TSUKUBA J. MATH.
Vol. 15 No. 1 (1991), 145—149

NOTE ON THE TAYLOR EXPANSION OF SMOOTH FUNCTIONS DEFINED ON SOBOLEV SPACES

By

Atsushi YOSHIKAWA*

§1. Introduction

It is well-known that the Sobolev spaces $H^{\sigma}(\boldsymbol{R}^{n})$ (with norm $\|\cdot\|_{\sigma}$) are multiplicative algebras when $\sigma>n/2$. Let $u\in H^{\sigma}(\boldsymbol{R}^{n})$ be real valued. If f is a rapidly decreasing function on the real line, i.e., $f\in \mathcal{S}(\boldsymbol{R})$, then we may speak of the composite function $f(u)$, which again belongs to $H^{\sigma}(\boldsymbol{R}^{n})$ provided $f(0)=0$ (See Rauch and Reed [1]). As for more precise results including higher order Taylor expansions, we have the following

THEOREM. *Suppose* $\sigma>(n/2)+1$, *and* u *and* $v\in H^{\sigma}(\boldsymbol{R}^{n})$ *are real valued. Let* $f\in \mathcal{S}(\boldsymbol{R})$. *Consider the* m*-th remainder*

$$R_{m}(f)(v;u)=f(v+u)-\sum_{k=0}^{m-1}\frac{1}{k!}f^{(k)}(v)u^{k} \tag{1.1}$$

of the Taylor expansion of $f(v+u)$ *around* $u=0$ $(m=1,2,\cdots)$. *Then* $R_{m}(f)(v;u)\in H^{\sigma}(\boldsymbol{R}^{n})$ *and, for* $0\leqq s\leqq\sigma$,

$$\|R_{m}(f)(v;u)\|_{s}\leqq A_{m,s}(1+\|v\|_{\mathrm{Max}(s,\sigma-1)}+\|v\|_{0}\|\nabla v\|_{\sigma-1}^{\mathrm{Max}(s,\sigma-1)}) \times\left(\frac{1}{m!}\|u^{m}\|_{s}+\frac{1}{(m+1)!}\|u\|_{(2m)}^{m}\|\nabla u\|_{\sigma-1}^{\mathrm{Max}(s,1)}\right), \tag{1.2}$$

where $A_{m,s}$ *is a positive constant independent of* u *and* v. *In the above,* ∇ *stands for the gradient operator, and* $\|w\|_{(p)}=\left(\int_{R}|w(x)|^{p}dx\right)^{1/p}$ *is the* L^{p}*-norm of a function* w *on* $\boldsymbol{R}^{n}$, $p>0$. *Note* $\|w\|_{(2)}=\|w\|_{0}$, *for* $H^{0}(\boldsymbol{R}^{n})=L^{2}(\boldsymbol{R}^{n})$.

REMARKS. (i) $\|u\|_{(2m)}$ makes sense for $u\in H^{\sigma}(\boldsymbol{R}^{n})$ since $\sigma>(n/2)+1$ and $H^{\sigma}(\boldsymbol{R}^{n})\subset H^{n(m-1)/2m}(\boldsymbol{R}^{n})\subset L^{2m}(\boldsymbol{R}^{n})$ by the Sobolev embedding theorem.

(ii) The constant $A_{m,s}$ admits the estimate

$$A_{m,s}\leqq C_{s}\frac{1}{2\pi}\int_{R}|\hat{f}(\tau)||\tau|^{m}(1+|\tau|^{s*})d\tau,$$

* Supported in part by Grant-in-Aid for Scientific Research, Ministry of Education, Science and Culture, Japanese Government, No. 62460005.
Received February 8, 1990. Revised June 1, 1990.

$s^*=1+\mathrm{Max}(s, 1)+\mathrm{Max}(s, \sigma-1)$. Here $\hat{f}(\tau)$ is the Fourier transform of f and C_s a positive constant independent of m and of f.

