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A CHARACTERIZATION OF REAL HYPERSURFACES OF QUATERNIONIC PROJECTIVE SPACE

By

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Abstract. We study a condition that allows us to characterize all real hypersurfaces of quaternionic projective space known until now.

1. Introduction.

Let M be a connected real hypersurface of the quaternionic projective space QP^m, $m\geqq 2$, endowed with the metric g of constant quaternionic sectional curvature 4. Let N be a unit local normal vector field on M and $U_k=-J_kN$, $k=1, 2, 3$, where $\{J_k\}_{k=1,2,3}$ is a local basis of the quaternionic structure of QP^m. Let us denote by $\boldsymbol{D}^{\perp}=\mathrm{Span}\{U_1, U_2, U_3\}$ and by $\boldsymbol{D}$ its orthogonal complement in TM.

Recently, [1], J. Berndt, generalizing previous results of A. Martínez and the author, [6], has proved that any real hypersurface of QP^m, $m\geqq 2$, such that $g(A\boldsymbol{D}, \boldsymbol{D}^{\perp})=\{0\}$, where A is the Weingarten endomorphism of M, is congruent to an open subset of either a tube of radius r, $0<r<\Pi/2$, over the canonically (totally geodesic) embedded quaternionic projective space QP^k, $k\in\{0, \cdots, m-1\}$ or a tube of radius r, $0<r<\Pi/4$, over the canonically (totally geodesic) embedded complex projective space CP^m.

In [5], A. Martínez introduced ruled real hypersurfaces of QP^m, obtaining several examples, as real hypersurfaces for which the distribution $\boldsymbol{D}$ is integrable. This is equivalent to the fact that $g(A\boldsymbol{D}, \boldsymbol{D})=\{0\}$, [7].

Moreover, if M is a real hypersurface of QP^m, TM is a subbundle of TQP^m over M and $T^{\circ}M=\{X\in TM\mid X\perp U_i,\ i=1, 2, 3\}$ is a subbundle of TM. Both TM and $T^{\circ}M$ have metric connections induced from TQP^m. The orthogonal complement of $T^{\circ}M$ in TQP^m is denoted by $N^{\circ}M$, which is also a subbundle of TQP^m with the induced metric connection.

Denote by $\bar{\nabla}$, ∇, ∇° and $\nabla^{\perp}$ the connections of TQP^m, TM, $T^{\circ}M$ and $N^{\circ}M$, respectively. Then we have

$$\nabla_XY=\nabla^{\circ}_XY+A_1(X)(Y) \tag{1.1}$$

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(1.2) $$\bar{\nabla}_X Y = \nabla^\circ_X Y + A_2(X)(Y)$$

for any $Y \in C^\infty(T^\circ M)$ and $X \in TM$, where A_1 and A_2 are the second fundamental forms of the subbundle $T^\circ M$ in TM and TQP^m, respectively. Set $A^\circ = A_2|_{T^\circ M}$, which is a smooth section of Hom($T^\circ M$, Hom($T^\circ M$, $N^\circ M$)). The connection on Hom($T^\circ M$, Hom($T^\circ M$, $N^\circ M$)) is also denoted by ∇°. The covariant derivative of A° is defined by

(1.3) $$(\nabla^\circ_X A^\circ)(Y)(Z) = \nabla^\perp_X A^\circ(Y)(Z) - A^\circ(\nabla^\circ_X Y)(Z) - A^\circ(Y)(\nabla^\circ_X Z)$$

for any $X \in TM$ and $Y, Z \in C^\infty(T^\circ M)$.

M is said to be η-parallel if $g((\nabla_X A)Y, Z) = 0$ for any $X, Y, Z \in \boldsymbol{D}$. It is easy to see from the expressions of their Weingarten endomorphisms, [6], that any tube of radius r, $0 < r < \Pi/2$, over QP^k, $k \in \{0, \cdots, m-1\}$ or of radius r, $0 < r < \Pi/4$ over CP^m is η-parallel. Also any ruled real hypersurface of QP^m is η-parallel.

On the other hand, we say that A° is η-parallel if $\nabla^\circ_X A^\circ \equiv 0$, for any $X \in C^\infty(T^\circ M)$.

