}^1\times\{0\}\cup\bigcup_{n\in \boldsymbol{N}} D\times\{1/n\}$. Then $H_1(X)\cong\boldsymbol{Z}$ and $\check{H}_1(X)=H_1^T(X)=0$.

(2) Let X be the example space due to H. B. Griffiths [23, p. 59], i.e., $X=Y\cup Y'$ the subspace of $\boldsymbol{R}^3$ where $Y=\{(x, y, z): 0\leqq z\leqq 1, (x-(1-z)/n)^2+y^2=(1-z)^2/n^2, n\in\boldsymbol{N}\}$ and Y' is the reflection of Y through the origin of $\boldsymbol{R}^3$. Then, $\pi_1(X)\neq 0$ but also $H_1(X)\neq 0$. On the other hand, $\check{H}_1(X)=H_1^T(X)=0$.

(3) Let $X=\tilde{\bigvee}_I \boldsymbol{S}^1$. Then $\check{H}_1(X)\cong\boldsymbol{Z}^I$ and $H_1^T(X)\cong\tilde{\prod}_I\boldsymbol{Z}$. The first author has shown that in case I is infinite $H_1(X)$ contains $\bigoplus_{2^{\aleph_0}}\boldsymbol{Q}$ as a subgroup and hence as a summand, where $\boldsymbol{Q}$ is the group of the rational [25, Theorem 4.14 and Theorem A.1]. It is an interesting question whether $H_1(X)$ is torsion-free or not.

(4) Since H_1^T is a factor of H_1, one may think that H_1^T gives us less information than H_1. However, it is not true. Let $\boldsymbol{H}$ be the so-called Hawaiian earing, i.e. $\boldsymbol{H}=X$ with $I=\boldsymbol{N}$ in (3), and let x be a point of $\boldsymbol{H}$ such that $\boldsymbol{H}$ is locally simply connected at x. Then $H_1(\boldsymbol{H})\cong A\oplus\bigoplus_{2^{\aleph_0}}\boldsymbol{Q}$ by (3). Take a simplicial complex Y with $y\in Y$ so that $H_1(Y)\cong\boldsymbol{Q}$. Let Z be the one point union $(\boldsymbol{H}, x)\vee(Y, y)$. Then Z is locally simply connected at the common point.

Therefore, $H_1(Z)\cong H_1(\boldsymbol{H})\oplus H_1(Y)\cong A\oplus(\oplus_{2^{\aleph_0}}\boldsymbol{Q})\oplus\boldsymbol{Q}\cong H_1(\boldsymbol{H})$. On the other hand, $H_1^T(Z)\cong H_1^T(\boldsymbol{H})\oplus H_1(Y)\cong \boldsymbol{Z}^N\oplus\boldsymbol{Q}$, which is apparently not isomorphic to $H_1^T(\boldsymbol{H})\cong\boldsymbol{Z}^N$.

5. Realizing the groups $C(X, \boldsymbol{Z})$

In this section, we show the following.

THEOREM 5.1. *Let X be a bounded subset of the real line $\boldsymbol{R}$. Then $H_n^T(\boldsymbol{R}^{n+1}\setminus X\times\{0\})\cong C(X, \boldsymbol{Z})$ canonically, where $0=(0, \cdots, 0)\in\boldsymbol{R}^n$.*

PROOF. We indentify $Y\subset\boldsymbol{R}$ with $Y\times\{0\}\subset\boldsymbol{R}^{n+1}$. For each $x\in X$, let $i_x:\boldsymbol{R}^{n+1}\setminus X\to\boldsymbol{R}^{n+1}\setminus\{x\}$ be the inclusion. Identifying $\boldsymbol{Z}$ with $H_n^T(\boldsymbol{R}^{n+1}\setminus\{x\})$ canonically, we define a homomorphism $\varphi: Z_n(\boldsymbol{R}^{n+1}\setminus X)\to\boldsymbol{Z}^X$ by $\varphi(z)(x)=(i_x)_*^T(z+\overline{B_n})$. Let $z=\sum_{k=0}^m\lambda_k u_k\in Z_n$, where $\lambda_k\in\boldsymbol{Z}$ and $u_k\in C(\Delta_n, \boldsymbol{R}^{n+1}\setminus X)$. Each $x\in X$ has a contractible neighborhood U in $\boldsymbol{R}^{n+1}$ such that $U\cap\operatorname{Im}u_k=\emptyset$ for every k. Therefore, $(i_x)_*^T(z+\overline{B_n})$ is constant in U as a function of x, which implies $\operatorname{Im}\varphi\subset C(X, Z)$. And clearly $\overline{B_n}\subset\operatorname{Ker}\varphi$. Then, it suffices to show that (i) $\operatorname{Ker}\varphi\subset\overline{B_n}$ and (ii) $C(X, \boldsymbol{Z})\subset\operatorname{Im}\varphi$.

(i) Let $z\in\operatorname{Ker}\varphi$ be written as above. For any open neighborhood V of z, there exists $\varepsilon>0$ such that $\rho(u_k, u_k')<\varepsilon$ $(0\leqq k\leqq m)$ implies $\sum_{k=0}^m\lambda_k u_k'\in V$, where ρ is the sup-metric induced from Euclidean metric. Choose $x_0<x_1<\cdots<x_M$ in $\boldsymbol{R}$ so that $X\subset(x_0, x_M)$ and $x_i-x_{i-1}<\varepsilon/n+1$ $(1\leqq i\leqq M)$ and $x_i\in X$ implies $(x_i-\delta, x_i+\delta)\subset X$ for some $\delta>0$ and $x_i, x_{i+1}\in X$ implies $[x_i, x_{i+1}]\subset X$. Let $K=\{i:(x_i, x_{i+1})\cap X\neq\emptyset\}$. For each $i\in K$, let $D_i=(x_i, x_{i+1})\times(-\varepsilon/n+1, \varepsilon/n+1)^n$ and take $y_i\in(x_i, x_{i+1})\cap X$ so that $x_i\in X$ implies $[x_i, y_i]\subset X$ and $x_{i+1}\in X$ implies $[y_i, x_{i+1}]\subset X$. Let

$$h:\boldsymbol{R}^{n+1}\setminus\{y_i: i\in K\}\longrightarrow\boldsymbol{R}^{n+1}/\bigcup_{i\in K}D_i$$

be the radial deformation retraction, i.e. for each $\boldsymbol{a}$ on the boundary of D_i, h maps the segment between y_i and $\boldsymbol{a}$ to the point $\boldsymbol{a}$. According to the choice of x_i and y_i, we have

$$h(\boldsymbol{R}^{n+1}\setminus X)=\boldsymbol{R}^{n+1}\setminus(X\cup\bigcup_{i\in K}D_i)\quad(\subset\boldsymbol{R}^{n+1}\setminus X).$$

which has a homotopy type of an n-dimensional bouquet. Since $(i_{y_i})_*^T(h_\#(z))=(i_{y_i})_*^T(z)=0$ for each $i\in E$, we can see $h_\#(z)\in B_n(h(\boldsymbol{R}^{n+1}\setminus X))\subset B_n(\boldsymbol{R}^{n+1}\setminus X)$. Since h moves points in distance less than ε, $\rho(u_k, h_\#(u_k))<\varepsilon$ for each k. Then $h_\#(z)=\sum_{k=0}^m\lambda_k h_\#(u_k)\in V$. Thus we have $\operatorname{Ker}\varphi\subset\overline{B_n}$.