(iii) Similar results are valid when $\sigma>n/2$ and $\sigma\geqq 1$. Then we have to replace (1.2) by

$$\|R_m(f)(v\,;u)\|_s\leqq A_{m,s,\varepsilon}(1+\|v\|_\sigma+\|v\|_0\|\nabla v\|_{\sigma-1}^{\sigma/\varepsilon}) \times\Big(\frac{1}{m!}\|u^m\|_s+\frac{1}{(m+1)!}\|u\|_{(2m)}^m\|\nabla u\|_{\sigma-1}^{\mathrm{Max}(s/\varepsilon,1)}\Big), \tag{1.3}$$

where

$$A_{m,s,\varepsilon}\leqq C_{s,\varepsilon}\frac{1}{2\pi}\int_{\boldsymbol{R}}|\hat{f}(\tau)|\,|\tau|^m(1+|\tau|^{s^*(\varepsilon)})d\tau\,,$$

$s^*(\varepsilon)=1+(\sigma/\varepsilon)+\mathrm{Max}(s/\varepsilon, 1)$, $0<\varepsilon<\sigma-(n/2)$, $\varepsilon\leqq 1$.

The proof of Theorem is carried out by extending the idea of Rauch and Reed [1] where they discussed the case of $m=1$ and $v=0$, $f(0)=0$. Observe

$$R_m(f)(v\,;u)=\frac{1}{2\pi}\int_{\boldsymbol{R}}e^{iv\tau}\Big(e^{iu\tau}-\sum_{k=0}^{m-1}\frac{(i\tau u)^k}{k!}\Big)\hat{f}(\tau)d\tau\,,$$

where $\hat{f}(\tau)=\int_{\boldsymbol{R}}e^{-i\tau t}f(t)dt$ is the Fourier transform of $f(t)$. Then, for $0\leqq s\leqq\sigma$

$$\|R_m(f)(v\,;u)\|_s=\frac{1}{2\pi}\int_{\boldsymbol{R}}\Big\|e^{iv\tau}\Big(e^{iu\tau}-\sum_{k=0}^{m-1}\frac{(i\tau u)^k}{k!}\Big)\Big\|_s|\hat{f}(\tau)|\,d\tau\,.$$

Therefore, in order to prove Theorem, we only have to verify the estimate:

$$\begin{aligned}&\Big\|e^{iv\tau}\Big(e^{iu\tau}-\sum_{k=0}^{m-1}\frac{(i\tau u)^k}{k!}\Big)\Big\|_s\\ &\leqq C_s(1+\|v\|_{\mathrm{Max}(s,\sigma-1)}+\|v\|_0\|\nabla v\|_{\sigma-1}^{\mathrm{Max}(s,\sigma-1)})\\ &\times\Big(\frac{1}{m!}\|u^m\|_s+\frac{1}{(m+1)!}\|u\|_{(2m)}^m\|\nabla u\|_{\sigma-1}^{\mathrm{Max}(s,1)}\Big)(1+|\tau|^{s^*})|\tau|^m\,,\end{aligned} \tag{1.4}$$

for real τ provided $u, v\in H^\sigma(\boldsymbol{R}^n)$, $\sigma>(n/2)+1$, are real valued. Here $s^*=1+\mathrm{Max}(s, 1)+\mathrm{Max}(s, \sigma-1)$ and C_s is a positive constant independent of u, v, τ and m.

For a verification of (1.4), we appeal to the following

LEMMA 1.1. *Suppose $\sigma>(n/2)+1$, and m a positive integer. Let $w\in H^\sigma(\boldsymbol{R}^n)$ be real valued. Then $e^{iw}-\sum_{k=0}^{m-1}(iw)^k/k!\in H^\sigma(\boldsymbol{R}^n)$ and*

$$\begin{aligned}&\Big\|e^{iw}-\sum_{k=0}^{m-1}\frac{(iw)^k}{k!}\Big\|_s\\ &\leqq C_s\Big(\frac{1}{m!}\|w^m\|_s+\frac{1}{(m+1)!}\|w\|_{(2m)}^m\|\nabla w\|_{\sigma-1}^{\mathrm{Max}(s,1)}\Big),\end{aligned} \tag{1.5}$$

for $0 \leqq s \leqq \sigma$. *Here* C_s *is a positive constant independent of* m *and* w.

A proof will be given in the next section.

