

**REALIZATIONS OF INVOLUTIVE AUTOMORPHISMS
 σ AND G^σ OF EXCEPTIONAL LINEAR LIE GROUPS
 G , PART III, $G = E_8$**

By

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M. Berger [1] classified involutive automorphisms σ of simple Lie algebras \mathfrak{g} and determined the type of the subalgebras \mathfrak{g}^σ of fixed points. In the preceding papers [Y1], [Y2], we found involutive automorphisms σ and realized the subgroups G^σ of fixed points explicitly for the connected exceptional universal linear Lie groups G of type G_2 , F_4 , E_6 and E_7 . In this paper we consider the case of type E_8 . Our results are as follows.

G	G^σ	σ			
E_8^c	$(SL(2, C) \times E_7^c) / \mathbf{Z}_2$	ν			
	$Ss(16, C)$	$\tilde{\lambda}\gamma$			
E_8^c	E_8	$\tau\tilde{\lambda}$			
E_8	$(SU(2) \times E_7) / \mathbf{Z}_2$	ν			
	$Ss(16)$	$\tilde{\lambda}\gamma$			
E_8^c	$E_{8(8)}$	$\tau\gamma$	$\tau\tilde{\lambda}\nu\gamma$	$\tau\sigma\gamma$	$\tau\nu\gamma$
$E_{8(8)}$	$(SL(2, \mathbf{R}) \times E_{7(7)}) / \mathbf{Z}_2 \times 2$	ν			
	$(SU(2) \times E_{7(-5)}) / \mathbf{Z}_2$		ν		
	$Ss(16)$	$\tilde{\lambda}\gamma$			
	$Sso(8, 8) \times 2$			$\tilde{\lambda}\gamma$	
	$Sso^*(16) \times 2$				$\tilde{\lambda}\gamma$
E_8^c	$E_{8(-24)}$	τ	$\tau\tilde{\lambda}\nu$	$\tau\tilde{\lambda}\gamma$	$\tau\tilde{\lambda}\nu\sigma$
$E_{8(-24)}$	$(SL(2, \mathbf{R}) \times E_{7(-25)}) / \mathbf{Z}_2 \times 2$	ν			
	$(SU(2) \times E_7) / \mathbf{Z}_2$		ν		
	$(SU(2) \times E_{7(-5)}) / \mathbf{Z}_2$			ν	
	$So(4, 12)$				σ
	$Sso^*(16) \times 2$		$\tilde{\lambda}\gamma$		

This paper is a continuation of [Y1], [Y2] and we use the same notations as them. So the numbering of sections and theorems starts from 5.1 and 5.1.1,

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respectively.

Groups E_8

5.1. The complex Lie algebra e_8^C

In the 248-dimensional C -vector space

$$e_8^C = e_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C,$$

we define the Lie bracket $[R_1, R_2]$ by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)] = (\Phi, P, Q, r, s, t)$$

where

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1, \\ P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1, \\ Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1, \\ r = -\frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1, \\ s = \frac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1, \\ t = -\frac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1, \end{cases}$$

then e_8^C becomes a simple C -Lie algebra of type E_8 [16], [17]. For $R \in e_8^C$, $\text{ad}R ((\text{ad}R)R_1 = [R, R_1], R_1 \in e_8^C)$ is denoted by $\Theta(R)$. Hereafter we often identify R and $\Theta(R)$, e_8^C and $\text{Der}_C(e_8^C) = \{\Theta(R) | R \in e_8^C\}$, respectively. The group E_8^C is defined to be the automorphism group of the Lie algebra e_8^C :

$$E_8^C = \{\alpha \in \text{Iso}_C(e_8^C) | \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}.$$

5.2. Involutions of the Lie group E_8^C

We arrange here main involutions used in this chapter E_8 . We define C -linear transformations $\gamma, \sigma, \lambda, \nu, \omega$ of e_8^C by

$$\begin{aligned} \gamma(\Phi, P, Q, r, s, t) &= (\gamma\Phi\gamma, \gamma P, \gamma Q, r, s, t), \\ \sigma(\Phi, P, Q, r, s, t) &= (\sigma\Phi\sigma, \sigma P, \sigma Q, r, s, t), \\ \lambda(\Phi, P, Q, r, s, t) &= (\lambda\Phi\lambda^{-1}, \lambda P, \lambda Q, r, s, t) \end{aligned}$$

where γ, σ, λ of right sides are the same ones as $\gamma \in G_2^C \subset F_4^C \subset E_6^C \subset E_7^C$, $\sigma \in F_4^C \subset E_6^C \subset E_7^C$, $\lambda \in E_7^C$,

$$\nu(\Phi, P, Q, r, s, t) = (\Phi, -P, -Q, r, s, t),$$

$$\omega(\Phi, P, Q, r, s, t) = (\Phi, Q, -P, -r, -t, -s).$$

Then $\gamma, \sigma, \lambda, \nu, \omega \in E_8^C$ and $\gamma^2 = \sigma^2 = \nu^2 = 1, \lambda^2 = \omega^2 = \nu$. (ν is nothing but the element -1 of the center of E_7^C (see Proposition 5.4.1), however we use ν in E_8^C to avoid confusions because -1 has many different meanings. Note that $\nu = \phi(-E), \omega = \phi(J)$ using ϕ of Proposition 5.4.2). We put

$$\tilde{\lambda} = \lambda\omega = \omega\lambda.$$

Then $\tilde{\lambda} \in E_8^C$ and $\tilde{\lambda}^2 = 1$. The complex conjugation in e_8^C is denoted by τ :

$$\tau(\Phi, P, Q, r, s, t) = (\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau s, \tau t).$$

These transformations $\gamma, \sigma, \nu, \tilde{\lambda}, \tau$ of e_8^C induce involutive automorphisms $\tilde{\gamma}, \tilde{\sigma}, \tilde{\nu}, \tilde{\lambda}, \tilde{\tau}$ of E_8^C :

$$\begin{aligned} \tilde{\gamma}(\alpha) &= \gamma\alpha\gamma, & \tilde{\sigma}(\alpha) &= \sigma\alpha\sigma, & \tilde{\nu}(\alpha) &= \nu\alpha\nu, \\ \tilde{\lambda}(\alpha) &= \tilde{\lambda}\alpha\tilde{\lambda}, & \tilde{\tau}(\alpha) &= \tau\alpha\tau, & & \alpha \in E_8^C. \end{aligned}$$

5.3. Lie groups of type E_8

We define \mathbf{R} -Lie algebras $e_{8(8)}, e_{8(-24)}$ by

$$e_{8(8)} = e_{7(7)} \oplus \mathfrak{P}' \oplus \mathfrak{P}' \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R} \quad (e_{7(7)} = (e_7^C)^{\tau r}),$$

$$e_{8(-24)} = e_{7(-25)} \oplus \mathfrak{P} \oplus \mathfrak{P} \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R} \quad (e_{7(-25)} = (e_7^C)^\tau)$$

(Theorem 4.5.2) with the Lie brackets as e_8^C , respectively. The connected linear Lie groups of type E_8 are obtained as

$$E_8^C = \{\alpha \in \text{Iso}_C(e_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\},$$

$$E_8 = \{\alpha \in \text{Iso}_C(e_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2], \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\},$$

$$E_{8(8)} = \{\alpha \in \text{Iso}_R(e_{8(8)}) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\},$$

$$E_{8(-24)} = \{\alpha \in \text{Iso}_R(e_{8(-24)}) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}$$

here $\langle R_1, R_2 \rangle = -(1/15)B_8(\tau\tilde{\lambda}R_1, R_2)$ where B_8 is the Killing form of e_8^C [18], [21]. E_8^C, E_8 are simply connected (see Appendix).

