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THE IGUSA LOCAL ZETA FUNCTION ASSOCIATED WITH THE NONREGULAR IRREDUCIBLE PREHOMOGENEOUS VECTOR SPACE

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Introduction.

Let K be a p-adic field, i.e., a finite algebraic extention of $\mathbf{Q}_p$, where p is a rational prime number. We denote by O_K the maximal compact subring of K, by πO_K the ideal of nonunits of O_K and put $q=\#(O_K/\pi O_K)$. We denote by $|\ |_K$ the absolute value on K normalized as $|\pi|_K=q^{-1}$. We normalize the Haar measure $dX=|dx_1\wedge\cdots\wedge dx_n|_K$ on K^n by $\mathrm{vol}(O_K^n)=\int_{O_K^n}dX=1$.

We denote by $O_K[x_1, \cdots, x_n]$ the polynomial ring of n variables over O_K. For a polynomial $f(X)=f(x_1, \cdots, x_n)\in O_K[x_1, \cdots, x_n]$, J. Igusa proved that the integral

$$Z_f(s)=\int_{O_K^n}|f(X)|_K^s dX \qquad (s\in\mathbf{C})$$

is a rational function of $t=q^{-s}$ (see [I-4][I-5]), and we call it the Igusa local zeta function (abbrev. I. L. zeta, in this paper) attached to $f(X)$ after J. P. Serre. When the polynomial $f(X)$ is a relative invariant of an irreducible prehomogeneous vector space, it has interesting properties, and it is explicitly calculated for some regular prehomogeneous vector spaces (see [I-3]).

We abbreviate "a prehomogeneous vector space" as a P. V. (see [S-K]). In this paper, we shall consider the nonregular case. M. Sato and T. Kimura proved the following proposition ([S-K], § 4, Proposition 18).

PROPOSITION. *There is a one-to-one correspondence between the relative invariants* $f(X)$ *of* $P=(G\times GL_n, \rho\otimes\Lambda_1, V(m)\otimes V(n))(m>n\geqq 1)$ *and the relative invariants* $f^*(X^*)$ *of* $P^*=(G\times GL_{m-n}, \rho^*\otimes\Lambda_1, V(m)^*\otimes V(m-n))$. *Moreover, there exists a positive integer* d *for each* $f(X)$ *such that* $\deg f(X)=nd$ *and* $\deg f^*(X^*)=(m-n)d$. *If* $f(X)$ *is irreducible, then* $f^*(X^*)$ *is also irreducible. Here,* ρ^* *denotes the contragredient representation of* ρ *on the dual space* $V^*(m)$ *of* $V(m)$.

Here, P and P^* are called the castling transforms of each other. All non-

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regular irreducible P. V.'s with relative invariants are obtained from $(GL_1\times Sp_{2n}\times SO_3, \square\otimes\Lambda_1\otimes\Lambda_1, V(6n))$ by a finite number of castling transformations. Moreover, J. Igusa proved the following formula (see [I-2]):

(1) $Z^*_{f^*}(s)/Z_f(s)=\prod_{n<j\leq m-n}(j)/(1-q^{-j}t^d)$ for $n<m-n$,

where $Z_f(s)$ (resp. $Z^*_{f^*}(s)$) is the I. L. zeta attached to $f(X)$ (resp. $f^*(X^*)$) and we put $(j)=(1-q^{-j})$ for $j\in\mathbf{Z}$. Therefore, it is enough to consider the I. L. zeta associated with the nonregular irreducible P. V. $(GL_1\times Sp_{2n}\times SO_3, \square\otimes\Lambda_1\otimes\Lambda_1, V(6n))$, i. e., we shall consider the explicit form of the I. L. zeta $Z_P(s)$ attached to the quartic invariant $P(X)$ of our P. V. and check some conjectures on the I. L. zeta's.

§1. Computation of $Z_P(s)$.

We shall review an integration formula given in [K] for the I. L. zeta $Z_f(s)$ attached to a polynomial $f(X)\in O_K[x_1, \cdots, x_n]$.