(ii) To see $C(X, \boldsymbol{Z})\subset\operatorname{Im}\varphi$, let $f\in C(X, \boldsymbol{Z})$. There exist pairwise disjoint open intervals I_k $(k\in \boldsymbol{N})$ such that $X\subset\bigcup_{k\in N}I_k$ and f is constant on each $X\cap I_k$. Let D_k be the $(n+1)$-ball with center x_k and its diametrical axis I_k, and C_k the boundary n-sphere of D_k. Take an n-simplex $E\subset\boldsymbol{R}^n\times\{0\}$ large enough so that E contains all $C_k\cap\boldsymbol{R}^n\times\{0\}$ in its interior. Take a homeomorphism $g:\Delta_n\to E$ and for each k let $u_k, v_k:\Delta_n\to E\cup D_k\setminus\{x_k\}$ be continuous maps such that:

(1) $u_k(\alpha)=v_k(\alpha)=g(\alpha)$ for $\alpha\notin g^{-1}(D_k\cap\boldsymbol{R}^n\times\{0\})$;

(2) u_k maps $g^{-1}(D_k\cap\boldsymbol{R}^n\times\{0\})$ to C_k;

(3) v_k maps $g^{-1}(D_k\cap\boldsymbol{R}^n\times\{0\})$ to $C_k\cap\boldsymbol{R}^n\times\{y: y\leqq 0\}$ homeomorphically;

(4) $(i_{x_k})^T_*(u_k-v_k+\overline{B}_n)=f(k)\in H^T_n(\boldsymbol{R}^{n+1}\setminus\{x_k\})$.

Finally, let $u, v:\Delta_n\to\boldsymbol{R}^{n+1}\setminus X$ be the maps defined by: $u(\alpha)=v(\alpha)=i(\alpha)$ for $\alpha\notin\bigcup_{k\in N}g^{-1}(D_k\cap\boldsymbol{R}^n\times\{0\})$; and $u(\alpha)=u_k(\alpha)$ and $v(\alpha)=v_k(\alpha)$ for $\alpha\in g^{-1}(D_k\cap\boldsymbol{R}^n\times\{0\})$ and each $k\in\boldsymbol{N}$. Then, $u-v\in Z_n$ and $\varphi(u-v)=f$.

REMARK 5.2. In case X is unbounded in the theorem, we get $H^T_n(\boldsymbol{R}^{n+1}\setminus X)\cong(f\in C(X,\boldsymbol{Z})$: supp f is bounded$\}$, where supp $f=\{x\in X: f(x)\neq 0\}$. Let $\boldsymbol{L}^+$ be Cantor's long ray [21, p. 643] and $X\subset\boldsymbol{L}^+$. Similarly we get an isomorphism $H^T_n(\boldsymbol{L}^+\times\boldsymbol{R}\setminus X\times\{0\})\cong\{f\in C(X,\boldsymbol{Z})$: supp f is bounded$\}$. Thus, the group in [9, Theorem 4.6] is naturally realized by $H^T_1(Y)$.

It will be shown in Proposition A.11 that $C(X,\boldsymbol{Z})$ is a σ-Reid$(\boldsymbol{Z})$ group for any scattered subspace X of $\boldsymbol{R}$. On the other hand neither $C(\boldsymbol{Q},\boldsymbol{Z})$ nor $C(\boldsymbol{R}\setminus\boldsymbol{Q},\boldsymbol{Z})$ belongs to the σ-Reid class of $\boldsymbol{Z}$ by the same reason as in [5]. Hence, neither $\boldsymbol{R}^2\setminus\boldsymbol{Q}$ nor $\boldsymbol{R}^2\setminus(\boldsymbol{R}\setminus\boldsymbol{Q})$ is quasi-homotopic to $\boldsymbol{R}^2\setminus X$ for any scattered subspace X of $\boldsymbol{R}$ by Theorem 5.1.

6. Spatial homomorphisms

For spaces X and Y, a homomorphism $h: H^T_n(X)\to H^T_n(Y)$ $(h: H_n(X)\to H_n(Y))$ is called *spatial* if there exists a continuous map $f: X\to Y$ such that $h=f^T_*$ $(h=f_*)$. In this section, we show that any homomorphism from $H^T_n(X)$ to $H^T_n(Y)$ is spatial for any $\boldsymbol{S}^n$-piled spaces X and Y. As we shall show later, this does not hold for singular homology groups. Recall that an $\boldsymbol{S}^n$-piled space X consists of copies of $\boldsymbol{S}^n$ with identified base points. By $\boldsymbol{e}$, we denote the base point of $\boldsymbol{S}^n$ and also of X. We call each copy of $\boldsymbol{S}^n$ a *basic component* of X. Our result of this section is concerned with slenderness of groups. A group A is *slender* if for any homomorphism $h:\boldsymbol{Z}^{\boldsymbol{N}}\to A$ there exists $n\in\boldsymbol{N}$ such that $h(e_m)=0$ for $m\geqq n$, where $e_m(m)=1$ and $e_m(i)=0$ for $i\neq m$ [11, XIII 94]. As is well known, it is also equivalent that for any homomorphism $h:\boldsymbol{Z}^{\boldsymbol{N}}\to A$ there exists

$n\in\boldsymbol{N}$ such that $h(\boldsymbol{Z}^{N\setminus n})=\{0\}$.

By a straightforward induction on the definition of piled spaces, we get the following

LEMMA 6.1. *Let X be an $\boldsymbol{S}^n$-piled space and X_λ ($\lambda\in\Lambda$) be all the basic components of X. If $\sigma_\lambda: X_\lambda\to X_\lambda$ ($\lambda\in\Lambda$) are base point preserving continuous maps, then the induced map $\sigma: X\to X$ (i.e., $\sigma|X_\lambda=\sigma_\lambda$ for each λ) is continuous.*

THEOREM 6.2. *Let X and Y be $\boldsymbol{S}^n$-piled spaces. Then any homomorphism from $H_n^T(X)$ to $H_n^T(Y)$ is spatial.*

PROOF. In case $X=Y=\boldsymbol{S}^n$, the theorem holds, since any endomorphism on $H_n^T(\boldsymbol{S}^n)$ ($\cong\boldsymbol{Z}$) is spatial. Our proof goes by induction on the definitions of X and Y. In the sequel, $q_j: Y\to Y_j$ ($j\in J$) denote the projections in both cases $Y=\tilde{\bigvee}_{j\in J}Y_j$ and $Y=\bigvee_{j\in J}Y_j$. Let $X=\bigvee_{i\in I}X_i$ or $X=\tilde{\bigvee}_{i\in I}X_i$. Our induction hypothesis is that any homomorphism from $H_n^T(X)$ to $H_n^T(Y_j)$ and one from $H_n^T(X_i)$ to $H_n^T(Y)$ are induced by base point preserving continuous maps. We remark the following property (+) of $\boldsymbol{S}^n$:

(+) There exist pairwise disjoint open sets O_m ($m\in\boldsymbol{N}$) in $\boldsymbol{S}^n$ and continuous maps $\sigma_m: \boldsymbol{S}^n\to\boldsymbol{S}^n$ ($m\in\boldsymbol{N}$) such that $\boldsymbol{e}\notin O_m$, $\sigma_m(\boldsymbol{S}^n\setminus O_m)=\{\boldsymbol{e}\}$ and $\sigma_m\underset{\boldsymbol{*}}{\simeq}\mathrm{id}$.