LEMMA 5.3.1. $(e_8^C)_{\tau r} \cong e_{8(8)}, (e_8^C)_\tau = e_{8(-24)}$.

THEOREM 5.3.2. $(E_8^C)^{\tau\tilde{\lambda}} = E_8, (E_8^C)^{\tau r} \cong E_{8(8)}, (E_8^C)^\tau = E_{8(-24)}$.

PROOF. As for $E_{8(8)}, E_{8(-24)}$, these are direct results of Lemma 5.3.1. E_8 is nothing but its definition.

Note that $\gamma \in G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$ ($E_7 = (E_7^C)^{\tau\lambda} \subset (E_8^C)^{\tau\lambda}$ (see Proposition 5.4.1) $= E_8$), $\sigma \in F_4 \subset E_6 \subset E_7 \subset E_8$, $\nu \in E_7 \subset E_8$, $\tilde{\lambda} \in E_8$. We know that the simply connected compact Lie group E_8 has two classes of involutive elements up to conjugation.

LEMMA 5.3.3. For $\sigma_1, \sigma_2 \in E_8$ such that $\sigma_1^2 = \sigma_2^2 = 1$, we have

$$\sigma_1 \sim \sigma_2 \iff (E_8)^{\sigma_1} \cong (E_8)^{\sigma_2} \iff (e_8)^{\sigma_1} \cong (e_8)^{\sigma_2},$$

moreover if and only if $\dim(e_8)^{\sigma_1} = \dim(e_8)^{\sigma_2}$.

PROOF is due to [19].

PROPOSITION 5.3.4. (1) $\tilde{\lambda}, \nu, \gamma, \nu\sigma$ are conjugate in E_8 with one another.
 (2) $\tilde{\lambda}\gamma$ and $\nu\gamma$ are conjugate in E_8 .

PROOF. We can easily calculate $\dim(e_8)^{\tilde{\lambda}} = \dim(e_8)^\nu = \dim(e_8)^\gamma = \dim(e_8)^{\nu\sigma} = 136$ and $\dim(e_8)^{\tilde{\lambda}\gamma} = \dim(e_8)^{\nu\gamma} = 120$, hence Proposition 5.3.4 follows from Lemma 5.3.3.

REMARK. The author can not find any element $\delta \in E_8$ which gives the conjugation: $\delta\tilde{\lambda} = \nu\delta$ etc..

5.4. Subgroups of type $A_1 \oplus E_7$ of Lie groups of type E_8

We consider a subgroup $(E_8^C)_{1,1^-,1_-}$ of E_8^C :

$$(E_8^C)_{1,1^-,1_-} = \{\alpha \in E_8^C \mid \alpha 1 = 1, \alpha 1^- = 1^-, \alpha 1_- = 1_-\}$$

where $1 = (0, 0, 0, 1, 0, 0)$, $1^- = (0, 0, 0, 0, 1, 0)$, $1_- = (0, 0, 0, 0, 0, 1) \in e_8^C$.

PROPOSITION 5.4.1. $(E_8^C)_{1,1^-,1_-} \cong E_7^C$.

PROOF ([17]). For $\beta \in E_7^C$ we correspond $\alpha \in E_8^C$,

$$\alpha(\Phi, P, Q, r, s, t) = (\beta\Phi\beta^{-1}, \beta P, \beta Q, r, s, t)$$

for $(\Phi, P, Q, r, s, t) \in e_8^C$. It is easy to verify $\alpha \in (E_8^C)_{1,1^-,1_-}$. Conversely let $\alpha \in (E_8^C)_{1,1^-,1_-}$. From the conditions $\alpha 1 = 1, \alpha 1^- = 1^-, \alpha 1_- = 1_-$, α has the form

$$\alpha = \begin{pmatrix} \beta_1 & \beta_{12} & \beta_{13} & 0 & 0 & 0 \\ \beta_{21} & \beta_2 & \beta_{23} & 0 & 0 & 0 \\ \beta_{31} & \beta_{32} & \beta_3 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{pmatrix}, \begin{matrix} \beta_1 \in \text{Hom}_C(e_7^C, e_7^C), \\ \beta_2, \beta_3, \beta_{23}, \beta_{32} \in \text{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C), \\ \beta_{21}, \beta_{31} \in \text{Hom}_C(e_7^C, \mathfrak{P}^C), \\ \beta_{12}, \beta_{13} \in \text{Hom}_C(\mathfrak{P}^C, e_7^C), \\ a_i \in \text{Hom}_C(e_7^C, C), \\ b_i, c_i \in \text{Hom}_C(\mathfrak{P}^C, C). \end{matrix}$$

From $[\alpha\Phi, 1]=\alpha[\Phi, 1]=0$ we have $\beta_{21}=\beta_{31}=0, a_2=a_3=0$, and from $[\alpha\Phi, 1^-]=\alpha[\Phi, 1^-]=0$ we have $a_1=0$. Put $P^-(= (0, P, 0, 0, 0, 0), Q_-(= (0, 0, Q, 0, 0, 0)$. From $[\alpha P^-, 1]=-\alpha P^-$ we have $\beta_{12}=\beta_{32}=0, b_1=b_2=b_3=0$. Similarly from $[\alpha Q_-, 1]=\alpha Q_-$ we have $\beta_{13}=\beta_{23}=0, c_1=c_2=c_3=0$. Operate α on $[P^-, Q_-]= (P \times Q, 0, 0, -(1/8)\{P, Q\}, 0, 0)$, then

$$\beta_1(P \times Q) = \beta_2 P \times \beta_3 Q, \quad \{\beta_2 P, \beta_3 Q\} = \{P, Q\}. \tag{i}$$

Again operate α on $[P^-, Q^-] = (1/4)\{P, Q\}1$, then

$$\{\beta_2 P, \beta_2 Q\} = \{P, Q\}. \tag{ii}$$

Operate α on $[\Phi, P^-] = (\Phi P)^-$, then

$$(\beta_1 \Phi)(\beta_2 P) = \beta_2(\Phi P). \tag{iii}$$

From (i), (ii) we have $\{\beta_2 P, \beta_3 Q\} = \{\beta_2 P, \beta_2 Q\}$ for all $P, Q \in \mathfrak{B}^c$, hence $\beta_2 = \beta_3$ (put $= \beta$). In (iii) put $\beta^{-1}P$ instead of P , then $\beta_1 \Phi = \beta \Phi \beta^{-1}$. Therefore, from (i) we have $\beta(P \times Q) \beta^{-1} = \beta_1(P \times Q) = \beta P \times \beta Q$. Thus $\beta \in E_7^c$.