PROPOSITION 1.1. (*Integratson formula,* [K]). *For* $1\leqq m\leqq n$, *put*

$$D_i=\{(x_1, \cdots, x_{i-1}, \lambda_i, x_{i+1}, \cdots, x_m)\in O_K^m \mid (x_1, \cdots, x_{i-1})\in\pi O_K^{i-1}\},$$

then we have

(2) $$Z_f(s)=\sum_{i=1}^m\int_{D_i\times O_K^{n-m}}|f(\lambda_i x_1, \cdots, \lambda_i x_{i-1}, \lambda_i, \lambda_i x_{i+1}, \cdots, \lambda_i x_m, x_{m+1}, \cdots, x_n)|_K^s$$
$$|\lambda_i^{m-1}dx_1\wedge\cdots\wedge dx_{i-1}\wedge d\lambda_i\wedge dx_{i+1}\wedge\cdots\wedge dx_m\wedge\cdots\wedge dx_n|_K.$$

LEMMA 1.2. *For positive integers d and m, we have*

$$\int_{O_K}|\lambda|_K^{ds+m-1}d\lambda=(1)/(1-q^{\ m}t^d).$$

This lemma is well-known and easily proved. When the polynomial $f(X)$ is homogeneous of degree d with respect to the m variables $x_1, \cdots, x_m$, then the integration formula (2) can be expressed as follows:

(3) $Z_f(s)=[(1)/(1-q^{-m}t^d)]\sum_{i=1}^m q^{-(i-1)}Z_{f,i}(s)$,

where we put

$$Z_{f,i}(s)=\int_{O_K^n}|f(\pi x_1, \cdots, \pi x_{i-1}, 1, x_{i+1}, \cdots, x_m, \cdots, x_n)|_K^s dX.$$

We identify the representation space $V(6n)$ of our P. V. with the totality of $2n\times 3$ matrices $M(2n, 3)$. Then the action $\rho=\square\otimes\Lambda_1\otimes\Lambda_1$ is given by $\rho(g)X=\alpha AX{}^tB$ with $g=(\alpha, A, B)\in GL_1\times Sp_{2n}\times SO_3$ and $X\in M(2n, 3)$.

For $X\in M(2n, 3)$, put

$$P(X)=(-1/2)tr\{({}^tXJX)^2\} \quad\text{with}\quad J=\begin{pmatrix} 0 & 1_n \\ -1_n & 0\end{pmatrix}.$$

This quartic form $P(X)$ is a relative invariant of our P. V. In fact, we have

$$\begin{aligned}P(\rho(g)X)&=(-1/2)tr\{[{}^t(\alpha AX{}^tB)J(\alpha AX{}^tB)]^2\}\\&=(-1/2)tr\{[\alpha^2B{}^tX({}^tAJA)XB^{-1})]^2\}\\&=(-1/2)tr\}[\alpha^2B({}^tXJX)B^{-1})]^2\}\\&=\alpha^4P(X)\qquad\text{for}\quad g=(\alpha,\,A,\,B)\in GL_1\times Sp_{2n}\times SO_3.\end{aligned}$$

Any relative invariant is of the form $cP(X)^m$ with some integer m and nonzero constant c. One can see easily that, for $X=\begin{pmatrix}x\\y\end{pmatrix}$ with $x=(x_{ij})$ and $y=(y_{ij})$ in $M(2n,3)$, we have

(4) $P(X)=\sum_{1\leq i<j\leq 3}[\sum_{k=1}^{n}d(k\,;\,i,\,j)]^2$,

where we put $d(k\,;\,i,\,j)=\det\begin{pmatrix}x_{ki} & y_{ki}\\x_{kj} & y_{kj}\end{pmatrix}$.

Now we apply the integration formula (3) to our local zeta function $Z_P(s)$ with respect to the $3n$ variables $x=(x_{ij})$. Then we obtain

(5) $Z_P(s)=[(1)/(1-q^{-3n}t^2)]\sum_{1\leq i\leq n,\,j=1,2,3}q^{-[3(i-1)+j-1]}Z_{P(i,j)}(s)$.

we introduce the notations $x_{(i,j)}$, $\bar{x}_{(i,j)}$ and $y_{(j)}$ for $1\leqq i\leqq n$, $j=1,2,3$, as follows:

$x_{(i,j)}$ is an $n\times 3$ matrix of the form $x_{(i,j)}=(x_1,\,x_2,\,x_3)$ with $x_k={}^t(\pi x_{1k},\,\cdots,\,\pi x_{i-1,k},\,t_{ik},\,x_{i+1,k},\,\cdots,\,x_{nk})$ where t_{ik} is πx_{ik} (resp. 1, x_{ik}) when $1\leqq k<j$ (resp. $k=j$, $j<k\leqq 3$).