The purpose of the present paper is to give a characterization of all the real hypersurfaces of QP^m, $m \geqq 2$, known until now by the following.

THEOREM. *Let M be a real hypersurface of QP^m, $m \geqq 2$. Then A° is η-parallel and A is η-parallel if and only if M is congruent to an open subset of either*

i) *A tube of radius r, $0 < r < \Pi/2$, over the canonically (totally geodesic) embedded quaternionic projective space QP^k, for some $k \in \{0, \cdots, m-1\}$, or*

ii) *A tube of radius r, $0 < r < \Pi/4$, over the canonically (totally geodesic) embedded complex projective space CP^m, or*

iii) *A ruled real hypersurface.*

REMARK. The conditions of η-paralleiness of A and A° for real hypersurfaces of complex projective space have been studied by M. Kimura and S. Maeda, [3], and S. Maeda and S. Udagawa, [4], respectively.

2. Preliminaries.

Let M be a real hypersurface (any real hypersurface is considered to be connected in the following) of QP^m, $m \geqq 2$, and $\{J_1, J_2, J_3\}$ a local basis of the quaternionic structure of QP^m, see [2], N a local normal unit vector field on M and $U_i = -J_i N$, $i = 1, 2, 3$.

Let X be a tangent vector field on M. We write $J_i X = \phi_i X + f_i(X) N$, $i =$

1, 2, 3, where $\phi_i X$ is the tangent component of $J_i X$, and $f_i(X)=g(X, U_i)$, $i=1, 2, 3$. As $J_i^2=-Id$, $i=1, 2, 3$, where Id denotes the identity endomorphism on TQP^m, we get

$$\phi_i^2 X=-X+f_i(X)U_i, \qquad i=1, 2, 3, \tag{2.1}$$

and

$$f_i(\phi_i X)=0, \qquad \phi_i U_i=0, \qquad i=1, 2, 3, \tag{2.2}$$

for any X tangent to M.

As $J_i J_j=-J_j J_i=J_k$, where (i, j, k) is a cyclic permutation of (1, 2, 3), we obtain

$$\phi_i X=\phi_j\phi_k X-f_k(X)U_j=-\phi_k\phi_j X+f_j(X)U_k \tag{2.3}$$

and

$$f_i(X)=f_j(\phi_k X)=-f_k(\phi_j X) \tag{2.4}$$

for any X tangent to M, where (i, j, k) is a cyclic permutation of (1, 2, 3). It is also easy to see that for any X, Y tangent to M and $i=1, 2, 3$,

$$g(\phi_i X, Y)+g(X, \phi_i Y)=0, \qquad g(\phi_i X, \phi_i Y)=g(X, Y)-f_i(X)f_i(Y) \tag{2.5}$$

and

$$\phi_i U_j=U_k=-\phi_j U_i, \tag{2.6}$$

(i, j, k) being a cyclic permutation of (1, 2, 3).

The formulae of Gauss and Weingarten of M in QP^m are given, respectively, by

$$\bar{\nabla}_X Y=\nabla_X Y+g(AX, Y)N \tag{2.7}$$

$$\bar{\nabla}_X N=-AX \tag{2.8}$$

for any X, Y tangent to M, A being the Weingarten endomorphism of the immersion. From the expression of the curvature tensor of QP^m, $m\geqq 2$, we have that the equation of Codazzi is given by

$$(\nabla_X A)Y-(\nabla_Y A)X=\sum_{i=1}^{3}\{f_i(X)\phi_i Y-f_i(Y)\phi_i X+2g(X, \phi_i Y)U_i\} \tag{2.9}$$

for any X, Y tangent to M.

The covariant derivatives of J_i, $i=1, 2, 3$, are given by $\bar{\nabla}_X J_i=p_j(X)J_k-p_k(X)J_j$ for any $X\in TQP^m$, where p_i, $i=1, 2, 3$, are local 1-forms on QP^m. Then from (2.7) and (2.8) we obtain

$$\nabla_X U_i=-p_j(X)U_k+p_k(X)U_j+\phi_i AX \tag{2.10}$$

and

(2.11) $(\nabla_X \phi_i)Y = p_j(X)\phi_k Y - p_k(X)\phi_j Y + f_i(Y)AX - g(AX, Y)U_i$

for any X, Y tangent to M, (i, j, k) being a cyclic permutation of (1, 2, 3).