PROPOSITION 5.4.2. $(E_8^c)^\nu$ has a subgroup $\phi(SL(2, C))$ which is isomorphic to the group $SL(2, C)$. Where $\phi(A), A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, C)$, is the C -linear transformation of e_8^c defined by

$$\phi\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & c1 & 0 & 0 & 0 \\ 0 & b1 & d1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+2bc & -ab & cd \\ 0 & 0 & 0 & -2ac & a^2 & -c^2 \\ 0 & 0 & 0 & 2bd & -b^2 & d^2 \end{pmatrix}.$$

PROOF. For $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \exp\begin{pmatrix} r & s \\ t & -r \end{pmatrix}$, we have

$$\begin{aligned} \phi(A) &= \exp \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r1 & s1 & 0 & 0 & 0 \\ 0 & t1 & -r1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & s \\ 0 & 0 & 0 & -2s & 2r & 0 \\ 0 & 0 & 0 & 2t & 0 & -2r \end{pmatrix} \\ &= \exp(\Theta(0, 0, 0, r, s, t)). \end{aligned}$$

LEMMA 5.4.3. $\phi: SL(2, C) \rightarrow (E_8^c)^\nu$ of Proposition 5.4.2 satisfies

$$\tau\phi(A)\tau = \phi(\tau A), \quad \omega\phi(A)\omega^{-1} = \phi({}^t A^{-1}),$$

$$\lambda\phi(A)\lambda^{-1}=\sigma\phi(A)\sigma=\gamma\phi(A)\gamma=\phi(A).$$

THEOREM 5.4.4. $(E_8^C)^\nu \cong (SL(2, C) \times E_7^C) / \mathbf{Z}_2$, $\mathbf{Z}_2 = \{(E, 1), (-E, -1)\}$.

PROOF. We define a mapping $\phi: SL(2, C) \times E_7^C \rightarrow (E_8^C)^\nu$ by

$$\phi(A, \beta) = \phi(A)\beta.$$

Obviously $\phi(A, \beta) \in (E_8^C)^\nu$. Since $\phi(A)$, $A \in SL(2, C)$ and $\beta \in E_7^C$ are commutative, ϕ is a homomorphism. $\text{Ker}\phi = \{(E, 1), (-E, -1)\} = \mathbf{Z}_2$. Since $(E_8^C)^\nu$ is connected (Lemma 0.7) and $\dim_C(\mathfrak{sl}(2, C) \oplus e_7^C) = 3 + 133 = 136 = \dim_C(e_8^C)^\nu$ (note that $(e_8^C)^\nu = \{(\Phi, 0, 0, r, s, t) \in e_8^C \mid \Phi \in e_7^C, r, s, t \in C\}$), ϕ is onto. Thus we have the required isomorphism.

THEOREM 5.4.5. (1) $(E_8)^\nu \cong (SU(2) \times E_7) / \mathbf{Z}_2 \cong (\tau\tilde{\lambda}\nu)^\nu \sim (E_{8(-24)})^\nu$.
 (2) $(E_{8(-24)})^\nu \sim (\tau\tilde{\lambda}\gamma)^\nu \cong (SU(2) \times E_{7(-5)}) / \mathbf{Z}_2 \cong (\tau\tilde{\lambda}\nu\gamma)^\nu \sim (E_{8(8)})^\nu$.

PROOF. (1) Let $\alpha \in (E_8)^\nu = ((E_8^C)^{\tau\tilde{\lambda}})^\nu = (\tau\tilde{\lambda})^\nu$. By Theorem 5.4.4, there exist $A \in SL(2, C)$, $\beta \in E_7^C$ such that $\alpha = \phi(A)\beta$. From the condition $\tau\tilde{\lambda}\alpha = \alpha\tau\tilde{\lambda}$, we have $\phi(\tau^t A^{-1})\tau\tilde{\lambda}\beta\tilde{\lambda}\tau = \phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases} \tau^t A^{-1} = A \\ \tau\tilde{\lambda}\beta\tilde{\lambda}\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \tau^t A^{-1} = -A \\ \tau\tilde{\lambda}\beta\tilde{\lambda}\tau = \nu\beta. \end{cases}$$

The latter case is impossible because $(\tau^t A)A = -E$ is false. In the first case, $(\tau^t A)A = E$, that is, $A \in SU(2)$. For $\beta \in E_7^C$, $\tau\tilde{\lambda}\beta\tilde{\lambda}\tau = \beta$ is $\tau\lambda\beta\lambda^{-1}\tau = \beta$, hence $\beta \in (E_7^C)^{\tau\lambda} = E_7$ (Theorem 4.3.2). Thus $(E_8)^\nu = \phi(SU(2) \times E_7) \cong (SU(2) \times E_7) / \mathbf{Z}_2$.

$$E_{8(-24)} = (E_8^C)^\tau \cong (E_8^C)^{\tau\tilde{\lambda}\nu}.$$

In fact, since $\tilde{\lambda} \sim \nu$ under some $\delta \in E_8$: $\delta\tilde{\lambda} = \nu\delta$, $\delta\tau\tilde{\lambda} = \tau\tilde{\lambda}\delta$ (Proposition 5.3.4), $(E_8^C)^\tau \ni \alpha \rightarrow \delta\alpha\delta^{-1} \in (E_8^C)^{\tau\tilde{\lambda}\nu}$ gives an isomorphism. Now $(E_{8(-24)})^\nu \sim (\tau\tilde{\lambda}\nu)^\nu = (\tau\tilde{\lambda})^\nu$.

$$(2) \quad E_{8(-24)} = (E_8^C)^\tau \cong (E_8^C)^{\tau\tilde{\lambda}\gamma}.$$

In fact, since $\tilde{\lambda} \sim \gamma$ under some $\delta \in E_8$: $\delta\tilde{\lambda} = \gamma\delta$, $\delta\tau\tilde{\lambda} = \tau\tilde{\lambda}\delta$ (Proposition 5.3.4), $(E_8^C)^\tau \ni \alpha \rightarrow \delta\alpha\delta^{-1} \in (E_8^C)^{\tau\tilde{\lambda}\gamma}$ gives an isomorphism. Let $\alpha \in ((E_8^C)^{\tau\tilde{\lambda}\gamma})^\nu = (\tau\tilde{\lambda}\gamma)^\nu$, $\alpha = \phi(A)\beta$, $A \in SL(2, C)$, $\beta \in E_7^C$. From $\tau\tilde{\lambda}\gamma\alpha = \alpha\tau\tilde{\lambda}\gamma$, we have $\phi(\tau^t A^{-1})\tau\tilde{\lambda}\gamma\beta\tilde{\lambda}\tau = \phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases} \tau^t A^{-1} = A \\ \tau\tilde{\lambda}\gamma\beta\tilde{\lambda}\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \tau^t A^{-1} = -A \\ \tau\tilde{\lambda}\gamma\beta\tilde{\lambda}\tau = \nu\beta. \end{cases}$$

As similar to (1), the latter case is impossible. In the first case, $A \in SU(2)$ and $\beta \in (E_7^C)^{\tau\tilde{\lambda}\gamma} = (E_7^C)^{\tau\lambda\gamma} = E_{7(-5)}$ (Theorem 4.3.2). Thus $(E_{8(-24)})^\nu \sim (\tau\tilde{\lambda}\gamma)^\nu \cong (SU(2) \times$

$$E_{7(-5)}/\mathbf{Z}_2.$$

$$E_{8(8)}=(E_8^C)^{\tau\gamma}\cong(E_8^C)^{\tau\tilde{\lambda}\nu\gamma}.$$

In fact, since $\tilde{\lambda}\gamma\sim\nu\gamma$ under some $\delta\in E_8$: $\delta\tilde{\lambda}\gamma=\nu\gamma\delta$, $\delta\tau\tilde{\lambda}=\tau\tilde{\lambda}\delta$ (Proposition 5.3.4), $(E_8^C)^{\tau\gamma}\ni\alpha\rightarrow\delta\alpha\delta^{-1}\in(E_8^C)^{\tau\tilde{\lambda}\nu\gamma}$ gives an isomorphism. Now $(E_{8(8)})^\nu\sim(\tau\tilde{\lambda}\nu\gamma)^\nu=(\tau\tilde{\lambda}\gamma)^\nu$.

