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SOME PROPERTIES OF CERTAIN MULTIVALENT FUNCTIONS

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By

Hitoshi SAITOH

1. Introduction

Let A_p be the class of functions of the form

$$f(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}z^{n} \qquad (p\in N=\{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U=\{z: |z|<1\}$.

A function $f(z)\in A_p$ is said to be in the class $R_p(\alpha)$ if it satisfies

$$\mathrm{Re}\{f^{(p)}(z)\}>\alpha \tag{1.2}$$

for some $\alpha(0\leqq\alpha<p!)$ and for all $z\in U$. Let the functions $F(z)$ and $G(z)$ be analytic in the unit disk U. Then the function $F(z)$ is said to be subordinate to $G(z)$ if there exists a function $w(z)$ analytic in U, with $w(0)=0$ and $|w(z)|<1(z\in U)$, such that $F(z)=G(w(z))(z\in U)$. We denote this subordination by $F(z)\prec G(z)$.

For $f(z)\in R_p(\alpha)$, Saitoh has proved that

$$f(z)\in R_p(\alpha)\Longrightarrow \frac{f^{(p-1)}(z)}{z}\prec 2\alpha-p!-\frac{2(p!-\alpha)}{z}\ln(1-z). \tag{1.3}$$

This is a generalization of the result for $p=1$ by Owa, Ma and Liu [5].

Let $S_p(\alpha)$ be the subclass of A_p consisting of functions which satisfy

$$f^{(p)}(z)\prec p!+(p!-\alpha)z \qquad (z\in U) \tag{1.4}$$

for some $\alpha(0\leqq\alpha<p!)$. Then, it is easy to see that

$$S_p(\alpha)\subset R_p(\alpha) \qquad (0\leqq\alpha<p!)$$

and that $f(z)\in A_p$ is in the class $S_p(\alpha)$ if and only if

$$|f^{(p)}(z)-p!|<p!-\alpha \qquad (z\in U) \tag{1.5}$$

for some $\alpha(0\leqq\alpha<p!)$.

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2. Some properties of the class $S_p(\alpha)$

We begin with the statement of the following lemma due to Jack [1] (also, due to Miller and Mocanu [4]).

LEMMA 1. *Let* $w(z)$ *be regular in* $\boldsymbol{U}$ *with* $w(0)=0$. *If* $|w(z)|$ *attains its maximum value in the circle* $|z|=r$ *at a point* $z_0\in\boldsymbol{U}$, *then we can write*

$$z_0w'(z_0)=kw(z_0),$$

where k is real and $k\geqq 1$.

An application of the above lemma leads to

THEOREM 1. *If* $f(z)\in\boldsymbol{A}_p$ *satisfies*

$$|\beta f^{(p)}(z)+(1-\beta)zf^{(p+1)}(z)-p!\beta|<p!-\alpha \qquad (z\in\boldsymbol{U}) \tag{2.1}$$

for some $\alpha(0\leqq\alpha<p!)$ and $\beta(0\leqq\beta\leqq 1)$, then $f(z)\in\boldsymbol{S}_p(\alpha)$.

PROOF. We define the function $w(z)$ by

$$w(z)=\frac{f^{(p)}(z)-p!}{p!-\alpha}. \tag{2.2}$$

Then $w(z)$ is regular in $\boldsymbol{U}$ and $w(0)=0$. Since

$$zf^{(p+1)}(z)=(p!-\alpha)zw'(z), \tag{2.3}$$

we have

$$\begin{aligned}\beta f^{(p)}(z)+(1-\beta)zf^{(p+1)}(z)-p!\beta \\ =(p!-\alpha)\{\beta w(z)+(1-\beta)zw'(z)\}.\end{aligned} \tag{2.4}$$

If there exists exists a point $z_0\in\boldsymbol{U}$ such that

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=1,$$

then Lemma 1 gives

$$z_0w'(z_0)=kw(z_0) \qquad (k\geqq 1).$$

Therefore, noting that $w(z_0)=e^{i\theta}$, we obtain that

$$\begin{aligned}|\beta f^{(p)}(z_0)+(1-\beta)z_0f^{(p+1)}(z_0)-p!\beta| \\ =(p!-\alpha)(\beta+(1-\beta)k) \\ \geqq p!-\alpha,\end{aligned} \tag{2.5}$$

which contradicts our condition (2.1). Thus $|w(z)|<1$ for all $z\in\boldsymbol{U}$, that is, (2.6) $$|w(z)|=\left|\frac{f^{(p)}(z)-p\,!}{p\,!-\alpha}\right|<1 \qquad (z\in \boldsymbol{U}).$$

This completes the proof of Theorem 1.