$\bar{x}_{(i,j)}$ is an $n\times 3$ matrix of the same as $x_{(i,j)}$ except the j-th column vector which is ${}^t(0,\,\cdots,\,\overset{i}{1},\,\cdots,\,0)$, and $y_{(j)}=(y_{rs})$ with $y_{rj}=0$ for all r. Then, with these notations, we may express

(6) $Z_{P(i,j)}(s)=\int_{O_K^{6n}}|P(x_{(i,j)}|\,y_{(j)})|_K^s dX$.

LEMMA 1.3. *In the right hand side of* (6), *we can replace* $P(x_{(i,j)}|\,y)$ *by a simpler polynomial* $P(\bar{x}_{(i,j)}|\,y_{(j)})$ *in a smaller number of variables, i.e., we have*

$$Z_{P(i,j)}(s)=\int_{O_K^{6n}}|P(\bar{x}_{(i,j)}|\,y_{(j)})|_K^s dX.$$

PROOF. We put

$$A^{(i,j)}=\begin{pmatrix}A_1 & 0\\ 0 & {}^tA_1^{-1}\end{pmatrix}\in M(2n),$$

where $A_1=(a_1,\,\cdots,\,a_i,\,\cdots,\,a_n)$ with $a_k={}^t(0,\,\cdots,\,\overset{k}{1},\,\cdots,\,0)$ for $k\neq i$, and $a_i={}^t(-\pi x_{1j},\,\cdots,\,-\pi x_{i-1,j},\,1,\,-x_{i+1,j},\,\cdots,\,-x_{nj})$.

Then this $2n\times 2n$ matrix $A^{(i,j)}$ is an element of Sp_{2n}. We define a map φ from O_K^{6n} to itself by $\varphi(X)=A^{(i,j)}X$, then the map φ gives a measure-preserving analytic homeomorphism of O_K^{6n} to itself, and $\varphi(x_{(i,j)}|y)$ is of the form $(\bar{x}_{(i,j)}|y)$. Since the relative invariant $P(X)$ is invariant under the action of Sp_{2n}, we have

$$Z_{P(i,j)}(s)=\int_{O_K^{6n}}|P(\bar{x}_{(i,j)}|y)|_K^s dX.$$

Similarly, we have

$$\int_{O_K^{6n}}|P(\bar{x}_{(i,j)}|y)|_K^s dX=\int_{O_K^{6n}}|P(\bar{x}_{(i,j)}|y_{(j)})|_K^s dX$$

by the action of $\bar{A}^{(i,j)}=\begin{pmatrix}1_n & 0\\ A_2 & 1_n\end{pmatrix}\in Sp_{2n}$, where $A_2=(b_1,\cdots,b_i,\cdots,b_n)$ with $b_k={}^t(0,\cdots,\overset{i}{-y_{kj}},\cdots,0)$ for $k\neq i$, and $b_i={}^t(-y_{1j},\cdots,-y_{ij},\cdots,-y_{nj})$.

Q. E. D.

Now we shall consider the partial integral $Z_{P(i,1)}(s)$. Applying the integration formula (3) to $Z_{P(i,j)}(s)$ with respect to the $2n$ variables y_{n2}, y_{n3}, $y_{n-1,2}$, $y_{n-1,3}$, $\cdots$, y_{12}, y_{13} in this order, we have

(7) $Z_{P(i,1)}(s)=[(1)/(1-q^{-2n}t^2)]\sum_{k=1}^{n}q^{-2(n-k)}[Z_{P(i,1)}(s)_{(k,2)}+q^{-1}Z_{P(i,1)}(s)_{(k,3)}]$.