THEOREM 5.4.6. (1) $(E_{8(8)})^\nu\cong(SL(2, \mathbf{R})\times E_{7(\tau)})/\mathbf{Z}_2\times 2$.

(2) $(E_{8(-24)})^\nu\cong(SL(2, \mathbf{R})\times E_{7(-25)})/\mathbf{Z}_2\times 2$.

PROOF. (1) Let $\alpha\in(E_{8(8)})^\nu=((E_8^C)^{\tau\gamma})^\nu=(\tau\gamma)^\nu$, $\alpha=\phi(A)\beta$, $A\in SL(2, \mathbf{C})$, $\beta\in E_7^C$. From $\tau\gamma\alpha=\alpha\tau\gamma$, we have $\phi(\tau A)\tau\gamma\beta\gamma\tau=\phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases} \tau A=A \\ \tau\gamma\beta\gamma\tau=\beta \end{cases} \quad \text{or} \quad \begin{cases} \tau A=-A \\ \tau\gamma\beta\gamma\tau=\nu\beta. \end{cases}$$

In the first case, $A\in SL(2, \mathbf{R})$, $\beta\in(E_7^C)^{\tau\gamma}=E_{7(\tau)}$ (Theorem 4.3.2). In the latter case, $A=(iI)B$, $B\in SL(2, \mathbf{R})$, $\beta=\iota\beta'$, $\beta'\in E_{7(\tau)}$ where $\iota\in E_7$ is one defined in 4.2. Thus $(E_{8(8)})^\nu\cong(SL(2, \mathbf{R})\times E_{7(\tau)}\cup(iI)SL(2, \mathbf{R})\times \iota E_{7(\tau)})/\mathbf{Z}_2=(SL(2, \mathbf{R})\times E_{7(\tau)})/\mathbf{Z}_2\times 2$. (The explicit form of $\phi(iI, \iota)$ is

$$\phi(iI, \iota)(\Phi, P, Q, r, s, t)=(\iota\Phi\iota^{-1}, \iota P, -\iota Q, r, -s, -t).$$

(2) Let $\alpha\in(E_{8(-24)})^\nu=((E_8^C)^\tau)^\nu=(\tau)^\nu$, $\alpha=\phi(A)\beta$, $A\in SL(2, \mathbf{C})$, $\beta\in E_7^C$. From $\tau\alpha=\alpha\tau$, we have $\phi(\tau A)\tau\beta\tau=\phi(A)\beta$ (Lemma 5.4.3). As similar to (1), $(E_{8(-24)})^\nu\cong(SL(2, \mathbf{R})\times E_{7(-25)}\cup(iI)SL(2, \mathbf{R})\times \iota E_{7(-25)})/\mathbf{Z}_2=(SL(2, \mathbf{R})\times E_{7(-25)})/\mathbf{Z}_2\times 2$.

5.5. Subgroups of type D_8 of Lie groups of type E_8

We define an \mathbf{R} -algebraic homomorphism $l: M(8, \mathbf{C})\rightarrow M(16, \mathbf{R})$ by

$$l((x+yi))=\left(\left(\begin{matrix} x & y \\ -y & x \end{matrix}\right)\right), \quad x, y\in\mathbf{R}$$

and $l: M(8, \mathbf{C}^c)\rightarrow M(16, \mathbf{C})$ is its complexification. These l satisfy

$$ll(X)=l(\bar{X})I, \quad Jl(X)=l(X)J, \quad {}^t(l(X))=l(X^*), \quad X\in M(8, K), \quad K=\mathbf{C}, \mathbf{C}^c.$$

PROPOSITION 5.5.1. (1) $l(\mathfrak{u}(8, \mathbf{C}^c))=\{B\in\mathfrak{so}(16, \mathbf{C})\mid JB=BJ\}$,

$$l(\mathfrak{e}(8, \mathbf{C}^c))I=\{B\in\mathfrak{so}(16, \mathbf{C})\mid JB=-BJ\}.$$

(2) Any element $B\in\mathfrak{so}(16, \mathbf{C})$ is uniquely expressed by

$$\begin{aligned} B &= l(D') + l(S)I & D' &\in \mathfrak{u}(8, \mathbf{C}^c), S \in \mathfrak{e}(8, \mathbf{C}^c) \\ &= l(D) + l(S)I + l(icE) & D &\in \mathfrak{su}(8, \mathbf{C}^c), S \in \mathfrak{e}(8, \mathbf{C}^c), c \in \mathbf{C}. \end{aligned}$$

PROOF. (1) If $D \in \mathfrak{u}(8, \mathbf{C}^c)$, then $Jl(D) = l(D)J$, ${}^t l(D) = l(D^*) = -l(D)$. Conversely, for $B \in \mathfrak{so}(16, \mathbf{C})$ such that $JB = BJ$, put $B = l(D)$, $D \in M(8, \mathbf{C}^c)$. Then $l(-D) = -B = {}^t B = {}^t l(D) = l(D^*)$, hence $D^* = -D$. Next, for $S \in \mathfrak{S}(8, \mathbf{C}^c)$, $Jl(S)I = l(S)JI = -l(S)IJ$, ${}^t l(S)I = {}^t I {}^t l(S) = Il(S^*) = -Il(\bar{S}) = -l(S)I$. Conversely, for $B \in \mathfrak{so}(16, \mathbf{C})$ such that $JB = -BJ$, consider BI . Then $JBI = BIJ$, hence we can put $BI = l(S)$, $S \in M(8, \mathbf{C}^c)$. Then $B = l(S)I$, ${}^t B = {}^t I {}^t l(S) = Il(S^*) = l({}^t S)$, hence $-S = {}^t S$, that is, $S \in \mathfrak{S}(8, \mathbf{C}^c)$.

(2) $B = (B - JBJ)/2 + (B + JBJ)/2$ and use the above (1).

We consider Lie subalgebras $\mathfrak{so}(4, 12)$, $\mathfrak{so}(8, 8)$, $\mathfrak{so}^*(16)$ (which are isomorphic to the ordinary ones) of $\mathfrak{so}(16, \mathbf{C})$ as

$$\begin{aligned}\mathfrak{so}(4, 12) &= \{B \in \mathfrak{so}(16, \mathbf{C}) \mid l(I_2)(\tau B)l(I_2) = B\}, \\ \mathfrak{so}(8, 8) &= \{B \in \mathfrak{so}(16, \mathbf{C}) \mid l(I_4')(\tau B)l(I_4') = B\}, \\ \mathfrak{so}^*(16) &= \{B \in \mathfrak{so}(16, \mathbf{C}) \mid JB = (\tau B)J\}\end{aligned}$$

where $I_4' = \text{diag}(1, 1, 1, 1, -1, -1, -1, -1)$, and corresponding the above we use the following notations.