Let us derive (1.4) for $\tau=1$ from (1.5), since then (1.4) for general τ follows via an elementary inequality:

$$(1+r^a X+r^{a+b}Y)(r^d Z+r^{c+d}W) \leqq r^d(1+r^{a+b+c})(1+X+Y)(Z+W),$$

for all $r>0$. Here a, b, c, d, X, Y, Z, W are all positive. Observe the identity:

$$e^{iv}\Big(e^{iu}-\sum_{k=0}^{m-1}\frac{(iu)^k}{k!}\Big)$$

$$=(e^{iv}-1)\Big(e^{iu}-\sum_{k=0}^{m-1}\frac{(iu)^k}{k!}\Big)+\Big(e^{iu}-\sum_{k=0}^{m-1}\frac{(iu)^k}{k!}\Big).$$

In view of Lemma 1.1, we only need to show

(1.6) $$\|(e^{iv}-1)w\|_s \leqq C_s(\|v\|_{\mathrm{Max}(s,\sigma-1)}+\|v\|_0\|\nabla v\|_{\sigma-1}^{\mathrm{Max}(s,\sigma-1)})\|w\|_s$$

for all $w \in H^s(\boldsymbol{R}^n)$, $0 \leqq s \leqq \sigma$, when v is real valued. Now by Lemma 1.1 and the Sobolev embedding theorem,

$$\|(e^{iv}-1)w\|_0 \leqq C\|e^{iv}-1\|_{\sigma-1}\|w\|_0 \leqq C(\|v\|_{\sigma-1}+\|v\|_0\|\nabla v\|_{\sigma-1}^{\sigma-1})\|w\|_0,$$

while, for $\sigma \geqq s \geqq \sigma-1$,

$$\|(e^{iv}-1)w\|_s \leqq C\|e^{iv}-1\|_s\|w\|_s \leqq C(\|v\|_s+\|v\|_0\|\nabla v\|_{\sigma-1}^s)\|w\|_s.$$

(1.6) then follows by interpolating $0 \leqq s \leqq \sigma-1$.

Remark. We also have $\|(e^{iv}-1)w\|_0 \leqq 2\|w\|_0$ since v is real valued. Thus, when $\|v\|_{\sigma-1}+\|v\|_0\|\nabla v\|_{\sigma-1}^{\sigma-1}$ is very large, we have

$$\|(e^{iv}-1)w\|_s \leqq C(\|v\|_{\sigma-1}+\|v\|_0\|\nabla v\|_{\sigma-1}^{\sigma-1})^{s/(\sigma-1)}\|w\|_s,$$

for $0 \leqq s \leqq \sigma-1$.

§ 2. Proof of Lemma 1.1

Our proof of Lemma 1.1 is based on the following simplified analogue of Proposition 4.1 of Rauch and Reed [1].

Lemma 2.1. *Suppose* $g \in H^\sigma(\boldsymbol{R}^n)$ *is real valued. Let* $0 \leqq s \leqq \sigma$. *Then*

(2.1) $$|\mathrm{Re}\,(i\langle D\rangle^s M_g\langle D\rangle^{-s}w,\, w)| \leqq B_s\|\nabla g\|_{\sigma-1}\|w\|_{-1}\|w\|_0,$$

for all $w \in H^0(\boldsymbol{R}^n)$ *provided* $\sigma>(n/2)+1$. *Here* B_s *is a positive constant independent of* w *and* g *and* $(\ ,\ )$ *the inner product of* $H^0(\boldsymbol{R}^n)$. *Recall* M_g *is the multi-*

plication operator by the function g, and $\langle D\rangle^s$ *is the pseudo-differential oterator with the full symbol* $\langle\xi\rangle^s=(1+|\xi|^2)^{s/2}$, $\xi\in\boldsymbol{R}^n$.

PROOF. Since g is real valued,

$$\mathrm{Re}\, i(\langle D\rangle^s M_g\langle D\rangle^{-s}w,\ w)=\mathrm{Re}\, i([\langle D\rangle^s,\ M_g]\langle D\rangle^{-s}w,\ w)\,.$$

Then (2.1) is shown by the classical estimate (See, e.g., [2]):

$$|(v,\ [\langle D\rangle^s,\ M_g]u)|\leqq C\|\nabla g\|_{\sigma-1}(\|v\|_0\|u\|_{s'}+\|v\|_{-t'}\|u\|_s)\,,$$

$$s'\geqq s-1,\quad 1\geqq t'\geqq 0,\quad \sigma-\frac{n}{2}>t',\quad \sigma-\frac{n}{2}>s-s',\quad \sigma\geqq 1,\quad s>0\,.$$

We can choose $s'=s-1$, $t'=1$ if $\sigma>(n/2)+1$. If we merely have $\sigma>n/2$, $\sigma\geqq 1$, then we choose $s'=s-\varepsilon$, $t'=\varepsilon$ for $\sigma-(n/2)>\varepsilon>0$, $1\geqq\varepsilon>0$.