Letting $\beta=0$ in Theorem 1, we have

COROLLARY 1. *If* $f(z)\in \boldsymbol{A}_p$ *satisfies*

(2.7) $$|zf^{(p+1)}(z)|<p\,!-\alpha \qquad (z\in \boldsymbol{U})$$

for some $\alpha(0\leqq\alpha<p\,!)$, *then* $f(z)\in \boldsymbol{S}_p(\alpha)$.

Further making $\beta=1/2$, Theorem 1 leads to

COROLLARY 2. *If* $f(z)\in \boldsymbol{A}_p$ *satisfies*

(2.8) $$|f^{(p)}(z)+zf^{(p+1)}(z)-p\,!|<2(p\,!-\alpha) \qquad (z\in \boldsymbol{U})$$

for some $\alpha(0\leqq\alpha<p\,!)$, *then* $f(z)\in \boldsymbol{S}_p(\alpha)$.

Next, we prove

THEOREM 2. *If* $f(z)$ *is in the class* $\boldsymbol{S}_p(\alpha)$, *then*

(2.9) $$\mathrm{Re}\left\{e^{i\beta}\frac{f^{(p-1)}(z)}{z}\right\}>0 \qquad (z\in \boldsymbol{U}),$$

where

(2.10) $$|\beta|\leqq\frac{\pi}{2}-\mathrm{Sin}^{-1}\left(\frac{p\,!-\alpha}{p\,!}\right).$$

PROOF. It follows from the definition of the class $\boldsymbol{S}_p(\alpha)$ that

$$f(z)\in \boldsymbol{S}_p(\alpha)\Longleftrightarrow |f^{(p)}(z)-p\,!|<p\,!-\alpha \qquad (z\in \boldsymbol{U})$$
$$\Longrightarrow \mathrm{Re}\{e^{i\beta}f^{(p)}(z)\}>0 \qquad (z\in \boldsymbol{U})$$
$$\Longrightarrow \mathrm{Re}\{e^{i\beta}f^{(p)}(z)-ip\,!\sin\beta\}>0 \qquad (z\in \boldsymbol{U}).$$

Defining the function $w(z)$ by

(2.11) $$e^{i\beta}\frac{f^{(p-1)}(z)}{z}-ip\,!\sin\beta=p\,!\cos\beta\cdot\frac{1+w(z)}{1-w(z)} \qquad (w(z)\neq 1),$$

we see that $w(z)$ is regular in $\boldsymbol{U}$ with $w(0)=0$. Since $\cos\beta>0$,

(2.12) $$e^{i\beta}f^{(p-1)}(z)-ip\,!\sin\beta\cdot z=p\,!\cos\beta\cdot\frac{1+w(z)}{1-w(z)}z,$$

and

(2.13) $$e^{i\beta}f^{(p)}(z)-ip\,!\sin\beta=p\,!\cos\beta\left\{\frac{1+w(z)}{1-w(z)}+\frac{2zw'(z)}{(1-w(z))^2}\right\},$$

we have

$$\operatorname{Re}\{e^{i\beta}f^{(p)}(z)-ip!\sin\beta\}=p!\cos\beta\cdot\operatorname{Re}\left\{\frac{1+w(z)}{1-w(z)}+\frac{2zw'(z)}{(1-w(z))^2}\right\} >0. \tag{2.14}$$

Suppose that there exists a point $z_0\in \boldsymbol{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1 \qquad (w(z_0)\neq 1).$$