$$\begin{aligned}\mathfrak{su}(2, 6) &= \{D \in \mathfrak{su}(8, \mathbf{C}^c) \mid I_2(\tau D)I_2 = D\}, \\ \mathfrak{S}(2, 6, \mathbf{C}) &= \{S \in \mathfrak{S}(8, \mathbf{C}^c) \mid I_2(\tau S)I_2 = S\}, \\ \mathfrak{su}(4, 4) &= \{D \in \mathfrak{su}(8, \mathbf{C}^c) \mid I_4'(\tau D)I_4' = D\}, \\ \mathfrak{S}(4, 4, \mathbf{C}) &= \{S \in \mathfrak{S}(8, \mathbf{C}^c) \mid I_4'(\tau S)I_4' = S\}.\end{aligned}$$

From Proposition 5.5.1, we have easily the following

PROPOSITION 5.5.2. *Any elements $B \in \mathfrak{so}(16)$, $\mathfrak{so}(4, 12)$, $\mathfrak{so}(8, 8)$, $\mathfrak{so}^*(16)$ are, respectively, expressed by $B = l(D) + l(S)I + l(i c E)$,*

- (1) case $\mathfrak{so}(16)$, $D \in \mathfrak{su}(8)$, $S \in \mathfrak{S}(8, \mathbf{C})$, $c \in \mathbf{R}$,
- (2) case $\mathfrak{so}(4, 12)$, $D \in \mathfrak{su}(2, 6)$, $S \in \mathfrak{S}(2, 6, \mathbf{C})$, $c \in \mathbf{R}$,
- (3) case $\mathfrak{so}(8, 8)$, $D \in \mathfrak{su}(4, 4)$, $S \in \mathfrak{S}(4, 4, \mathbf{C})$, $c \in \mathbf{R}$,
- (4) case $\mathfrak{so}^*(16)$, $D \in \mathfrak{su}(8)$, $S \in i\mathfrak{S}(8, \mathbf{C})$, $c \in \mathbf{R}$.

Recall the \mathbf{C} -linear isomorphism $\chi: \mathfrak{P}^c = \mathfrak{S}^c \oplus \mathfrak{S}^c \oplus \mathbf{C} \oplus \mathbf{C} \rightarrow \mathfrak{S}(8, \mathbf{C}^c)$,

$$\chi(X, Y, \xi, \eta) = \left(k \left(gX - \frac{\xi}{2} E \right) \right) J + i \left(k \left(g(\gamma Y) - \frac{\eta}{2} E \right) \right) J$$

which is used to define the homomorphism $\phi: SU(8, \mathbf{C}^c) \rightarrow (E_{\gamma}^c)^{\lambda r}$, $\phi(A)P = \gamma^{-1}(A(\chi P)^t A)$, $P \in \mathfrak{P}^c$ (Theorem 4.5.3).

LEMMA 5.5.3. *For $S \in \mathfrak{S}(8, \mathbf{C}^c)$, we have*

$$\gamma\chi^{-1}S=\chi^{-1}(I_2SI_2), \quad \tau\chi^{-1}S=\chi^{-1}(I_2(\tau S)I_2), \quad \sigma\chi^{-1}S=\chi^{-1}(I_4'SI_4').$$

PROOF. $\lambda\gamma\chi^{-1}S=\lambda\gamma(X, Y, \xi, \eta)=\lambda(\gamma X, \gamma Y, \xi, \eta)$

$$\begin{aligned} &=k\left(g(\gamma X)-\frac{\xi}{2}E\right)J+ik\left(gY-\frac{\eta}{2}E\right)J \\ &=k\left(I_1(gX)I_1-\frac{\xi}{2}E\right)J+ik\left(I_1(g(\gamma Y))I_1-\frac{\eta}{2}E\right)J \\ &=I_2\left(k\left(gX-\frac{\xi}{2}E\right)J+ik\left(g(\gamma Y)-\frac{\eta}{2}E\right)J\right)I_2 \\ &=I_2\lambda(X, Y, \xi, \eta)I_2=I_2SI_2. \end{aligned}$$

Using $g(\tau X)=I_1(\tau(gX))I_1$, $g(\sigma X)=I_2'(gX)I_2'$, $I_2'=\text{diag}(1, 1, -1, -1)$, other formulae are similarly obtained.

LEMMA 5.5.4. For $S_1, S_2 \in \mathfrak{S}(8, \mathbf{C}^c)$, we have

- (1) $\text{tr}(S_1\bar{S}_2-S_2\bar{S}_1)=4i\{\chi^{-1}S_1, \chi^{-1}S_2\}$.
- (2) $\phi_*\left(S_1\bar{S}_2-S_2\bar{S}_1-\frac{1}{8}\text{tr}(S_1\bar{S}_2-S_2\bar{S}_1)E\right)=4(\lambda\gamma\chi^{-1}S_1 \times \chi^{-1}S_2 - \lambda\gamma\chi^{-1}S_2 \times \chi^{-1}S_1)$

where $\phi_*: \mathfrak{sp}(4, \mathbf{H}^c) \rightarrow (\mathfrak{e}_6^c)^{\lambda\gamma}$ is one defined in 3.5.

PROOF is in [21, Proposition 6].

THEOREM 5.5.5. $(\mathfrak{e}_8^c)^{\lambda\gamma} \cong \mathfrak{so}(16, \mathbf{C})$.

PROOF. $(\mathfrak{e}_8^c)^{\lambda\gamma} = \{\Theta \in \text{Der}_c(\mathfrak{e}_8^c) \mid \tilde{\lambda}\gamma\Theta = \Theta\tilde{\lambda}\gamma\}$
 $= \{\Theta(\Phi, \lambda\gamma Q, Q, 0, s, -s) \mid \Phi \in (\mathfrak{e}_7^c)^{\lambda\gamma}, Q \in \mathfrak{P}^c, s \in \mathbf{C}\}$.

We define a mapping $\zeta: \mathfrak{so}(16, \mathbf{C}) \rightarrow (\mathfrak{e}_8^c)^{\lambda\gamma}$ by

$$\zeta(l(D)+l(S)I+l(icE))=\Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c)$$

where $D \in \mathfrak{su}(8, \mathbf{C}^c)$, $S \in \mathfrak{S}(8, \mathbf{C}^c)$, $c \in \mathbf{C}$ (Proposition 5.5.1) and $\phi_*: \mathfrak{su}(8, \mathbf{C}^c) \rightarrow (\mathfrak{e}_7^c)^{\lambda\gamma}$, $\chi: \mathfrak{P}^c \rightarrow \mathfrak{S}(8, \mathbf{C}^c)$ are ones defined in the beginning. Clearly ζ is bijective.

We have to prove that ζ is a homomorphism, that is,

$$\zeta[X, Y]=[\zeta X, \zeta Y], \quad X, Y=l(D), l(S)I, l(icE).$$

For example, to prove $\zeta[l(S_1)I, l(S_2)I]=[\zeta(l(S_1)I), \zeta(l(S_2)I)]$, we use Lemma 5.5.4. The details of calculations are in [21].

THEOREM 5.5.6. (1) $(\mathfrak{e}_{8(8)})^{\lambda\gamma} \cong \mathfrak{so}(16) \cong (\mathfrak{e}_8)^{\lambda\gamma}$.