3. Proof of the theorem.

We first obtain the following Propositions.

PROPOSITION 1.

a) $A_1(X)(Y) = -\sum_{i=1}^3 g(\phi_i AX, Y)U_i$

b) $A_2(X)(Y) = g(AX, Y)N - \sum_{i=1}^3 g(\phi_i AX, Y)U_i$

c) $(\nabla_X^\circ \phi_i)(Y) = p_k(X)\phi_j Y - p_j(X)\phi_k Y$

d) $\nabla_X^\perp N = -\sum_{i=1}^3 g(AX, U_i)U_i$

e) $\nabla_X^\perp U_i = p_k(X)U_j - p_j(X)U_k + g(AX, U_i)N - g(AX, U_k)U_j + g(AX, U_j)U_k$

for any $X \in TM$, $Y \in C^\infty(T^\circ M)$, (i, j, k) *being a cyclic permutation of* (1, 2, 3).

The proof is straightforward bearing in mind (1.1), (1.2) and (2.1) to (2.11).

From (1.3) and Proposition 1 we have

PROPOSITION 2. *For any* $X \in TM$, $Y, Z \in C^\infty(T^\circ M)$,

$$(\nabla_X^\circ A^\circ)(Y)(Z) = \psi(X, Y, Z)N + \sum_{i=1}^3 \psi(X, Y, \phi_i Z)U_i$$

where

(3.1) $$\psi(X, Y, X) = g((\nabla_X A)Y, Z) - \sum_{i=1}^3 \{f_i(AZ)g(\phi_i AX, Y) + f_i(AY)g(\phi_i AX, Z) + f_i(AX)g(\phi_i AY, Z)\}.$$

From Proposition 2 A° is η-parallel if and only if $\psi(X, Y, Z)=0$ for any $X, Y, Z \in C^\infty(T^\circ M)$, that is,

(3.2) $$g((\nabla_X A)Y, Z) = \sum_{i=1}^3 \{f_i(AZ)g(\phi_i AX, Y) + f_i(AY)g(\phi_i AX, Z) + f_i(AX)g(\phi_i AY, Z)\}.$$

If A is η-parallel, $g((\nabla_X A)Y, Z)=0$. From this and (3.2), if $g(A\boldsymbol{D}, \boldsymbol{D}^\perp) = \{0\}$, we obtain that M is locally congruent to 1) on 2) in Theorem, [1].

Let us now suppose that $g(A\boldsymbol{D}, \boldsymbol{D}^\perp) \neq \{0\}$. From (3.2) we see that

(3.3) $$\sum_{i=1}^3 f_i(AZ)g((A\phi_i + \phi_i A)X, Y) = 0$$

for any $X, Y, Z \in C^\infty(T^\circ M)$. We can have the following cases:

CASE 1. $(AU_i)^D$, $i=1, 2, 3$, are linearly independent, where $(AU_i)^D$ denotes the $\boldsymbol{D}$-component of AU_i. From (3.3) we have that $g((A\phi_i+\phi_i A)X, Y)=0$ for any $X, Y\in\boldsymbol{D}$, $i=1, 2, 3$. But this is equivalent to $\boldsymbol{D}$ to be integrable and then M must be ruled.

CASE 2. $(AU_i)^D$, $i=1, 2, 3$, are linearly dependent. We distinguish the following subcases:

CASE 2-i. $(AU_2)^D=(AU_3)^D=0$ and $(AU_1)^D\neq 0$. We suppose that $AU_1=\gamma X_1+\beta_1 U_1+\beta_2 U_2+\beta_3 U_3$. From (3.3) we obtain $g((\phi_1 A+A\phi_1)X, Y)=0$ for any $X, Y\in C^\infty(T^\circ M)$. Thus for any $X\in C^\infty(T^\circ M)$, $\phi_1 AX+A\phi_1 X=\alpha_1(X)U_1+\alpha_2(X)U_2+\alpha_3(X)U_3$. From (3.3) taking $Z=X_1$ we have $\gamma g(\phi_1 AX, Y)=0$ for any $X, Y\in C^\infty(T^\circ M)$. Thus either $\gamma=0$ which is impossible or $g(\phi_1 AX, Y)=0$. But also $g(\phi_1 AX, U_i)=0$, $i=1, 2, 3$. Thus $\phi_1 AX=0$ for any $X\in C^\infty(T^\circ M)$. This means that for such an X, $A\phi_1 X=\alpha_1(X)U_1+\alpha_2(X)U_2+\alpha_3(X)U_3$. From (2.1) it follows that $-AX=\alpha_1(\phi_1 X)U_1+\alpha_2(\phi_1 X)U_2+\alpha_3(\phi_1 X)U_3$ for any $X\in C^\infty(T^\circ M)$. Thus $g(A\boldsymbol{D}, \boldsymbol{D})=\{0\}$ and M is ruled.

CASE 2-ii. Let us suppose that $(AU_3)^D=0$ and $(AU_1)^D$, $(AU_2)^D$ are linearly independent. From (3.3) we have $g((\phi_i A+A\phi_i)X, Y)=0$, $i=1, 2$, for any $X, Y\in C^\infty(T^\circ M)$.

From (3.2) we also get $g(\phi_i AX, Y)=0$, $i=1, 2$, for any $X, Y\in C^\infty(T^\circ M)$. Thus $g(A\phi_1 X, Y)=0$ for any $X, Y\in C^\infty(T^\circ M)$. Then $A\phi_1 X=g(A\phi_1 X, U_1)U_1+g(A\phi_1 X, U_2)U_2$ for any $X\in C^\infty(T^\circ M)$. From (2.1), for such an X, $-AX=-f_1(AX)U_1-f_2(AX)U_2$ and then $g(A\boldsymbol{D}, \boldsymbol{D})=\{0\}$, thus M is ruled.

CASE 2-iii. Let us suppose that $(AU_3)^D=0$, $(AU_1)^D$, $(AU_2)^D\neq 0$ and linearly dependent. Thus we can write $AU_1=\mu_1 W+\gamma_1 U_1+\gamma_2 U_2+\gamma_3 U_3$, $AU_2=\mu_2 W+\gamma_2 U_1+\beta_2 U_2+\beta_3 U_3$, when $W\in\boldsymbol{D}$ is a unit vector field. From (3.3) we get $\mu_1 g(\phi_1 AX, Y)+\mu_2 g(\phi_2 AX, Y)=0$ for any $X, Y\in C^\infty(T^\circ M)$. Taking scalar products with U_i, $i=1, 2, 3$, it is easy to show that $\mu_1\phi_1 AX+\mu_2\phi_2 AX=0$ for any $X\in C^\infty(T^\circ M)$. Thus either $\mu_1=\mu_2=0$, which is impossible, or $\phi_1 AX$ and $\phi_2 AX$ are linearly dependent for any $X\in C^\infty(T^\circ M)$. Therefore we can have the following situations:

a) $AX=0$ for any $X\in C^\infty(T^\circ M)$. In this case M is ruled.

b) $AX\neq 0$ for some $X\in C^\infty(T^\circ M)$, $X\perp W$. Then from (2.3), $\phi_1 AX$ and $\phi_2 AX$ are orthogonal, wich is a contradiction with the fact that they are linearly dependent.

c) $AX=0$ for any $X\in C^\infty(T^\circ M)$, $X\perp W$, $AW\neq 0$ and $\mu_1\phi_1 AW+\mu_2\phi_2 AW=0$. Thus

(3.4) $$AW=g(AW, W)W+\mu_1U_1+\mu_2U_2.$$

But then we have $\mu_1g(\phi_1AW, \phi_1AW)+\mu_2g(\phi_2AW, \phi_1AW)=0$. From (2.3), this implies that $\mu_1g(AW, AW)-\mu_1^3-\mu_1\mu_2^2=0$. Thus either $\mu_1=0$ which is impossible or $g(AW, AW)=\mu_1^2+\mu_2^2$. But from (3.4) $g(AW, AW)=g(AW, W)^2+\mu_1^2+\mu_2^2$. Then $g(AW, W)=0$, which implies that $g(A\boldsymbol{D}, \boldsymbol{D})=\{0\}$ and M is ruled.