In the right hand of (7), we put

$$Z_{P(i,1)}(s)_{(k,h)}=\int_{O_K^{6n}}|A_{k,h}(y)+B_{k,h}(x,y)^2|_K^s dX,$$

where we put

$$A_{k,h}(y)\begin{cases}=y_{i2}^2+y_{i3}^2\ (i<k\leqq n)\\ =1+y_{i3}^2\ (k=i,\ h=2),\ 1+\pi^2y_{i2}^2\ (k=i,\ h=3)\\ =\pi^2y_{i2}^2+\pi^2y_{i3}^2\ (1\leqq k<i,\ h=2,3)\end{cases}$$

and

$$B_{k,h}(x,y)=\pi\sum_{l=i}^{k-1}d(l;2,3)+\sum_{l=i}^{k-1}d(l;2,3)$$
$$+d(k;2,3)_h+\pi\sum_{l=k+1}^{n}d(l;2,3)\quad(i<k\leqq n,\ h=2,3)$$
$$B_{k,h}(x,y)=\pi\sum_{l=i}^{k-1}d(l;2,3)$$
$$+d(k;2,3)_h+\pi\sum_{l=k+1}^{n}d(l;2,3)\quad(k=i,\ h=2,3)$$
$$B_{k,h}(x,y)=\pi\sum_{l=i}^{k-1}d(l;2,3)+\pi d(k;2,3)_h+\pi\sum_{h=i}^{n}d(h;2,3)\quad(1\leqq k<i).$$

Here we define

$$d(k\,;\,2,3)_2=\det\begin{pmatrix} x_{k2} & 1 \\ x_{k3} & y_{k3}\end{pmatrix} \quad\text{and}\quad d(k\,;\,2,3)_3=\det\begin{pmatrix} x_{k2} & \pi y_{k2} \\ x_{k3} & 1\end{pmatrix}.$$

Now we put

$$Z_1(s)=\int_{O_K^3}|x^2+y^2+z^2|_K^s|dx\wedge dy\wedge dy|_K,$$

$$Z_2(s)=\int_{O_K^2}|1+x^2+y^2|_K^s|dx\wedge dy|_K,$$

and

$$Z_3(s)=\int_{O_K^2}|1+x^2+\pi^2y^2|_K^s|dx\wedge dy|_K.$$

Then, by a suitable change of variables, we have

$$Z_{P(i,1)}(s)_{(k,l)}=Z_1(s) \qquad (i+1\leqq k\leqq n,\ l=2,3),$$

$$Z_{P(i,1)}(s)_{(i,2)}=Z_2(s), \qquad Z_{P(i,1)}(s)_{(i,3)}=Z_3(s),$$

and

$$Z_{P(i,1)}(s)_{(k,l)}=t^2Z_1(s) \qquad (1\leqq k\leqq i-1,\ l=2,3).$$

For example, we put

$$\tilde{x}_{k3}=B_K(y)=-x_{k3}+\cdots, \qquad \text{for}\quad i+1\leqq k\leqq n,$$

then we have

$$Z_{P(i,1)}(s)_{(k,2)}=\int_{O_K^3}|y_{i2}^2+y_{i3}^2+\tilde{x}_{k3}^2|_K^s|dy_{i2}\wedge dy_{i3}\wedge d\tilde{x}_{k3}|_K$$
$$=Z_1(s).$$

Therefore, we have, from the formula (7),

(8) $Z_{P(i,1)}(s)=[(1)/(1-q^{-2n}t^2)]\times[A\cdot Z_1(s)+q^{-2(n-j)}(Z_2(s)+q^{-1}Z_3(s))]$

where we put

$$A=[(2n-2i)+(2i-2)q^{-2s-2(n-i+1)}]/(1).$$

Similarly, we have

(9) $Z_{P(i,2)}(s)=[(1)/(1-q^{-2n}t^2)]\times[A\cdot Z_1(s)+q^{-2(n-i)}(Z_2(s)+q^{-1})].$

(10) $Z_{P(i,3)}(s)=[(1)/(1-q^{-2n}t^2)]\times[A\cdot Z_1(s)+q^{-2(n-i)}(Z_3(s)+q^{-1})].$

Therefore, by the formulas (5), (8), (9) and (10), we have

(11) $Z_P(s)=[(1)^2/(1-q^{-2n}t^2)(1-q^{-3n}t^2)]\times$

$$[B\cdot Z_1(s)+q^{-2(n-1)}\cdot C\cdot(Z_2(s)+q^{-1}Z_3(s)+q^{-2})],$$

where we put

$$B=\{[(1)-(3)q^{-2(n-1)}+(2)q^{-3n+2}]+[(2)q^{-1}-(3)q^{-n}+(1)q^{-3n}]q^{-2n}t^2\}/(1)^3$$

and $C=(1+q^{-1})(n)/(1)$.