(2) $(\mathfrak{e}_{8(-24)})^{\lambda\gamma} \cong \mathfrak{so}(4, 12)$.

$$(3) \quad (e_{8(8)})^{\lambda r} \sim ((e_8^c)^{\tau\sigma r})^{\lambda r} \cong \mathfrak{so}(8, 8).$$

$$(4) \quad (e_{8(8)})^{\lambda r} \sim ((e_8^c)^{\tau\nu r})^{\lambda r} \cong \mathfrak{so}^*(16) \cong ((e_8^c)^{\tau\lambda\nu})^{\lambda r} \sim (e_{8(-24)})^{\lambda r}.$$

PROOF. (1) Let $\Theta \in (e_{8(8)})^{\lambda r} = ((e_8^c)^{\tau r})^{\lambda r}$. By Theorem 5.5.5, there exist $D \in \mathfrak{su}(8, \mathbf{C}^c)$, $S \in \mathfrak{S}(8, \mathbf{C}^c)$, $c \in \mathbf{C}$ such that

$$\Theta = \Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c).$$

From the condition $\tau\gamma\Theta(R)\gamma\tau = \Theta(R)$, $R \in e_8^c$, that is, $\Theta(\tau\gamma R) = \Theta(R)$, we have

$$\tau\gamma\phi_*(D)\gamma\tau = \phi_*(D), \quad \tau\gamma\chi^{-1}S = \chi^{-1}S, \quad \tau c = c.$$

Hence $D \in \mathfrak{su}(8)$ (Theorem 4.5.5), $\chi^{-1}(\tau S) = \chi^{-1}S$ (Lemma 5.5.3), $\tau S = S$, that is, $S \in \mathfrak{S}(8, \mathbf{C})$ and $c \in \mathbf{R}$. Therefore, $\Theta \in \zeta(l(\mathfrak{su}(8)) + l(\mathfrak{S}(8, \mathbf{C}))I + l(\mathbf{iRE})) = \zeta(\mathfrak{so}(16))$ (Proposition 5.5.2. (1)). Thus $(e_{8(8)})^{\lambda r} \cong \mathfrak{so}(16)$. $(e_8)^{\lambda r} = ((e_8^c)^{\tau\lambda})^{\lambda r} = ((e_8^c)^{\tau r})^{\lambda r}$.

(2) Let $\Theta \in (e_{8(-24)})^{\lambda r} = ((e_8^c)^{\tau})^{\lambda r}$, $\Theta = \Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c)$ as in (1). From $\tau\Theta = \Theta\tau$, we have

$$\tau\phi_*(D)\tau = \phi_*(D), \quad \tau\chi^{-1}S = \chi^{-1}S, \quad \tau c = c.$$

Hence $D \in \mathfrak{su}(2, 6)$ (Theorem 4.5.5), $\chi^{-1}(I_2(\tau S)I_2) = \chi^{-1}S$ (Lemma 5.5.3), $I_2(\tau S)I_2 = S$, that is, $S \in \mathfrak{S}(2, 6, \mathbf{C})$ and $c \in \mathbf{R}$. Therefore, from Proposition 5.5.2. (2), we have $(e_{8(-24)})^{\lambda r} \cong \mathfrak{so}(4, 12)$.

$$(3) \quad E_{8(8)} = (E_8^c)^{\tau r} \cong (E_8^c)^{\tau\sigma r}$$

because $\gamma \sim \sigma\gamma$ under $\delta \in F_4$ (Proposition 2.2.3) $\subset E_6 \subset E_7 \subset E_8$: $\delta\gamma = \sigma\gamma\delta$, $\delta\tau = \tau\delta$. Now let $\Theta \in ((e_8^c)^{\tau\sigma r})^{\lambda r}$, $\Theta = \Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c)$ as in (1). From $\tau\sigma\gamma\Theta = \Theta\tau\sigma\gamma$, we have

$$\tau\sigma\gamma\phi_*(D)\gamma\sigma\tau = \phi_*(D), \quad \tau\sigma\gamma\chi^{-1}S = \chi^{-1}S, \quad \tau c = c.$$

Hence $D \in \mathfrak{su}(4, 4)$ (Theorem 4.5.7), $\chi^{-1}(I_4'(\tau S)I_4') = \chi^{-1}S$ (Lemma 5.5.3), $I_4'(\tau S)I_4' = S$, that is, $S \in \mathfrak{S}(4, 4, \mathbf{C})$ and $c \in \mathbf{R}$. Therefore, from Proposition 5.5.2. (3), we have $(e_{8(8)})^{\lambda r} \sim ((e_8^c)^{\tau\sigma r})^{\lambda r} \cong \mathfrak{so}(8, 8)$.

$$(4) \quad E_{8(8)} = (E_8^c)^{\tau r} \cong (E_8^c)^{\tau\nu r}.$$

In fact, consider $\delta: e_8^c \rightarrow e_8^c$,

$$\delta(\Phi, P, Q, r, s, t) = (\Phi, iP, -iQ, r, -s, -t)$$

(δ is $\phi(iI)$ of Proposition 5.4.2), then $\delta \in E_8$, $\delta\gamma = \gamma\delta$, $\delta\tau = \tau\delta$, and $(E_8^c)^{\tau r} \ni \alpha \rightarrow \delta\alpha\delta^{-1} \in (E_8^c)^{\tau\nu r}$ gives an isomorphism. Now let $\Theta \in ((e_8^c)^{\tau\nu r})^{\lambda r}$, $\Theta = \Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c)$ as in (1). From $\tau\nu\gamma\Theta = \Theta\tau\nu\gamma$, we have

$$\tau\gamma\phi_*(D)\gamma\tau = \phi_*(D), \quad -\tau\gamma\chi^{-1}S = \chi^{-1}S, \quad \tau c = c.$$

Hence $D \in \mathfrak{su}(8)$ (Theorem 4.5.5), $-\chi^{-1}(\tau S) = \chi^{-1}S$ (Lemma 5.5.3), $-\tau S = S$, that is, $S \in i\mathfrak{C}(8, C)$ and $c \in \mathbf{R}$. Therefore, from Proposition 5.5.2.(4), we have $(e_{8(8)})^{\tilde{\lambda}\gamma} \sim ((e_8^C)^{\tau\nu\gamma})^{\tilde{\lambda}\gamma} \cong \mathfrak{so}^*(16)$. $(e_{8(-24)})^{\tilde{\lambda}\gamma} \sim ((e_8^C)^{\tau\tilde{\lambda}\nu})^{\tilde{\lambda}\gamma}$ (Theorem 5.4.5.(1)) $= ((e_8^C)^{\tau\nu\gamma})^{\tilde{\lambda}\gamma}$.

THEOREM 5.5.7. (1) $(E_8^C)^{\tilde{\lambda}\gamma} \cong Ss(16, C)$ ($= Spin(16, C)/\mathbf{Z}_2$ and not $SO(16, C)$).
 (2) $(E_8)^{\tilde{\lambda}\gamma} \cong Ss(16)$ ($= Spin(16)/\mathbf{Z}_2$ and not $SO(16)$).

PROOF is in [21]. The outline of the proof is as follows. The group $(E_8^C)^{\tilde{\lambda}\gamma}$ is connected (Lemma 0.7) and the center $z((E_8^C)^{\tilde{\lambda}\gamma})$ is $\mathbf{Z}_2 = \{1, \tilde{\lambda}\gamma\}$, hence $(E_8^C)^{\tilde{\lambda}\gamma}$ is isomorphic to one of $SO(16, C)$ or $Ss(16, C)$. $(E_8^C)^{\tilde{\lambda}\gamma}$ has the 128-dimensional irreducible C -representation $(e_8^C)_{-\tilde{\lambda}\gamma}$, however $SO(16, C)$ has no 128-dimensional irreducible C -representation. Therefore $(E_8^C)^{\tilde{\lambda}\gamma}$ must be $Ss(16, C)$. As for $(E_8)^{\tilde{\lambda}\gamma}$, the argument is similar to $(E_8^C)^{\tilde{\lambda}\gamma}$.