CASE 3. $(AU_i)^D\neq 0$, $i=1, 2, 3$ and are linearly dependent. We distinguish the following subcases.

CASE 3-i. $AU_1=\mu_1X_1+\alpha_1U_1+\alpha_2U_2+\alpha_3U_3$, $AU_2=\mu_2X_1+\alpha_2U_1+\gamma_2U_2+\gamma_3U_3$, $AU_3=\mu_3X_1+\alpha_3U_1+\gamma_3U_2+\delta_3U_3$, where $X_1\in\boldsymbol{D}$ is a unit vector field.

From (3.2), taking $Z=X_1$ and $X, Y\in C^\infty(T^\circ M)$ orthogonal to X_1 we have

(3.5) $$\mu_1g(\phi_1AX, Y)+\mu_2g(\phi_2AX, Y)+\mu_3g(\phi_3AX, Y)=0.$$

From (3.3) and (3.5) we obtain that $\mu_1\phi_1AX+\mu_2\phi_2AX+\mu_3\phi_3AX\in\boldsymbol{D}^\perp$ for any $X\in C^\infty(T^\circ M)$, and taking scalar products with U_i, $i=1, 2, 3$ we get

(3.6) $$\mu_1\phi_1AX+\mu_2\phi_2AX+\mu_3\phi_3AX=0, \qquad X\in C^\infty(T^\circ M).$$

Thus either $\mu_1=\mu_2=\mu_3=0$ which is impossible, or ϕ_iAX, $i=1, 2, 3$, are linearly dependent for any $X\in C^\infty(T^\circ M)$.

Then we have the following possibilities:

a) $AX=0$ for any $X\in\boldsymbol{D}$ and M is ruled.

b) $AX\neq 0$ for some $X\in C^\infty(T^\circ M)$, $X\perp X_1$. From (2.3), ϕ_iAX, $i=1, 2, 3$, are mutually orthogonal, which is a contradiction because from (3.6) they are linearly dependent.

c) $AX=0$ for any $X\in C^\infty(T^\circ M)$, $X\perp X_1$ and $AX_1\neq 0$. From (3.6) we get $\mu_1g(\phi_1AX_1, \phi_1AX_1)+\mu_2g(\phi_2AX_1, \phi_1AX_1)+\mu_3g(\phi_3AX_1, \phi_1AX_1)=0$. Then from (2.3) and (2.5), we obtain from this that $\mu_1g(AX_1, AX_1)-\mu_1^3-\mu_1\mu_2^2-\mu_1\mu_3^2=0$. Thus either $\mu_1=0$ and this case cannot occur or $g(AX_1, AX_1)=\mu_1^2+\mu_2^2+\mu_3^2$. But on the other hand, $g(AX_1, AX_1)=\mu_1^2+\mu_2^2+\mu_3^2+g(AX_1, X_1)^2$. Therefore $g(AX_1, X_1)=0$ and M is ruled. This finishes Case 3-i.

CASE 3-ii. $AU_1=\mu_1X_1+\alpha_1U_1+\alpha_2U_2+\alpha_3U_3$, $AU_2=\mu_2X_2+\alpha_2U_1+\beta_2U_2+\beta_3U_3$, $AU_3=\delta_1X_1+\delta_2X_2+\alpha_3U_1+\beta_3U_2+\gamma_3U_3$, where $X_1, X_2\in\boldsymbol{D}$ are orthonormal. From (3.2) we have

(3.7) $$\mu_1\phi_1AX+\delta_1\phi_3AX=\alpha(X)X_2$$

for any $X\in C^\infty(T^\circ M)$ and $X\perp X_2$ and

(3.8) $$\mu_2\phi_2AX+\delta_2\phi_3AX=\beta(X)X_1$$

for any $X\in C^{\infty}(T^{\circ}M)$ and $X\perp X_1$. Thus if $X\perp X_i$, $i=1, 2$, ϕ_jAX, $j=1, 2, 3$, are mutually orthogonal and from (3.7) and (3.8) we obtain $\delta_1\delta_2g(\phi_3AX, \phi_3AX)=0$. That is, we have the following possibilities:

a) $AX=0$ for any $X\in C^{\infty}(T^{\circ}M)$ orthogonal to X_1 and X_2. From (3.7) we get $\mu_1g(\phi_1AX_1, \phi_1X_2)+\delta_1g(\phi_3AX_1, \phi_1X_2)=0$. This implies