Then, applying Lemma 1, and letting $w(z_0)=e^{i\theta}$, we obtain

$$\begin{aligned}\operatorname{Re}\{e^{i\beta}f^{(p)}(z_0)-ip!\sin\beta\}\\ &=p!\cos\beta\cdot\operatorname{Re}\left\{\frac{1+e^{i\theta}}{1-e^{i\theta}}+\frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right\}\\ &=p!\cos\beta\cdot\frac{k}{\cos\theta-1}\\ &<0,\end{aligned} \tag{2.15}$$

which contradicts (2.14). Therefore, $|w(z)|<1$ for all $z\in\boldsymbol{U}$, which implies that

$$\begin{aligned}\operatorname{Re}\left\{e^{i\beta}\frac{f^{(p-1)}(z)}{z}-ip!\sin\beta\right\}\\ &=\operatorname{Re}\left\{e^{i\beta}\frac{f^{(p-1)}(z)}{z}\right\}\\ &>0.\end{aligned} \tag{2.16}$$

Taking $\alpha=0$ in Theorem 2, we have

COROLLARY 3. *If $f(z)$ is in the class $\boldsymbol{S}_p(\alpha)$, then*

$$\operatorname{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\}>0 \qquad (z\in\boldsymbol{U}). \tag{2.17}$$

3. A Subclass $F_{p,b}(\alpha)$

Let $\boldsymbol{G}(\alpha)$ be the class of functions $g(z)$ of the form

$$g(z)=1+\sum_{n=1}^{\infty}g_nz^n \tag{3.1}$$

which are analytic in $\boldsymbol{U}$ and satisfy

$$\operatorname{Re}\{g(z)\}>\alpha \qquad (z\in\boldsymbol{U}) \tag{3.2}$$

for some $\alpha(0\leqq\alpha<1)$. Further, let $\boldsymbol{G}_b(\alpha)$ be the subclass of $\boldsymbol{G}(\alpha)$ consisting of functions $g(z)$ of the form (3.1) satisfying

$$g_1=2b(1-\alpha)\equiv g'(0) \qquad (0\leqq b\leqq 1). \tag{3.3}$$

For the above class $G_b(\alpha)$, McCarty ([2], [3]) has shown that

LEMMA 2. ([2]). *If* $g(z)\in G_b(\alpha)$, *then*

$$\left|\frac{g'(z)}{g(z)}\right|\leqq\frac{2(1-\alpha)}{1-r^2}\left\{\frac{b+2r+br^2}{1+2b(1-\alpha)r+(1-2\alpha)r^2}\right\}\quad(r=|z|<1). \tag{3.4}$$

LEMMA 3 ([3]). *If* $g(z)\in G_b(\alpha)$, *then*

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\}\geqq\begin{cases}\dfrac{-2(1-\alpha)r(b+2r+br^2)}{(1+2\alpha br+(2\alpha-1)r^2)(1+2br+r^2)} & (R'\leqq R_b)\\ \dfrac{2\sqrt{\alpha A_1}-A_1-\alpha}{1-\alpha} & (R'\geqq R_b),\end{cases} \tag{3.5}$$

where

$$R_b=A_b-D_b, \tag{3.6}$$

$$A_b=\frac{(1+br)^2-(2\alpha-1)(b+r)^2r^2}{(1+2br+r^2)(1-r^2)}, \tag{3.7}$$

$$D_b=\frac{2(1-\alpha)(b+r)(1+br)r}{(1+2br+r^2)(1-r^2)}, \tag{3.8}$$

and

$$R'=\sqrt{\alpha A_1}. \tag{3.9}$$

Let $F_{p,b}(\alpha)$ be the subclass of $R_p(\alpha)$ consisting of functions $f(z)\in R_p(\alpha)$ satisfying

$$a_{p+1}=\frac{2b(p\,!-\alpha)}{(p+1)\,!}\equiv\frac{f^{(p+1)}(0)}{(p+1)\,!}, \tag{3.10}$$

where $0\leqq\alpha<p\,!$ and $0\leqq b\leqq1$.