According to Theorem 5.5.6, we use the following notations.

$$So^*(16) = ((E_8^C)^{\tau\tilde{\lambda}\nu})^{\tilde{\lambda}\gamma} \sim (E_{8(-24)})^{\tilde{\lambda}\gamma}, \quad Sso^*(16) = (So^*(16))_0,$$

$$So(4, 12) = ((E_8^C)^\tau)^{\tilde{\lambda}\gamma} = (E_{8(-24)})^{\tilde{\lambda}\gamma},$$

$$So(8, 8) = ((E_8^C)^{\tau\sigma r})^{\tilde{\lambda}\gamma} \sim (E_{8(8)})^{\tilde{\lambda}\gamma}, \quad Sso(8, 8) = (So(8, 8))_0.$$

THEOREM 5.5.8. $(E_{8(-24)})^{\tilde{\lambda}\gamma} \sim (\tau\tilde{\lambda}\nu)^{\tilde{\lambda}\gamma} = So^*(16) = Sso^*(16) \times 2 \cong (\tau\nu\gamma)^{\tilde{\lambda}\gamma} \sim (E_{8(8)})^{\tilde{\lambda}\gamma}$.
 The Cartan decomposition of $So^*(16)$ is

$$So^*(16) \simeq ((SO(2) \times SU(8))/\mathbf{Z}_4 \times 2) \times \mathbf{R}^{56}$$

where $\mathbf{Z}_4 = \{(E, E), (E, -E), (-E, iE), (-E, -iE)\}$.

PROOF. The maximal compact subgroup $So^*(16)_K$ of $So^*(16)$ is

$$\begin{aligned} So^*(16)_K &= So^*(16) \cap E_8 = ((E_8^C)^{\tau\tilde{\lambda}\nu})^{\tilde{\lambda}\gamma}{}^{\tau\tilde{\lambda}} = ((\tau\tilde{\lambda}\nu)^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}} = ((\nu)^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}} \\ &= ((\tau\tilde{\lambda})^\nu)^{\tilde{\lambda}\gamma} = ((E_8)^\nu)^{\tilde{\lambda}\gamma} \cong ((SU(2) \times E_7)/\mathbf{Z}_2)^{\tilde{\lambda}\gamma} \\ &= ((\tau\tilde{\lambda})^{\tilde{\lambda}\gamma})^\nu = ((E_8)^{\tilde{\lambda}\gamma})^\nu = (Ss(16))^\nu. \end{aligned}$$

Let $\alpha \in So^*(16)_K = ((E_8)^\nu)^{\tilde{\lambda}\gamma}$. By Theorem 5.4.5, there exist $A \in SU(2)$, $\beta \in E_7$ such that $\alpha = \phi(A)\beta$. From the condition $\tilde{\lambda}\gamma\alpha = \alpha\tilde{\lambda}\gamma$, we have $\phi({}^tA^{-1})\tilde{\lambda}\gamma\beta\gamma\tilde{\lambda} = \phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases} {}^tA^{-1} = A \\ \tilde{\lambda}\gamma\beta\gamma\tilde{\lambda} = \beta \end{cases} \quad \text{or} \quad \begin{cases} {}^tA^{-1} = -A \\ \tilde{\lambda}\gamma\beta\gamma\tilde{\lambda} = \nu\beta. \end{cases}$$

In the first case, $A \in SO(2)$, $\beta \in (E_7)^{\tilde{\lambda}\gamma} = (E_7)^{\lambda\gamma} \cong SU(8)/\mathbf{Z}_2$ (where $\mathbf{Z}_2 = \{E, -E\}$) (Theorem 4.5.5). In the latter case, $A = (iI)B$, $B \in SO(2)$, $\beta = {}^t\beta'$, $\beta' \in (E_7)^{\lambda\gamma}$. Thus $So^*(16)_K \cong (SO(2) \times (E_7)^{\lambda\gamma} \cup (iI)SO(2) \times {}^t(E_7)^{\lambda\gamma})/\mathbf{Z}_2$ (where $\mathbf{Z}_2 = \{(E, 1), (-E,$

$-1\}) = (SO(2) \times (E_7)^{\lambda r}) / \mathbf{Z}_2 \times 2 \cong (SO(2) \times SU(8) / \mathbf{Z}_2) / \mathbf{Z}_2 \times 2$ (where $\mathbf{Z}_2 = \{(E, E), (-E, iE)\mathbf{Z}_2\} \cong (SO(2) \times SU(8)) / \mathbf{Z}_4 \times 2$. (As for the explicit form of $\psi(iI, \iota)$, see Theorem 5.4.6). $(E_{8(8)})^{\lambda r} \sim (\tau \nu \gamma)^{\lambda r}$ (Theorem 5.5.6. (4)) $= (\tau \tilde{\lambda} \nu)^{\lambda r}$.

THEOREM 5.5.9. $(E_{8(-24)})^{\lambda r} \sim (\tau \tilde{\lambda} \nu \sigma)^\sigma = So(4, 12)$ is connected. The Cartan decomposition of $So(4, 12)$ is

$$So(4, 12) \cong (SU(2) \times SU(2) \times Spin(12)) / (\mathbf{Z}_2 \times \mathbf{Z}_2) \times \mathbf{R}^{48}$$

where $\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(E, E, 1), (E, -E, -\sigma)\} \times \{(E, E, 1), (-E, -E, \sigma)\} = \{(E, E, 1), (E, -E, -\sigma), (-E, -E, \sigma), (-E, E, -1)\}$.

PROOF. $E_{8(-24)} = (E_8^C)^\tau \cong (E_8^C)^{\tau \tilde{\lambda} \nu \sigma}$.

In fact, since $\tilde{\lambda} \sim \nu \sigma$ under some $\delta \in E_8$: $\delta \tilde{\lambda} = \nu \sigma \delta$, $\delta \tau \tilde{\lambda} = \tau \tilde{\lambda} \delta$ (Proposition 5.3.4), $(E_8^C)^\tau \ni \alpha \rightarrow \delta \alpha \delta^{-1} \in (E_8^C)^{\tau \tilde{\lambda} \nu \sigma}$ gives an isomorphism. Consider the subgroup $((E_8^C)^{\tau \tilde{\lambda} \nu \sigma})^\sigma = (\tau \tilde{\lambda} \nu \sigma)^\sigma = (\tau \tilde{\lambda} \nu)^\sigma$ of $(\tau \tilde{\lambda} \nu \sigma)$. Since $\dim((E_8^C)^{\tau \tilde{\lambda} \nu \sigma}) = 120$, the type of $(\tau \tilde{\lambda} \nu)^\sigma$ must be D_8 . Moreover, as is shown in the following, the type of the maximal compact subgroup of $(\tau \tilde{\lambda} \nu)^\sigma$ is $D_2 \oplus D_6$, hence the type of the Lie algebra of $(\tau \tilde{\lambda} \nu)^\sigma$ is $\mathfrak{so}(4, 12)$. Hence we put here $So(4, 12) = (\tau \tilde{\lambda} \nu)^\sigma$. Now the maximal compact subgroup $So(4, 12)_K$ of $So(4, 12)$ is

$$\begin{aligned} So(4, 12)_K &= So(4, 12) \cap E_8 = ((E_8^C)^{\tau \tilde{\lambda} \nu \sigma})^{\tau \tilde{\lambda}} = ((\tau \tilde{\lambda} \nu)^\sigma)^{\tau \tilde{\lambda}} = ((\tau \tilde{\lambda})^\nu)^\sigma \\ &= ((E_8)^\nu)^\sigma \cong ((SU(2) \times E_7) / \mathbf{Z}_2)^\sigma. \end{aligned}$$