(*Case* 1) $X=\boldsymbol{S}^n$. Let $Y=\bigvee_{j\in J}Y_j$. Since $H_n^T(Y)\cong\bigoplus_{j\in J}H_n^T(Y_j)$ canonically by Theorem 4.6 and $H_n^T(\boldsymbol{S}^n)\cong\boldsymbol{Z}$, there exists a finite subset F of J such that $\mathrm{Im}\,h\subset\bigoplus_{j\in F}H_n^T(Y_j)$. The induction hypothesis implies the existence of base point preserving continuous maps $f_j: X\to Y_j$ ($j\in F$) such that $(f_j)_*^T=(q_j)_*^T\circ h$. Take O_j and $\sigma_j: X\to X$ indexed by $j\in J$ with the properties assured by (+). Define $f: X\to Y$ by:

$$f(s)=\begin{cases} f_j\circ\sigma_j(s) & \text{for } s\in O_j \ (j\in F);\\ \boldsymbol{e} & \text{otherwise.}\end{cases}$$

Then f is continuous, $\mathrm{Im}\,f_*^T\subset\bigoplus_{j\in F}H_n^T(Y_j)$ and $(q_j)_*^T\circ f_*=(q_j\circ f)_*^T=(q_j\circ f\circ\sigma_j)_*^T=(f_j)_*^T=(q_j)_*^T\circ h$ for each $j\in F$. Hence $f_*^T=h$.

Next, let $Y=\tilde{\bigvee}_{j\in J}Y_j$. Since $H_n^T(Y)\cong\tilde{\prod}_{j\in J}H_n^T(Y_j)$ canonically by Theorem 4.6, there exists a countable subset C of J such that $\mathrm{Im}\,h\subset\prod_{j\in C}H_n^T(y_j)$. As in the case of $\bigvee_{j\in J}Y_j$, take f_j, O_j and σ_j ($j\in C$) and define $f: X\to Y$ by:

$$f(s)=\begin{cases} f_j\circ\sigma_j(s) & \text{for } s\in O_j \ (j\in C);\\ \boldsymbol{e} & \text{otherwise.}\end{cases}$$

Since $q_j \circ f$ is continuous for all $j \in J$, f is continuous and we can conclude $f_*^T = h$ as above.

(*Case* 2) $X = \bigvee_{i \in I} X_i$. By induction hypothesis, there exist base point preserving continuous maps $f_i : X_i \to Y$ $(i \in I)$ such that $h | H_n^T(X_i) = (f_i)_*^T$. Define a continuous map $f : X \to Y$ by: $f | X_i = f_i$ for $i \in I$. Then it is easy to see that $h = f_*^T$.

(*Case* 3) $X = \tilde{\bigvee}_{i \in I} X_i$. When $Y = \boldsymbol{S}^n$, $H_n^T(Y)$ $(\cong \boldsymbol{Z})$ is a slender group. Applying Theorem 1(1) of [8] in a similar way as in the proof of Lemma A.2, we get a finite subset E of I and $\bar{h} : \bigoplus_{i \in E} H_n^T(X_i) \to H_n^T(Y)$ such that $h = \bar{h} \circ \pi_E$, where $\pi_E : \tilde{\prod}_{i \in I} H_n^T(X_i) \to \bigoplus_{i \in E} H_n^T(X_i)$ is the projection. We get the desired continuous map f through the projection as in Case 2.

Let $Y = \bigvee_{j \in J} Y_j$. By Lemma A.2 and the torsion-freeness of $H_n^T(Y)$, there exist finite subsets E of I and F of J such that $h(\tilde{\prod}_{i \in I \setminus E} H_n^T(X_j)) \subset \bigoplus_{j \in F} H_n^T(Y_j)$. Let $X_{i\lambda}$ $(\lambda \in \Lambda_i)$ be all the basic components of X_i for each i. Then, $H_n^T(X_{i\lambda})$ $(\lambda \in \Lambda_i, i \in I)$ correspond to all the basic components of $H_n^T(X)$. (See Definition A.9.) Since $\tilde{\bigvee}_{i \in I \setminus E} X_i$ is a retract of X, by induction hypothesis there exists a base point preserving continuous map $g_j : \tilde{\bigvee}_{i \in I \setminus E} X_i \in Y_j$ for each j such that $(g_j)_*^T = (q_j)_*^T \circ h | \tilde{\prod}_{i \in I \setminus E} H_n^T(X_i)$. We get $O_{i\lambda j} \subset X_{i\lambda}$ $(j \in F)$ and $\sigma_{i\lambda j} : X_{i\lambda} \to X_{i\lambda}$ with the properties assured by (+). Then there exists a continuous map $\sigma_j : \tilde{\bigvee}_{i \in I \setminus E} X_i \to \tilde{\bigvee}_{i \in I \setminus E} X_i$ induced by $\sigma_{i\lambda j}$ $(i \in I \setminus E, \lambda \in \Lambda_i)$ for each $j \in F$ by Lemma 6.1. Define $f_E : \tilde{\bigvee}_{i \in I \setminus E} X_i \to \bigvee_{j \in F} Y_j$ $(\subset \bigvee_{j \in J} Y_j)$ by:

$$f_E(s) = \begin{cases} g_j \circ \sigma_{i\lambda j}(s) & \text{for } s \in O_{i\lambda j} \ (i \in I \setminus E, \lambda \in \Lambda_i, j \in F); \\ \boldsymbol{e} & \text{otherwise.} \end{cases}$$

Then the continuity of f_E follows from the continuity of σ_j $(j \in F)$. We have $h | \tilde{\prod}_{i \in I \setminus E} H_n^T(X_i) = (f_E)_*^T$ by the definition and Lemma A.10, since $(q_j)_*^T \circ h | H_n^T(X_{i\lambda}) = (g_j)_*^T | H_n^T(X_{i\lambda}) = (q_j)_*^T \circ (f_E)_*^T$ for $j \in J$. For each $i \in E$, there is a base point preserving continuous map $f_i : X_i \to Y$ such that $h | H_n^T(X_i) = (f_i)_*^T$ by induction hypothesis. Combining f_E and f_i $(i \in E)$, we get the desired map.