Now, we have

THEOREM 3. *If* $f(z)\in F_{p,b}(\alpha)$, *then*

$$\left|\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right|\leqq\frac{2(p\,!-\alpha)r}{1-r^2}\left\{\frac{b+2r+br^2}{p\,!+2b(p\,!-\alpha)r+(p\,!-2\alpha)r^2}\right\} \tag{3.11}$$
$$(r=|z|<1).$$

PROOF. Note that $f(z)\in F_{p,b}(\alpha)$ implies

$$\frac{f^{(p)}(z)}{p\,!}=1+2b\left(1-\frac{\alpha}{p\,!}\right)z+\cdots \tag{3.12}$$

and $0\leqq b\leqq1$, $0\leqq\alpha<p\,!$. It follows from (3.12) that $f(z)\in F_{p,b}(\alpha)$ if and only if $f^{(p)}(z)/p\,!\in G_b(\alpha/p\,!)$. Therefore, (3.11) follows Lemma 2.

Also, using Lemma 3, we have

THEOREM 4. *If* $f(z)\in \boldsymbol{F}_{p,b}(\alpha)$, *then*

$$(3.13)\qquad \mathrm{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}\geqq\begin{cases}\dfrac{-2(p\,!-\alpha)r(b+2r+br^2)}{(p\,!+2\alpha br+(2\alpha-p\,!)r^2)(1+2br+r^2)} & (T'\leqq T_b)\\[2ex] \dfrac{2\sqrt{p\,!\alpha B_1}-p\,!B_1-\alpha}{p\,!-\alpha} & (T'\geqq T_b),\end{cases}$$

where

$$(3.14)\qquad T_b=B_b-C_b,$$

$$(3.15)\qquad B_b=\frac{p\,!(1+br)^2-(2\alpha-p\,!)(b+r)^2r^2}{p\,!(1+2br+r^2)(1-r^2)},$$

$$(3.16)\qquad C_b=\frac{2(p\,!-\alpha)(b+r)(1+br)r}{p\,!(1+2br+r^2)(1-r^2)},$$

and

$$(3.17)\qquad T'=\sqrt{\frac{\alpha B_1}{p\,!}}.$$

4. Generalization of Saitoh's result

Finally, we give the generalization theorem of (1.3) which was recently proved by Saitoh [6].

THEOREM 5. *If* $f(z)\in \boldsymbol{A}_p$ *satisfies*

$$(4.1)\qquad \mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>\alpha \qquad (z\in \boldsymbol{U})$$

for some $\alpha(0\leqq\alpha<p\,!/(p-j)\,!)$, then

$$(4.2)\qquad \frac{\displaystyle\int_0^z \frac{f^{(j)}(t)}{t^{p-j}}dt}{z}<2\alpha-q-\frac{2(q-\alpha)}{z}\ln(1-z),$$

where $1\leqq j\leqq p$ and $q=p\,!/(p-j)\,!$.

PROOF. Let define $F(z)$ by

$$(4.3)\qquad F'(z)=\frac{f^{(j)}(z)}{qz^{p-j}}=1+c_1z+c_2z^2+\cdots.$$

It is easy to see that

$$(4.4)\qquad \mathrm{Re}(F'(z))>\beta \qquad \left(\beta=\frac{\alpha}{q},\ 0\leqq\beta<1\right)$$

and

$$(4.5)\qquad F(z)=\frac{1}{q}\int_0^z \frac{f^{(j)}(t)}{t^{p-j}}dt.$$

Therefore, applying the result by Owa, Ma and Liu [5], we have

(4.6)
$$\frac{\int_0^z \frac{f^{(j)}(t)}{t^{p-j}}dt}{qz} \prec 2\beta-1-\frac{2(1-\beta)}{z}\ln(1-z),$$

or

(4.7)
$$\frac{\int_0^z \frac{f^{(j)}(t)}{t^{p-j}}dt}{z} \prec 2\alpha-q-\frac{2(q-\alpha)}{z}\ln(1-z).$$

REMARK.

(i) Letting $j=p$ in Theorem 5, we have the result (1.3) by Saitoh [6].

(ii) Letting $j=p=1$ in Theorem 5, we have the result by Owa, Ma and Liu [5].

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Department of Mathematics
Gunma College of Technology
Toriba, Maebashi, Gunma 371
Japan