(3.9) $$\mu_1g(AX_1, X_2)-\delta_1g(AX_1, X_1)g(X_1, \phi_2X_2)=0.$$

But as $\phi_1AX_1=g(AX_1, X_1)\phi_1X_1+g(AX_1, X_2)\phi_1X_2-\delta_1U_2$ and $\phi_3AX_1=g(AX_1, X_1)\phi_3X_1+g(AX_1, X_2)\phi_3X_2+\mu_1U_2$, from (3.7), $\mu_1g(AX_1, X_1)\phi_1X_1+\mu_1g(AX_1, X_2)\phi_1X_2+\delta_1g(AX_1, X_1)\phi_3X_1+\delta_1g(AX_1, X_2)\phi_3X_2$ is proportional to X_2, and taking the scalar product with ϕ_3X_2 we obtain

(3.10) $$\mu_1g(AX_1, X_1)g(X_1, \phi_2X_2)+\delta_1g(AX_1, X_2)=0$$

Thus, from (3.9) and (3.10), if $g(AX_1, X_2)^2+g(AX_1, X_1)^2g(X_1, \phi_2X_2)^2\neq 0$, $\mu_1=\delta_1=0$, this is impossible. Therefore we suppose that $g(AX_1, X_2)^2+g(AX_1, AX_1)^2g(X_1, \phi_2X_2)^2=0$. On the other hand, as for (3.9) and (3.10) we can obtain

(3.11) $$-\mu_2g(AX_2, X_2)g(X_2, \phi_1X_1)+\delta_2g(AX_1, X_2)=0,$$

(3.12) $$\mu_2g(AX_1, X_2)+\delta_2g(AX_2, X_2)g(X_2, \phi_1X_2)=0.$$

Then if $g(AX_1, X_2)=g(AX_1, X_1)=0$, $AX_1=\mu_1U_1+\delta_1U_3$, and $AX_2=g(AX_2, X_2)X_2+\mu_2U_2+\delta_2U_3$ and if $g(AX_2, X_2)=0$, M is ruled. If $g(AX_2, X_2)\neq 0$, from (3.11) and (3.12), if $g(X_2, \phi_1X_1)\neq 0$, then $\mu_2=\delta_2=0$ which cannot occur. Thus we suppose that $g(AX_2, X_2)\neq 0$ and $g(X_2, \phi_1X_1)=0$. Then $A\phi_1X_1=A\phi_1X_2=0$ and $\phi_1AX_1=\delta_1U_2$. Then $g((\phi_1A+A\phi_1)X_1, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$. From (3.3), $\delta_1g((\phi_3A+A\phi_3)X_1, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$. Thus either $\delta_1=0$ or $g((\phi_3A+A\phi_3)X_1, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$.

If $\delta_1=0$, $\mu_1g((\phi_1A+A\phi_1)X_2, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$. Bearing in mind that $A\phi_1X_2=0$, $\mu_1g(\phi_1AX_2, \phi_1X_2)=0$ which implies that $\mu_1=0$ and this case cannot occur.

If $g((\phi_3A+A\phi_3)X_1, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$, as $\phi_3AX_1\in \boldsymbol{D}^{\perp}$, we get $g(A\phi_3X_1, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$. Then we can write $\phi_3X_1=g(\phi_3X_1, X_2)X_2+W$ where $W\perp X_i$, $i=1, 2$ is a unit vector field. Thus $A\phi_3X_1=g(\phi_3X_1, X_2)g(AX_2, X_2)X_2+U$, $U\in \boldsymbol{D}^{\perp}$. Then $g(\phi_3X_1, X_2)=0$ which implies that $A\phi_3X_1=A\phi_3X_2=0$. From (3.3) $\mu_2g((\phi_2A+A\phi_2)X_1, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$. As μ_2 must be nonnull, $g((\phi_2A+A\phi_2)X_1, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$. As $\phi_2AX_1\in \boldsymbol{D}^{\perp}$, $g(A\phi_2X_1, X)=0$ for any $X\in C^{\infty}(T^{\circ}M)$. But $\phi_2X_1=g(\phi_2X_1, X_2)X_2+\mu X$, where X is orthogonal to X_1 and X_2. Then $A\phi_2X_1=g(\phi_2X_1, X_2)g(AX_2, X_2)X_2+W'$, where $W'\in \boldsymbol{D}^{\perp}$. Thus $g(\phi_2X_1, X_2)=0$ and this implies that $A\phi_2X_1=A\phi_2X_2=0$. Then