Let $Y = \tilde{\bigvee}_{j \in J} Y_j$. Then there exist base point preserving continuous maps $g_j : X \to Y_j$ $(j \in J)$ such that $g_j(x) = y_j$ and $(q_j)_*^T \circ h = (g_j)_*^T$. As before for each $i \in I$ and $\lambda \in \Lambda_i$, there exists a countable subset $C_{i\lambda}$ of J such that $h(H_n^T(X_{i\lambda})) \subset \prod_{j \in C_{i\lambda}} H_n^T(Y_j)$. We get $O_{i\lambda j} \subset X_{i\lambda}$ $(j \in C_{i\lambda})$ and $\sigma_{i\lambda j} : X_{i\lambda} \to X_{i\lambda}$ as before. Define $f : X \to Y$ by:

$$f(s) = \begin{cases} g_j \circ \sigma_{i\lambda j}(s) & \text{for } s \in O_{i\lambda j} \ (i \in I, \lambda \in \Lambda_i, j \in C_{i\lambda}); \\ \boldsymbol{e} & \text{otherwise.} \end{cases}$$

For each j, define a continuous map $\varphi_j: X\to X$ by:

$$\varphi_j|X_{i\lambda}=\sigma_{i\lambda j} \quad \text{if } j\in C_{i\lambda};\ \varphi_j(X_{i\lambda})=\{x\} \text{ otherwise.}$$

Then $q_j\circ f=g_j\circ\varphi_j$ for each $j\in J$, hence f is continuous. It follows that

$$(q_j)_*^T\circ f_*^T|H_n^T(X_{i\lambda})=(g_j\circ\varphi_j)_*^T|H_n^T(X_{i\lambda})=(g_j)_*^T|H_*^n(X_{i\lambda})=(q_j)_*^T\circ h|H_n^T(X_{i\lambda})$$

$$\text{for } i\in I,\ \lambda\in\Lambda_i \text{ and } j\in C_{i\lambda} \quad \text{and}$$

$$(q_j)_*^T\circ f_*^T|H_n^T(X_{i\lambda})=0=(q_j)_*^T\circ h|H_n^T(X_{i\lambda}) \quad \text{otherwise.}$$

Therefore $h=f_*^T$ by Lemma A.10.

Next we show that slenderness of groups can be characterized using the notion of spatial homomorphisms. Let $\boldsymbol{H}=\tilde{\bigvee}_{i\in N}\boldsymbol{S}_i^1$ be the Hawaiian earring, where $\boldsymbol{S}_i^1$ is a copy of $\boldsymbol{S}^1$ for $i\in\boldsymbol{N}$, and $\boldsymbol{K}(A, 1)$ be an Eilenberg-Maclane complex [23].

THEOREM 6.3. *The following statements are equivalent for a group* A.

(1) *A is slender;*

(2) *For a path connected space X with $H_1(X)\cong A$, any homomorphism $h: H_1(\boldsymbol{H})\to H_1(X)$ is spatial;*

(2^T) *For a path connected space X with $H_1^T(X)\cong A$, any homomorphism $h: H_1^T(\boldsymbol{H})\to H_1^T(X)$ is spatial;*

(3) *Any homomorphism $h: H_1(\boldsymbol{H})\to H_1(\boldsymbol{K}(A, 1))$ is spatial;*

(3^T) *Any homomorphism $h: H_1^T(\boldsymbol{H})\to H_1^T(\boldsymbol{K}(A, 1))$ is spatial.*

PROOF. The implications (2)→(3) and (2^T)→(3^T) are obvious. Since $H_1(\boldsymbol{K}(A, 1))=H_1^T(\boldsymbol{K}(A, 1))$ by Corollary 2.2, (3)→(3^T) follows from Proposition 1.1.

(1)→(2^T): By Theorem 4.6, $H_1^T(\boldsymbol{H})\cong\prod_{i\in N}H_1^T(\boldsymbol{S}_i^1)$ ($\cong\boldsymbol{Z}^N$) naturally. There exists $n\in\boldsymbol{N}$ such that $h(\prod_{i\geq n}H_1^T(\boldsymbol{S}_i^1))=\{0\}$. Fix a point $x\in X$. Since X is path connected, there exist continuous maps $f_i: \boldsymbol{S}_i^1\to X$ ($i<n$) such that $f_i(\boldsymbol{e})=x$ and $(f_i)_*^T=h|H_1^T(\boldsymbol{S}_i^1)$. Define $f: \boldsymbol{H}\to X$ by: $f|\boldsymbol{S}_i^1=f_i$ ($i<n$) and $f(\bigvee_{i\geq n}\boldsymbol{S}_i^1)=\{x\}$. Then $h=f_*^T$.

(1)→(2): Here we use some results and notation in [25]. By Griffith' theorem [25, Theorem A.1], $\pi_1(\boldsymbol{H}, \boldsymbol{e})\cong\circledast_N\boldsymbol{Z}$ hence $H_1(\boldsymbol{H})\cong Ab(\circledast_N\boldsymbol{Z})$. Since $H_1^T(\boldsymbol{H})\cong\boldsymbol{Z}^N$ canonically, $\operatorname{Ker}\sigma_{\boldsymbol{H}}\cong C_N/(\circledast_N\boldsymbol{Z})'$, which is complete mod-$U$ by Theorems 3.3 and 4.7 of [25]. Consequently $h(\operatorname{Ker}\sigma_{\boldsymbol{H}})=\{0\}$, hence, there exists a homomorphism $\bar{h}: H_1^T(\boldsymbol{H})\to H_1(X)$ such that $h=\bar{h}\circ\sigma_{\boldsymbol{H}}$. Similarly as above we define f_i's so that $(f_i)_*=h|H_1(\boldsymbol{S}_i^1)$ and also f. Since h only depends on the direct summands $H_1(\boldsymbol{S}_i^1)$ ($1\leqq i<n$) of $H_1(\boldsymbol{H})$, we get $h=f_*$.

$(3^T)\to(1)$: Let $h: \boldsymbol{Z}^N\to A$ be a homomorphism, where we identify $\boldsymbol{Z}^N$ and A with $H_1^T(\boldsymbol{H})$ and $H_1^T(\boldsymbol{K}(A, 1))$ respectively. Let $f: \boldsymbol{H}\to\boldsymbol{K}(A, 1)$ be a continuous map with $f_*^T=h$. Since $\boldsymbol{K}(A, 1)$ is locally contractible, there exists $n\in\boldsymbol{N}$ such that $f_*^T(H_1^T(\boldsymbol{S}_i^1))=\{0\}$ for $i<n$. Hence A is slender.

REMARK 6.4. (1) Here we show that Theorem 6.2 does not hold for singular homology groups. By [25, Theorem 4.14], $H_1(\boldsymbol{H})$ contains a subgroup isomorphic to $\boldsymbol{Q}$. Since $H_n^T(\boldsymbol{H})\cong\boldsymbol{Z}^N$ is a torsion-free abelian group of cardi nality $2^{\aleph_0}$ and $\sigma_{\boldsymbol{H}}: H_1(\boldsymbol{H})\to H_1^T(\boldsymbol{H})$ is an epimorphism, there exist $2^{2^{\aleph_0}}$-many endomorphisms on $H_1(\boldsymbol{H})$. On the other hand, there exist only $2^{\aleph_0}$-many continuous maps from $\boldsymbol{H}$ to itself. Hence, not all endomorphisms are induced by continuous maps.