Now let us proceed to a verification of Lemma 1.1. The case when $m=1$ is essentially due to Rauch and Reed [1]. By slightly modifying their ideas, a proof of Lemma 1.1 for general m is obtained. Thus, to verify (1.5), we first reproduce a part of the discussions of Rauch and Reed [1], and then indicate our modification. Let

$$E_m(w)=e^{iw}-\sum_{k=0}^{m-1}\frac{(iw)^k}{k\,!}\,,\qquad m=1,\ 2,\ \cdots,$$

and

$$W_m(t)=\langle D\rangle^s E_m(tw)\,.$$

A straightforward computation yields to

$$\frac{d}{dt}W_m(t)=i\langle D\rangle^s M_w\langle D\rangle^{-s}W_m(t)+\frac{t^{m-1}}{(m-1)\,!}\langle D\rangle^s(iw)^m,$$

with $W_m(0)=0$. Taking the inner product of the both hand sides with $W_m(t)$, and using Lemma 2.1 we have,

$$\frac{d}{dt}\|W_m(t)\|_0\leqq B_s\|\nabla w\|_{\sigma-1}\|W_m(t)\|_{-1}+\frac{t^{m-1}}{(m-1)\,!}\|w^m\|_s\,. \tag{2.2}$$

Our idea is to employ the logarithmic convexity of the Sobolev scale. Thus, suppose $s>1$. Then

$$\|W_m(t)\|_{-1}=\|E_m(tw)\|_{s-1}\leqq\|E_m(tw)\|_0^{1-\theta}\|E_m(tw)\|_s^{\theta}\,,$$

$\theta=1-1/s$. Therefore, for any $\delta>0$,

$$\|W_m(t)\|_{-1}\leqq\delta^{\theta}\frac{t^m}{m\,!}\|w\|_{(2m)}^m+C_\theta\delta^{\theta-1}\|W_m(t)\|_0\,.$$

Here we have used the fact $\|E_m(tw)\|_0\leqq(t^m/m\,!)\|w\|_{(2m)}^m$, which is also a consequence of realness of w. It follows

$$\frac{d}{dt}\|W_m(t)\|_0 \leqq C_\theta B_s \|\nabla w\|_{\sigma-1}\delta^{\theta-1}\|W_m(t)\|_0$$

$$+B_s\delta^\theta\|\nabla w\|_{\sigma-1}\frac{t^m}{m!}\|w\|^m_{(2m)}+\frac{t^{m-1}}{(m-1)!}\|w^m\|_s .$$

Since Lemma 1.1 is trivial when $w=0$, we assume $w\neq 0$ so that $\nabla w\neq 0$. Choose $\delta=\|\nabla w\|^s_{\sigma-1}$. Then

$$\frac{d}{dt}\|W_m(t)\|_0 \leqq C_\theta B_s\|W_m(t)\|_0$$

$$+B_s\|\nabla w\|^s_{\sigma-1}\|w\|^m_{(2m)}\frac{t^m}{m!}+\|w^m\|_s\frac{t^{m-1}}{(m-1)!}.$$

Therefore, integrating from $t=0$ to $t=1$, we have

$$\|E_m(w)\|_s=\|W_m(1)\|_0\leqq B_s e^{C_\theta B_s}\frac{1}{(m+1)!}\|w\|^m_{(2m)}\|\nabla w\|^s_{\sigma-1}+e^{C_\theta B_s}\frac{1}{m!}\|w^m\|_s .$$

On the other hand, if $s\leqq 1$, then

$$\|W_m(t)\|_{-1}=\|\langle D\rangle^{s-1}E_m(tw)\|_0\leqq\|E_m(tw)\|_0\leqq\frac{t^m}{m!}\|w\|^m_{(2m)} .$$

Thus, (2.2) yields to

$$\frac{d}{dt}\|W_m(t)\|_0\leqq B_s\frac{t^m}{m!}\|\nabla w\|_{\sigma-1}\|w\|^m_{(2m)}+\frac{t^{m-1}}{(m-1)!}\|w^m\|_s ,$$

whence

$$\|E_m(w)\|_s\leqq\frac{B_s}{(m+1)!}\|\nabla w\|_{\sigma-1}\|w\|^m_{(2m)}+\frac{1}{m!}\|w^m\|_s .$$

References

[1] J. Reed and M. Rauch, Nonlinear microlocal analysis of semilinear hyperbolic system in one space dimension, Duke Math. J., **49** (1982), 337-475.

[2] A. Yoshikawa, On expansions of commutators acting in the Sobolev scale (preprint).