$\mu_1\phi_1 AX_2+\delta_1\phi_3 AX_2=\mu_1 g(AX_2, X_2)\phi_1 X_2+\delta_1 g(AX_2, X_2)\phi_3 X_2+\mu_1\mu_2 U_3-\mu_1\delta_2 U_2-\delta_1\mu_2 U_1$ and $\mu_1 A\phi_1 X_2+\delta_1 A\phi_3 X_2=0$. From (3.3) adding these expressions, the result cannot have component in $\boldsymbol{D}$, thus $\mu_1=\delta_1=0$ and we have a contradiction.

Let us now suppose that $g(AX_1, X_2)=g(X_1, \phi_2 X_2)\neq 0$. Then we can write $g(AX_1, X_1)X_1+\mu_1 U_1+\delta_1 U_3=AX_1$ and $AX_2=g(AX_2, X_2)X_2+\mu_2 U_2+\delta_2 U_3$. Also we get $A\phi_2 X_2=A\phi_2 X_1=0$. From (3.11) and (3.12) we have $\mu_2 g(AX_2, X_2)g(X_2, \phi_1 X_1)=0=\delta_2 g(AX_2, X_2)g(X_2, \phi_1 X_1)$. The possible situations have been already studied except if $g(X_2, \phi_1 X_1)=0$. This implies that $A\phi_1 X_1=A\phi_1 X_2=0$. From (3.3) $\mu_1 g(\phi_1 AX_1, \phi_1 X_1)+\delta_1 g(\phi_3 AX_1, \phi_1 X_1)+\delta_1,\ g(A\phi_3 X_1, \phi_1 X_1)=0$. This implies that $\mu_1 g(AX_1, X_1)=0$. Thus $g(AX_1, X_1)=0$, and M is ruled.

b) $\delta_1=0$. As μ_1 must be nonnull, $\phi_1 AX=\alpha'(X)X_2$ for any $X\in C^\infty(T^\circ M)$, $X\perp X_2$. We now can write $J_1 AX=\phi_1 AX+f_1(AX)N=\alpha'(X)X_2+f_1(AX)N$. Thus $AX=-\alpha'(X)\phi_1 X_2+f_1(AX)U_1$ for any $X\in C^\infty(T^\circ M)$, $X\perp X_2$. From (3.3) we also know that $g((\phi_1 A+A\phi_1)X, Y)=0$ for any $X, Y\in C^\infty(T^\circ M)$. In particular, $A\phi_1 X_2=-\alpha'(\phi_1 X_2)\phi_1 X_2+f_1(A\phi_1 X_2)U_1$.

If $X\perp \mathrm{Span}\{X_1, X_2, \phi_1 X_2\}$, $g(X, A\phi_1 X_2)=g(AX, \phi_1 X_2)=0$. This means that $AX=0$ for any $X\in C^\infty(T^\circ M)$ and $X\perp \mathrm{Span}\{X_1, X_2, \phi_1 X_2\}$.