(2) In case $X=\tilde{\prod}_{i\in I}\boldsymbol{S}_i^1$, any endomorphism h of $H_1(X)$ $(=H_1^T(X))$ is spatial. We show this as follows. Since $\pi_1(X)\cong\tilde{\prod}_I\boldsymbol{Z}$, $H_1(X)\cong\tilde{\prod}_I\boldsymbol{Z}$. Let $p_i: X\to\boldsymbol{S}_i^1$ $(i\in I)$ and $p_F: X\to\prod_{i\in F}\boldsymbol{S}_i^1$ $(F\subset I)$ be the projections, where we consider $\boldsymbol{S}_i^1$ and $\prod_{i\in F}\boldsymbol{S}_i^1$ as subspaces of X. $\mathrm{Hom}(\tilde{\prod}_{i\in I}A_i, \boldsymbol{Z})\cong\bigoplus_{i\in I}\mathrm{Hom}(A_i, \boldsymbol{Z})$ [8], hence for any $i\in I$ there exists a finite subset F_i of I such that $(p_i)_*\circ h=(p_i)_*\circ h\circ(p_{F_i})_*$. Since any homomorphism from $H_1(\prod_{j\in F_i}\boldsymbol{S}_j^1)$ to $H_1(\boldsymbol{S}_i^1)$ is spatial, there exists a continuous map $f_i: \prod_{j\in I}\boldsymbol{S}_j^1\in\boldsymbol{S}_i$ such that $(p_i)_*\circ h=(f_i)_*$ $(=(f_i)_*^T)$ for each i. Define a continuous map $f: X\to X$ by: $p_i\circ f=f_i$ for every i. Then $h=f_*$.

In case $X=\prod_I\boldsymbol{S}^1$, the situation is a little different. (Recall that this is a canonical compact abelian group.) If the cardinality of I is less than the least measurable cardinal, then every endomorphism of $H_1(X)$ is spatial, since $\mathrm{Hom}(\prod_{i\in I}A_i, \boldsymbol{Z})\cong\bigoplus_{i\in I}\mathrm{Hom}(A_i, \boldsymbol{Z})$ [11, § 94]. Otherwise, there exist non-spatial endomorphisms of $H_1(X)$ $(\cong H_1^T(X))$. To see this, let p_i and p_F be the projections as above. Take a non-principal countably complete ultrafilter $\mathfrak{F}$ on I and define a homomorphism $h: H_1(X)\to H_1(\boldsymbol{S}_{i_0}^1)$ $(\subset H_1(X))$ by:

$$h(u+B_1)=a \qquad \text{iff } \{i\in I: p_i\circ u+B_1=a\}\in\mathfrak{F}\,,$$

where $u\in C(\Delta_1, X)\cap Z_1(X)$ and $a\in H_1(\boldsymbol{S}_{i_0}^1)$ $(\cong\boldsymbol{Z})$. Suppose that $h=f_*$ for some continuous map $f: X\to X$. Then $(p_{i_0}\circ f)_*=f_*$ holds. We define $u\in C(\Delta_1, \boldsymbol{S}^1)\cap Z_1(\boldsymbol{S}^1)$ by: $u(x, y)(i)=(\cos 2\pi x, \sin 2\pi x)$ for $(x, y)\in\Delta_1$ and $i\in I$. Then $h(u+B_1)\neq 0$. Let $\varepsilon>0$ be so small that $\rho(v, w)<\varepsilon$ implies that $v\underset{h}{\sim}w$ for $v, w\in C(\Delta_1, \boldsymbol{S}^1)\cap Z_1(\boldsymbol{S}^1)$. Choose basic open sets $V_1, \cdots, V_m\subset X$ so that $\mathrm{Im}\,u\subset\bigcup_{j=1}^m V_j$ and $\sup\{\rho(p_{i_0}\circ f(x), p_{i_0}\circ f(y)): x, y\in V_j\}<\varepsilon$ for $1\leqq j\leqq m$. Then there exists a finite $F\subset I$ such that every V_j depends on $\prod_{i\in F}\boldsymbol{S}_i^1$. Let $v=p_F\circ u$. Then $\mathrm{Im}\,v\subset\bigcup_{j=1}^m V_j$ and $p_{i_0}\circ f\circ u\underset{h}{\sim}p_{i_0}\circ f\circ v$, hence $h(u+B_1)=h(v+B_1)$. However, $p_i\circ v\in B_1$

for $i \notin F$ and hence $(p_{i_0} \circ f)_*(v+B_1)=h(v+B_1)=0$ by the definition of h. This contradicts to $h(u+B_1) \neq 0$.

(3) Let $\boldsymbol{T}=\boldsymbol{S}_1^1 \times \boldsymbol{S}_2^1$ be the torus with base point $\boldsymbol{e}$. Then any endomorphism of $H_1(\boldsymbol{T})$ $(=H_1^T(\boldsymbol{T}))$ is spatial, but Theorem 6.2 does not hold for $\boldsymbol{T}$-piled spaces. To see this, take free generators e_1, e_2, e_1^i, e_2^i $(i=1, 2)$ so that $H_1(\boldsymbol{T}) \cong \langle e_1, e_2 \rangle$ and $H_1(\boldsymbol{T} \vee \boldsymbol{T}) \cong \langle e_1^1, e_2^1 \rangle \oplus \langle e_1^2, e_2^2 \rangle$ naturally. Let $h: H_1(\boldsymbol{T}) \to H_1(\boldsymbol{T} \vee \boldsymbol{T})$ be the homomorphism such that $h(e_i)=e_i^i$ for $i=1, 2$. Suppose that $h=f_*$ $(=f_*^T)$ for some continuous map $f: \boldsymbol{T} \to \boldsymbol{T} \vee \boldsymbol{T}$. Let $p: \boldsymbol{T} \vee \boldsymbol{T} \to \boldsymbol{S}^1 \vee \boldsymbol{S}^1$ be the projection so that p projects the first torus to the first coordinate $\boldsymbol{S}^1$ and the second to the second. Then $p \circ f$ induces a homomorphism from $\pi_1(\boldsymbol{T})$ to $\pi_1(\boldsymbol{S}^1 \vee \boldsymbol{S}^1)$ $(\cong \boldsymbol{Z} * \boldsymbol{Z})$ and $\operatorname{Im} p_* h \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$.

(Diagram)
$$\begin{array}{ccccc} \pi_1(T) & \longrightarrow & \pi_1(T \vee T) & \longrightarrow & \pi_1(S^1 \vee S^1) \cong Z * Z \\ \eta \downarrow & & \downarrow & & \downarrow \\ H_1(T) & \underset{f_*}{\longrightarrow} & H_1(T \vee T) & \underset{p_*}{\longrightarrow} & H_1(S^1 \vee S^1) \cong Z \oplus Z \end{array}$$

Since any nonzero abelian subgroup of $\boldsymbol{Z} * \boldsymbol{Z}$ is isomorphic to $\boldsymbol{Z}$ and the diagram commutes, $\operatorname{Im} p_* \circ f_* \circ \eta$ is isomorphic to $\boldsymbol{Z}$ or trivial, hence so is $\operatorname{Im} p_* \circ f_*$ because η is surjective. This is a contradiction.