LEMMA 1.4. *We have the following formlas.*

(a) $Z_1(s)=(1)(1-q^{-3}t)/(1-q^{-1}t)(1-q^{-3}t^2)$.

(b) $Z_2(s)+q^{-1}Z_3(s)+q^{-3}=(1-q^{-3}t)/(1-q^{-1}t)$.

PROOF. (a) This is a classical result. (b) Applying the integration formula (3) to $Z_1(s)$ with respect to all variables x, y and z, we have

$$Z_1(s)=[(1)/(1-q^{-3}t^2)]\times[Z_2(s)+q^{-1}Z_3(s)+q^{-2}].$$

Q. E. D.

Combining the formula (11) with Lemma 1.4., we obtain the following theorem.

THEOREM. *Let $Z_P(s)$ be the I. L. zeta associated with the (reduced) nonregular irreducible P. V.*

$$(GL_1\times Sp_{2n}\times SO_3,\ \square\otimes\Lambda_1\otimes\Lambda_1,\ V(6n)),$$

then we have

$$Z_P(s)=[(1)(1-q^{-3}t)/(1-q^{-1}t)(1-q^{-3}t^2)]\times(2n)/(1-q^{-2}t^2).$$

§ 2. Some Remarks on $Z_P(s)$.

We shall give some remarks on $Z_P(s)$.

A. A factor $(1)(1-q^{-3}t)/(1-q^{-1}t)(1-q^{-3}t^2)$ of $Z_P(s)$ is the I. L. zeta $Z_1(s)$ associated with $(GL_1\times SO_3,\ \square\otimes\Lambda_1,\ V(3))$.

B. The I. L. zeta $Z_f(s)$ can be defined for any p-adic field K, and the I. L. zeta $Z_f(s)$ depends on the choice of the p-adic fields K. If there exists a rational function $Z_f(u, v)\in\mathbf{Q}(u, v)$ satisfying

$$Z_f(s)=Z_f(q^{-1}, q^{-s}),$$

which is independent of the choice of the p-adic fields K, then we call it "the universal p-adic zeta function" for the polynomial $f(X)$ (see [I-1]). J. Igusa proved that the universal p-adic zeta function $Z_f(u, v)$ for the relative invariant $f(X)$ of some irreducible P. V.'s satisfies the functional equation:

$$Z_f(u^{-1}, v^{-1})=v^{\deg f}Z_f(u, v).$$

In [I-1], J. Igusa also conjectured that if the universal p-adic zeta function $Z_f(u, v)$ exists for a homogeneous polynomial $f(X)$ with coefficients in a number field, then $Z_f(u, v)$ satisfies the above functional equation.

Our calculation of $Z_P(s)$ shows that Igusa's conjecture holds for the nonregular irreducible case. In fact, if we put

$$Z_P(u, v)=[(1-u)(1-u^3v)/(1-uv)(1-u^3v^2)]\times[(1-u^{2n})/(1-u^{2n}v^2)],$$

then this rational function $Z_f(u, v)$ is the universal p-adic zeta function for $P(X)$, and satisfies the functional equation:

$$Z_P(u^{-1}, v^{-1})=v^4Z_P(u, v).$$

C. T. Kimura, F. Sato and X. Zhu have proved that any real poles of the I. L. zeta associated with an irreducible reduced regular P. V. is a special root of the b-function $b(s)$ (see [K-S-Z], §2, Main theorem 2.1.). In the nonregular case, the b-function (in the sense of the Bernstein polynomial) associated with $(GL_1\times Sp_{2n}\times SO_3, \square\otimes\Lambda_1\otimes\Lambda_1, V(6n))$ is

$$b(s)=(s+1)(s+3/2)(s+2n/2)(s+(2n+1)/2).$$

Therefore, all poles $\{1, 3/2, 2n/2\}$ of $Z_P(s)$ are roots of the b-function $b(s)$.

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