As $\mu_1 A\phi_1 X_2+\mu_1\phi_1 AX_2\in \boldsymbol{D}^\perp$ we obtain

$$\begin{aligned} &AX_1=-\alpha'(X_1)\phi_1 X_2+\mu_1 U_1,\\ (3.13)\qquad &A\phi_1 X_2=-g(AX_2, X_2)\phi_1 X_2+\mu_1 g(\phi_1 X_2, X_1)U_1,\\ &AX_2=g(AX_2, X_2)X_2+\mu_2 U_2+\delta_2 U_3. \end{aligned}$$

We can write $\phi_1 X_1=g(\phi_1 X_1, X_2)X_2+W$, where $W\perp \mathrm{Span}\{X_1, X_2, \phi_1 X_2\}$. Thus $A\phi_1 X_1=g(\phi_1 X_1, X_2)AX_2=g(\phi_1 X_1, X_2)g(AX_2, X_2)X_2+\mu_2 g(\phi_1 X_1, X_2)U_2+\delta_2 g(\phi_1 X_1, X_2)U_3$.

Then $A\phi_1 X_1+\phi_1 AX_1=\{g(\phi_1 X_1, X_2)g(AX_2, X_2)+\alpha'(X_1)\}X_2+\delta T$, where $T\in\boldsymbol{D}^\perp$. As this field cannot have component in $\boldsymbol{D}$, we obtain

$$(3.14)\qquad \alpha'(X_1)=-g(\phi_1 X_1, X_2)g(AX_2, X_2).$$

Moreover, $\phi_2 X_2=g(\phi_2 X_2, X_1)X_1+hT'$, $T'\perp \mathrm{Span}\{\phi_1 X_2, X_1, X_2\}$. Thus $A\phi_2 X_2=g(\phi_2 X_2, X_1)AX_1=-g(\phi_2 X_2, X_1)\alpha'(X_1)\phi_1 X_2+\mu_1 g(\phi_2 X_2, X_1)U_1$. Then from (3.13) $g(A\phi_2 X_2, \phi_1 X_2)=-g(\phi_2 X_2, X_1)\alpha'(X_1)=g(\phi_2 X_2, A\phi_1 X_2)=0$. Therefore, either $\alpha'(X_1)=0$ or $g(\phi_2 X_2, X_1)=0$.

Analogously $A\phi_3 X_2=g(\phi_3 X_2, X_1)AX_1=-g(\phi_3 X_2, X_1)\alpha'(X_1)\phi_1 X_2+\mu_1 g(\phi_3 X_2, X_1)U_1$. Then, $g(A\phi_3 X_2, \phi_1 X_2)=-g(\phi_3 X_2, X_1)\alpha'(X_1)=g(\phi_3 X_2, A\phi_1 X_2)=0$.

Suppose that $g(\phi_2 X_2, X_1)=g(\phi_3 X_2, X_1)=0$. Then $A\phi_2 X_2=A\phi_3 X_2=0$. In this case, $\mu_2\phi_2 AX_2+\delta_2\phi_3 AX_2$ has not component in $\boldsymbol{D}$. Thus $\mu_2 g(AX_2, X_2)=\delta_2 g(AX_2,$

$X_2)=0$. Thus either $\mu_2=\delta_2=0$, which is impossible, or $g(AX_2, X_2)=0$. This and (3.14) imply that $\alpha'(X_1)=0$ and M is ruled.

Otherwise, $\alpha'(X_1)=0$ implies that either $g(AX_2, X_2)=0$ and M is ruled or $g(X_2, \phi_1X_1)=0$. In this case $AX=0$ for any $X\in C^\infty(T^\circ M)$, $X\perp\mathrm{Span}\{X_1, X_2, \phi_1X_2\}$, $AX_1=\mu_1U_1$, $A\phi_1X_2=-g(AX_2, X_2)\phi_1X_2$, $AX_2=g(AX_2, X_2)X_2+\mu_2U_2+\delta_2U_3$. Then $\phi_1AX_1=A\phi_1X_1=0$, $A\phi_2X_2=\mu_1g(\phi_2X_2, X_1)U_1$, $\phi_2AX_2=g(AX_2, X_2)\phi_2X+\delta_2U_1$. From (3.3) we obtain $\mu_2g(AX_2, X_2)+\delta_2g(\phi_3AX_2, \phi_2X_2)+\delta_2g(A\phi_3X_2, \phi_2X_2)=\mu_2g(AX_2, X_2)=0$. Thus $g(AX_2, X_2)=0$ and M is ruled.

c) $\delta_2=0$. This case is similar to Case b) and M must be ruled.

This finishes the proof, because the converse is trivial.

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