(4) Let X be a connected 2-simplicial complex and Y a path connected space. Then the standard method shows that any homomorphism $\pi_1(X)$ to $\pi_1(Y)$ is induced by a continuous map from X to Y. On the other hand, there exists a 2-simplicial complex X with $\pi_1(X) \cong \boldsymbol{Z}^N$. Then $H_1^T(X) \cong H_1(X) \cong \boldsymbol{Z}^N$. However, we cannot replace $\boldsymbol{H}$ by such an X in Thoerem 6.3. In addition, though $H_1^T((\boldsymbol{S}^1)^N) \cong \boldsymbol{Z}^N$, we cannot replace $\boldsymbol{H}$ by $(\boldsymbol{S}^1)^N$ as the preceding (3) shows.

A. Appendix

All groups in the sequel are abelian groups. Here we prove a hierarchy theorem for the σ-Reid class of certain abelian groups, which is a version of Theorem 5 of [5] and Theorem 1 of [6] and corresponds to the class of X-piled spaces.

DEFINITION A.1. Let A be a group. The *σ-Reid class* of A is defined as the smallest class which contains A and also $\tilde{\prod}_{i \in I} X_i$ and $\bigoplus_{i \in I} X_i$ when each X_i $(i \in I)$ belongs to the class. A group in the σ-Reid class of A is called a *σ-Reid(A) group*. Along the inductive definition, we define *types* for σ-Reid(A) groups. Orders are the ordinals of types as in Definition 4.8.

(1) $\mathrm{ty}(A)=(0, \boldsymbol{P})=(0, \boldsymbol{S})$;

(2) Let X_i $(i\in I)$ be σ-Reid(A) groups of type $\mathrm{ty}(X_i)$, where $|I|\geqq 2$.

Case (a): $\sup^+\{\mathrm{o}(X_i): i\in I\}=1$.

$$\mathrm{ty}(\oplus_{i\in I}X_i)=\begin{cases}(1, \boldsymbol{S}) & \text{if } I \text{ is infinite;}\\ (0, \boldsymbol{M}) & \text{otherwise.}\end{cases}$$

$$\mathrm{ty}(\tilde{\prod}_{i\in I}X_i)=\begin{cases}(1, \boldsymbol{P}) & \text{if } I \text{ is infinite;}\\ (0, \boldsymbol{M}) & \text{otherwise.}\end{cases}$$

Case (b): $\sup^+\{o(X_i): i\in I\}=\mu+1$ for $\mu\geqq 1$.

$$\mathrm{ty}(\oplus_{i\in I}X_i)=\begin{cases}(\mu+1, \boldsymbol{S}) & \text{if } I_P \text{ is infinite;}\\ (\mu, \boldsymbol{S}) & \text{if } I_P \text{ is empty;}\\ (\mu, \boldsymbol{M}) & \text{otherwise,}\end{cases}$$

$$\mathrm{ty}(\tilde{\prod}_{i\in I}X_i)=\begin{cases}(\mu+1, \boldsymbol{P}) & \text{if } I_S \text{ is infinite;}\\ (\mu, \boldsymbol{P}) & \text{if } I_S \text{ is empty;}\\ (\mu, \boldsymbol{M}) & \text{otherwise,}\end{cases}$$

where $I_P=\{i: \mathrm{ty}(X_i)=(\mu, \boldsymbol{P}) \text{ or } (\mu, \boldsymbol{M})\}$ and $I_S=\{i: \mathrm{ty}(X_i)=(\mu, \boldsymbol{S}) \text{ or } (\mu, \boldsymbol{M})\}$.

Case (c): $\sup^+\{\mathrm{o}(X_i): i\in I\}=\mu$ a limit ordinal.

$$\mathrm{ty}(\oplus_{i\in I}X_i)=(\mu, \boldsymbol{S}) \quad \text{and} \quad \mathrm{ty}(\tilde{\prod}_{i\in I}X_i)=(\mu, \boldsymbol{P}).$$

(Refer the remarks before Definition 4.8.)

By induction along the definition of σ-Reid(A) groups, we get the following Lemmas.

LEMMA A.2. *Let X be a σ-*Reid(A) *group and $(\mu, \Gamma)\leqq \mathrm{ty}(X)$. Then there exists a σ-*Reid(A) *group of type (μ, Γ) which is isomorphic to a summand of X.*

LEMMA A.3. *If a σ-*Reid(A) *group X is of type $(\mu, \boldsymbol{M})$ $(\mu\geqq 1)$ then there exist σ-*Reid(A) *groups Y_1 and Y_2 such that $Y_1\oplus Y_2\cong X$ and* $\mathrm{ty}(Y_1)=(\mu, \boldsymbol{P})$ *and* $\mathrm{ty}(Y_2)=(\mu, \boldsymbol{S})$.

Since the functors D and $n!$ not only commute with direct sums but also with σ-products, we get the next lemma.

LEMMA A.4. *If X is a σ-*Reid(A) *group of type (μ, Γ), then there exists a*

σ-Reid$(A/D(A))$ group of type (μ, Γ) which is isomorphic to $X/D(X)$ and also exists a σ-Reid$(n!A)$ group of type (μ, Γ) which is isomorphic to $n!X$.

The next lemma is a version of [14], but there exists no trap of the measurable cardinality, because we only deal with σ-products. (Direct products cause a trap of the measurable cardinality [11] and [6]. Using Π instead of $\tilde{\bigvee}$ in in the definition of piled spaces, we get a version of piled spaces. In this case, H_n^T corresponds to groups in the Reid class, which is obtained by using direct products instead of σ-products.) We investigate σ-Reid(A) groups for a group A with the properties $(\boldsymbol{P})$ or $(\boldsymbol{S})$ (Definition 4.9).

LEMMA A.5. *Let A_i $(i\in I)$ be groups and G_j $(j\in J)$ be reduced groups. For any homomorphism $h:\tilde{\prod}_{i\in I}A_i\to\bigoplus_{j\in J}G_j\,(=G)$, there exist finite sets $I'\subset I$, $J'\subset J$ and $n\in\boldsymbol{N}$ such that*

$$h(n\cdot\tilde{\prod}_{i\in I\setminus I'}A_i)\subset\bigoplus_{j\in J'}G_j\,.$$

PROOF. Since we want to apply Theorem 1(3) of [8], we use the same notion and notation. There exists a quasi-sheaf (S, ρ) over $\boldsymbol{P}(I)$ such that $S^{\wedge}$ is isomorphic to $\tilde{\prod}_{i\in I}A_i$. (See [8, Definition 1].) By Theorem 1(3) of [8], there exist countably complete maximal filters $F_1, \cdots, F_m$ of $\boldsymbol{P}(I)$ and an integer $n>0$ and a finite subset J' of J such that $h(n\cdot K_{F_1\cdots F_m})\subset\bigoplus_{j\in J'}G_j$. If F is a non-principal countably complete maximal filter of $\boldsymbol{P}(I)$ $K_F=\tilde{\prod}_{i\in I}A_i$ holds. Therefore we may assume that $F_1, \cdots, F_m$ are principal. There exist $\alpha_1, \cdots, \alpha_m\in I$ such that $K_{F_k}=\tilde{\prod}_{i\neq\alpha_k}A_i$. Let $I'=\{\alpha_1, \cdots, \alpha_m\}$. Then $K_{F_1\cdots F_k}=\bigcap_{k=1}^m K_{F_k}=\tilde{\prod}_{i\in I\setminus I'}A_i$ and we get the conclusion.