Let $\alpha \in So(4, 12)_K = ((E_8)^\nu)^\sigma$. By Theorem 5.4.5, there exist $A \in SU(2)$, $\beta \in E_7$ such that $\alpha = \phi(A)\beta$. From the condition $\sigma \alpha = \alpha \sigma$, we have $\phi(A)\sigma \beta \sigma = \phi(A)\beta$ (Lemma 5.4.3). Hence $\sigma \beta \sigma = \beta$, that is, $\beta \in (E_7)^\sigma \cong (SU(2) \times Spin(12)) / \mathbf{Z}_2$ (where $\mathbf{Z}_2 = \{(E, 1), (-E, -\sigma)\}$) (Theorem 4.6.14). Thus $((\tau \tilde{\lambda})^\nu)^\sigma \cong (SU(2) \times (E_7)^\sigma) / \mathbf{Z}_2$ (where $\mathbf{Z}_2 = \{(E, 1), (-E, -1)\} \cong (SU(2) \times (SU(2) \times Spin(12)) / \mathbf{Z}_2) / \mathbf{Z}_2$ (where $\mathbf{Z}_2 = \{(E, (E, 1)), (-E, (-E, \sigma)\mathbf{Z}_2)\} \cong (SU(2) \times SU(2) \times Spin(12)) / (\mathbf{Z}_2 \times \mathbf{Z}_2)$.

REMARK. The maximal compact subgroup $So(4, 12)_K$ of $So(4, 12) = (\tau)^{\lambda r}$ is

$$So(4, 12)_K = ((\tau)^{\lambda r})^{\tau \tilde{\lambda}} = ((\tau)^{\tilde{\lambda}})^r = ((\tau \tilde{\lambda})^{\tilde{\lambda}})^r = ((\tau \tilde{\lambda})^{\tilde{\lambda} r})^r = (Ss(16))^r.$$

The author can not give any isomorphism between $((\tau \tilde{\lambda})^{\tilde{\lambda}})^r$ and $((\tau \tilde{\lambda})^\nu)^\sigma$ directly.

Before determine the maximal compact subgroup $So(8, 8)_K$ of $So(8, 8)$, recall the construction of the spinor group $Spin(n)$ using the Clifford algebra $C(\mathbf{R}^n_-)$ [15]. Let $C(\mathbf{R}^n_-)$ be the Clifford algebra generated by e_1, \dots, e_n with relations $e_i^2 = -1$. Let \mathbf{R}^n be the \mathbf{R} -vector space spanned by e_1, \dots, e_n and put $S^{n-1} = \{a \in \mathbf{R}^n \mid a^2 = -1\}$. Now the spinor group $Spin(n)$ is defined by

$$Spin(n) = \{a_1 \cdots a_{2m} \mid a_i \in S^{n-1}, m=1, 2, \dots\}.$$

We use here the notation $Spin(n) = Spin(e_1, \dots, e_n)$. In the case of $n \equiv 0 \pmod{4}$, the center $z(Spin(n))$ of $Spin(n)$ is $\mathbf{Z}_2 \times \mathbf{Z}_2 = \{1, -1\} \times \{1, e_1 \cdots e_n\} = \{1, -1, e_1 \cdots e_n, -e_1 \cdots e_n\}$ and we know

$$Spin(n)/\{1, -1\} = SO(n), \quad Spin(n)/\{1, e_1 \cdots e_n\} = Ss(n).$$

THEOREM 5.5.10. $(E_{8(8)})^{\lambda r} \sim (\tau\sigma\gamma)^{\lambda r} = So(8, 8) = Sso(8, 8) \times 2$. The Cartan decomposition of $So(8, 8)$ is

$$So(8, 8) \simeq ((Spin(8) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2) \times 2) \times \mathbf{R}^{64}$$

where $\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1), (-1, -1)\} \times \{(1, 1), (e_1 \cdots e_8, e_9 \cdots e_{16})\} = \{(1, 1), (-1, -1), (e_1 \cdots e_8, e_9 \cdots e_{16}), (-e_1 \cdots e_8, -e_9 \cdots e_{16})\}$.

PROOF. Consider the group $(E_8^C)^{\tau\sigma r} = (\tau\sigma\gamma) \cong E_{8(8)}$ (Theorem 5.5.6.(2)). J. Sekiguchi [20] shows that $(\tau\sigma\gamma)/((\tau\sigma\gamma)^{\lambda r})_0$ is simply connected and the fundamental group of $(\tau\sigma\gamma)/(\tau\sigma\gamma)^{\lambda r}$ is \mathbf{Z}_2 , hence $Sso(8, 8) = (\tau\sigma\gamma)^{\lambda r}$ must have two connected components. The maximal compact subgroup $So(8, 8)_K$ of $So(8, 8)$ is

$$\begin{aligned} So(8, 8)_K &= So(8, 8) \cap E_8 = (((E_8^C)^{\tau\sigma r})^{\lambda r})^{\tau\bar{\lambda}} = ((\tau\sigma\gamma)^{\lambda r})^{\tau\bar{\lambda}} = ((\tau\sigma\gamma)^{\tau r})^{\tau\bar{\lambda}} \\ &= ((\sigma)^{\tau r})^{\tau\bar{\lambda}} = ((\sigma)^{\lambda r})^{\tau\bar{\lambda}} = ((\tau\bar{\lambda})^{\lambda r})^\sigma = (Ss(16))^\sigma. \end{aligned}$$

We shall find a subgroup of type $D_4 \oplus D_4$ in $Ss(16)$. In the spinor group $Spin(16) = Spin(e_1, \dots, e_{16})$, consider two subgroups $Spin(8) = Spin(e_1, \dots, e_8)$, $Spin(8) = Spin(e_9, \dots, e_{16})$. We define a mapping

$$Spin(8) \times Spin(8) \xrightarrow{\phi} Spin(16) \xrightarrow{\pi} Ss(16)$$

by $\phi(\alpha, \beta) = \alpha\beta$ and let π be the natural projection. Then the image of $\pi\phi$ is isomorphic to the group $(Spin(8) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$.

REMARK. The author can not find any element of $So(8, 8)$ which does not be contained in $Sso(8, 8)$, and can not realize the subgroup $(Spin(8) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ in $So(8, 8)$ concretely.

Appendix

The Cartan decompositions of the exceptional linear Lie groups of type E_8 are given as follows.

E_8 : simply connected compact Lie group of type E_8 ,

$$E_8^C \simeq E_8 \times \mathbf{R}^{248},$$

$$E_{8(8)} \simeq Ss(16) \times \mathbf{R}^{128},$$

$$E_{8(-24)} \simeq (SU(2) \times E_7) / \mathbf{Z}_2 \times \mathbf{R}^{112}.$$

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