By using Lemma A.5 instead of Lemma 4 of [5], we prove the following lemma.

LEMMA A.6. *Let A be a reduced group. If any σ-Reid$(n!A)$ group of type (μ, Γ) is not isomorphic to a summand of any σ-Reid$(n!A)$ group of type (μ, ∇) for each $n\in\boldsymbol{N}$, then any σ-Reid$(n!A)$ group of type $(\mu+1, \nabla)$ is not isomorphic to a summand of any σ-Reid$(n!A)$ group of type $(\mu+1, \Gamma)$ for each $n\in\boldsymbol{N}$, where $\Gamma=\boldsymbol{P}$ and $\nabla=\boldsymbol{S}$, or $\Gamma=\boldsymbol{S}$ and $\nabla=\boldsymbol{P}$ respectively.*

PROOF. Suppose that a σ-Reid$(m!A)$ group X of type $(\mu+1, \nabla)$ is isomorphic to a summand of a σ-Reid$(m!A)$ group Y of type $(\mu+1, \Gamma)$.

First consider the case that $\Gamma=\boldsymbol{P}$ and $\nabla=\boldsymbol{S}$. Then by Lemma A.3, we may assume that $X=\bigoplus_{i\in I}X_i$ and $Y=\tilde{\prod}_{j\in J}Y_j$ where $\{i:\mathrm{ty}(X_i)=(\mu, \boldsymbol{P})\}$ is infinite and $\mathrm{ty}(Y_j)\leqq(\mu, \boldsymbol{S})$ for every $j\in J$. There exist $h:X\to Y$ and $\sigma:Y\to X$ such that $\sigma\circ h=\mathrm{id}$. By Lemma A.5, there exist $n\in\boldsymbol{N}\,(n>m)$ and finite sets $I'\subset I$, $J'\subset J$ such that $\sigma(n\cdots(m+1)\cdot\tilde{\prod}_{j\in J\setminus J'}Y_j)\subset\bigoplus_{i\in I'}X_i$. Then $\bigoplus_{i\in I\setminus I'}n\cdots(m+1)X_i$ is isomorphic to a summand of $\prod_{j\in J'}n\cdots(m+1)Y_j$. The latter group is isomorphic to a σ-Reid$(n!A)$ group of type less than or equal to $(\mu, \boldsymbol{S})$. There exists an $i\in I\setminus I'$ such that $\mathrm{ty}(X_i)=(\mu, \boldsymbol{P})$. Then $n\cdots(m+1)X_i$ is isomorphic to a σ-Reid $(n!A)$ group of type $(\mu, \boldsymbol{P})$ and thus we get a contradiction.

Next consider the case that $\Gamma=\boldsymbol{S}$ and $\nabla=\boldsymbol{P}$. Then as the former case, we may let $X=\tilde{\prod}_{i\in I}X_i$ and $Y=\bigoplus_{j\in J}Y_j$ where $\{i:\mathrm{ty}(X_i)=(\mu, \boldsymbol{S})\}$ is infinite and $\mathrm{ty}(Y_j)\leqq(\mu, \boldsymbol{P})$ for every $j\in J$. Let h and σ be as before. By Lemma A.5, there exist $n\in\boldsymbol{N}\,(n>m)$ and finite sets $I'\subset I$, $J'\subset J$ such that $h(n\cdots(m+1)\cdot\tilde{\prod}_{i\in I\setminus I'}X_i)\subset\bigoplus_{j\in J'}Y_j$. This implies that $\tilde{\prod}_{i\in I\setminus I'}n\cdots(m+1)X_i$ is isomorphic to a summand of $\bigoplus_{j\in J'}n\cdots(m+1)Y_j$, which induces a contradiction as the dual case.

THEOREM A.7. *Let X and Y be σ-*Reid(A) *groups and X be isomorphic to a summand of Y. Then,* $\mathrm{ty}(X)\leqq\mathrm{ty}(Y)$ *if A satisfies (Γ) and*

(a) $\mathrm{ty}(Y)=(n, \nabla)$ *for an odd $n\in\boldsymbol{N}$;*

(b) $\mathrm{ty}(Y)=(n, \Gamma)$ *for an even $n\in\boldsymbol{N}$; or*

(c) $o(Y)$ *is infinite,*

where (1) $\Gamma=\boldsymbol{P}$ *and* $\nabla=\boldsymbol{S}$*; or* (2) $\Gamma=\boldsymbol{S}$ *and* $\nabla=\boldsymbol{P}$*, respectively.*

Consequently, if A satisfies both $(\boldsymbol{S})$ *and* $(\boldsymbol{P})$ *and* $o(Y)\geqq 1$*, then* $\mathrm{ty}(X)\leqq\mathrm{ty}(Y)$*. Therefore, isomorphic σ-*Reid(A) *groups of nonzero order have the same type.*

PROOF. If X is isomorphic to a summand of Y, then $X/D(X)$ is isomorphic to a summand of $Y/D(Y)$. Therefore we may assume $D(A)=0$ by Lemma A.4.

(Finite case): From Lemma A.6, it suffices to show the case of $n=1$. We use the notation in the proof of Lemma A.6. Suppose that $\mathrm{ty}(X)\not\leqq\mathrm{ty}(Y)$. If A satisfies $(\boldsymbol{S})$, $\mathrm{ty}(Y)=(1, \boldsymbol{P})$ and $\mathrm{ty}(X)\geqq(1, \boldsymbol{S})$. By Lemma A.2, we may assume that $X=\bigoplus_I A$ and $Y=\tilde{\prod}_J A$ for infinite index sets I and J. By Lemma A.5, there exist $m\in\boldsymbol{N}$ and finite sets $I'\subset I$, $J'\subset J$ such that $\sigma(m!\cdot\tilde{\prod}_{J\setminus J'}A)\subset\bigoplus_{I'}m!A$. This implies that $\bigoplus_{I\setminus I'}(m!A)$ is isomorphic to a summand of $(m!A)^{|J'|}$, which contradicts to $(\boldsymbol{S})$. In case A satisfies $(\boldsymbol{P})$, we can perform the proof similarly.

(Infinite case): Suppose that the lemma holds in both cases $\mathrm{ty}(Y)=(\mu, \boldsymbol{P})$ and $\mathrm{ty}(Y)=(\mu, \boldsymbol{S})$. Then it holds in cases $\mathrm{ty}(Y)=(\mu+1, \boldsymbol{P})$ and $\mathrm{ty}(Y)=(\mu+1, \boldsymbol{S})$ by Lemmas A.6 and A.2. Then it also holds for the case $\mathrm{ty}(Y)=(\mu, \boldsymbol{M})$. The only remaining cases are $\mathrm{ty}(Y)=(\mu, \boldsymbol{P})$ and $\mathrm{ty}(Y)=(\mu, \boldsymbol{S})$ for a limit μ. We can prove similarly to Lemma A.6 by using Lemma A.2.

CORRLLARY A.8. *Let A be a group of finite rank. If A is not isomorphic to a direct sum of a divisible group and a bounded group, then isomorphic σ-*Reid(A) *groups have the same type.*

PROOF. Since $A/D(A)$ is a group of finite rank and unbounded, $m\cdot A/D(A)$ is a nonzero group of finite rank for each $m\in\boldsymbol{N}$. Therefore A satisfies $(\boldsymbol{P})$ and $(\boldsymbol{S})$. Moreover, A^m is isomorphic to A only when $m=1$. Now the corollary follows from Theorem A.7.

There exist many groups of infinite rank which satisfy either $(\boldsymbol{P})$ or $(\boldsymbol{S})$. For example, unbounded almost slender groups satisfy $(\boldsymbol{P})$ and unbounded, reduced Fuchs-44-groups satisfy $(\boldsymbol{S})$. We refer the reader to [7], [10] and [15] foɹ those groups.

Recall that a group is cotorsion-free if it does not contain a nonzero cotorsion group, and that a homomorphic image of an algebraically compact group is cotorsion [11]. A slender group is cotorsion-free and the class of cotorsion-free groups is closed under direct products and subgroups. To prove a lemma about homomorphisms between σ-Reid(A) groups, we introduce a concept corresponding to basic components of X-piled spaces.

DEFINITION A.9. Basic components of σ-Reid(A) groups are inductively defined as follows:

(1) A itself is the only basic component of A;

(2) Let $A_{i\lambda}$ $(\lambda\in\Lambda_i)$ be all the basic components of a σ-Reid(A) group X_i for each $i\in I$. Then $A_{i\lambda}$ $(\lambda\in\Lambda_i,\ i\in I)$ are all the basic components of both $\tilde{\prod}_{i\in I}X_i$ and $\bigoplus_{i\in I}X_i$.

Note that a basic component of a σ-Reid(A) group X is isomorphic to A and is a subgroup of X. The next is a lemma for Theorem 6.2.

LEMMA A.10. *Let A be a cotorsion-free, and X and Y be σ-*Reid(A) *groups and g, $h\in$*Hom(X, Y). *If $g|A_\lambda=h|A_\lambda$ for each basic component A_λ of X, then $g=h$.*

PROOF. We prove by induction on the definition a σ-Reid(A) group X.

(1) In case $X=A$, there is nothing to prove.

(2) In case $X=\bigoplus_{i\in I}X_i$, it is clear from Hom$(X, Y)\cong\prod_{i\in I}Hom(X_i, Y)$ and induction hypothesis. In case $X=\tilde{\prod}_{i\in I}X_i$, we can see that $g|\bigoplus_{i\in I}X_i=h|\bigoplus_{i\in I}X_i$

by the preceding case. Since $\tilde{\prod}_{i\in I}X_i/\bigoplus_{i\in I}X_i$ is algebraically compact [11, Theorem 42.1] and Y is cotorsion-free, we conclude $g=h$.

In the remaining part of this appendix, we state about $C(X, \boldsymbol{Z})$ for a scattered subspace X of $\boldsymbol{R}$. First recall definitions concerning scatteredness. For a space X, let X' be the subset of X consisting of all accumulation points. Let $X_0=X$ and define $X_\alpha=X'_\beta$ for $\alpha=\beta+1$ and $X_\alpha=\bigcap_{\beta<\alpha}X_\beta$ for a limit ordinal α. We say that X is *scattered* if $X_\alpha=\emptyset$ for some α. For a scattered space X, let $r(X)$ be the least ordinal α such that $X_\alpha=\emptyset$ and $r(x)=\max\{\alpha : x\in X_\alpha\}$ for $x\in X$.

PROPOSITION A.11. *Let A be a group with the discrete topology.*

(1) *Any σ-*Reid(A) *group defined by using only index sets of countable cadinalities is isomorphic to $C(X, A)$ for some scattered subspace X of $\boldsymbol{R}$.*

(2) *For any scattered subspace X of $\boldsymbol{R}$, $C(X, A)$ is isomorphic to a σ-*Reid(A) *group defined by using index sets of countable cardinalities.*

PROOF. (1) To prove by induction, it suffices to realize $\bigoplus_{n\in N}C(X_n, A)$ and $\prod_{n\in N}C(X_n, A)$ for scattered subspaces X_n of $\boldsymbol{R}$. The only nontrivial case is $\bigoplus_{n\in N}C(X_n, A)$. We may assume $X_n\subset(1/n+1, 1/n)$ and $C(X_n, A)\neq 0$. Let $X=\{0\}\cup\bigcup_{n\in N}X_n\subset\boldsymbol{R}$. Then X is scattered and $C(X, A)\cong A\oplus\bigoplus_{n\in N}C(X_n, A)$. Since every $C(X_n, A)$ contains a summand isomorphic to A, $C(X, A)\cong\bigoplus_{n\in N}C(X_n, A)$.

(2) This is shown by induction on $r(X)$. We remark that for any scattered subspace X of $\boldsymbol{R}$, $r(X)$ is countable and hence X is countable and 0-dimensional. If $r(X)=1$, X is discrete and hence $C(X, A)$ is isomorphic to A^X, which is a σ-Reid(A) groups. In case $r(X)=\beta+1$, X_β is discrete. We can take clopen subsets U_x of X $(x\in X_\beta)$ so that $\{x\}=U_x\cap X_\beta$, $U_x\cap U_y=\emptyset$ and $\bigcup_{x\in X_\beta}U_x=X$. Now we work in U_x. Take clopen subsets V_n $(n\in\boldsymbol{N})$ of U_x so that $V_m\cap V_n=\emptyset$ $(m\neq n)$, $\bigcup_{n\in N}V_n\cup\{x\}=U_x$ and $\bigcup_{n\geq n}V_n\cup\{x\}$ $(m\in\boldsymbol{N})$ form neighborhood bases of x. Then $C(U_x, A)\cong A\oplus\bigoplus_{n\in N}C(V_n, A)$, which is isomorphic to a σ-Reid(A) group because $r(V_n)\leq\beta$. Therefore $C(X, A)\cong\prod_{x\in X_\beta}C(U_x, A)$ is also isomorphic to a σ-Reild(A) group. In case $r(X)$ is limit, let $X=\{x_n : n\in\boldsymbol{N}\}$. Since $r(x_n)<r(X)$, we can successively take clopen subsets U_n of X so that $U_i\cap U_j=\emptyset$ $(i\neq j)$, $r(U_n)<r(X)$ and $x_n\in\bigcup_{i=1}^n U_i$. Then $\bigcup_{n\in N}U_n=X$ and hence $C(X, A)\cong\prod_{n\in N}C(U_n, A)$, which is isomorphic to a σ-Reid(A) group by the induction hypothesis.

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