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THE DIRICHLET PROBLEM WITH L^{p} BOUNDARY DATA IN DOMAINS WITH DINI CONTINUOUS NORMAL

By

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In a series of articles [1-4, 6-8, 15, 16, 22-26], Mikhailov, Chabrowski, and others have used energy inequality methods to study the Dirichlet problem with L^{p} boundary values in C^{2} domains. Here we show to adapt this method to $C^{1,\mathrm{Dini}}$ domains. Certain important elements of our analysis are present in work of Petrushko [26] for $C^{1,\alpha}$ domains, but our approach was motivated by different considerations. The key idea in the energy method (concerning boundary values) is that the traces of the solution of a suitable elliptic equation on "parallel" surfaces to the boundary should have a limit as these surfaces converge to the boundary. For C^{2} domains this convergence is readily understood because there is a natural C^{1} diffeomorphism between a level surface of the distance function (at least near the boundary) and the boundary itself. Petrushko provided special local diffeomorphisms between level surfaces of a regularized distance and the $(n-1)$ dimensional ball which fit together nicely. Here we use essentially any regularized distance and our conditions on the coefficients of the equation are weaker than Petrushko's.

A second approach to the Dirichlet problem with L^{p} boundary data is given by the methods of singular integrals. See [9-12, 17, 18]. This approach has the advantage of considering weaker regularity hypotheses on the leading coefficients of the elliptic operator and on the domain; however, none of the papers mentioned here considers lower order terms (in fact, several are concerned only with Laplace's equation) and some deep and subtle machinery is required. In addition a different definition of trace is used; the solution of the differential equation now approaches its boundary values nontangentially a.e. with respect to the elliptic measure induced by the differential equation. A key step, then, is the verification that this measure is absolutely continuous with respect to ordinary surface measure. Some connections between the two definitions of trace appear in [1].

An intermediate approach was recently proposed by Gushchin [14] to show

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that solutions of the differential equations considered here have traces. He defines a space $C_{n\ 1}(\bar{\Omega})$ for a domain Ω, via Carleson measures, with two important properties. Every element of $C_{n-1}(\bar{\Omega})$ has a trace on $\partial\Omega$ and $W^{1,2}(\Omega)$ is a subsapce of $C_{n-1}(\bar{\Omega})$. From an energy estimate simitar to our (3.3) below, Gushchin uses properties of the nontangential maximal function and area integrals to infer an estimate on the $C_{n-1}(\bar{\Omega})$ norm of solutions of the differential equation and hence the existence of traces in either sense.

We begin in Section 1 with some definitions and the introduction of various regularized distance functions, which play a key role in our considerations. Section 2 deals with the definition of an L^2 trace for suitable solutions of differential equations. An energy inequality and its application to the Dirichlet problem appear in Section 3. Extensions to L^p boundary data are considered in Section 4.

1. Regularized distance and a mollification.

In this section we introduce two regularized distance functions (as defined in [19]) and we show how to extend functions which are continuous on the boundary. Throughout we denote by Ω an open subset of $\boldsymbol{R}^n$ with nonempty boundary and we write ν for the inner unit normal to $\partial\Omega$.

As in [19], we defined the signed distance function d by

$$d(x)=\begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x\in\Omega \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x\notin\Omega . \end{cases}$$

A *regularized distance* is a function $\rho\in C^2(\boldsymbol{R}^n\setminus\partial\Omega)\cap C^{0,1}(\boldsymbol{R}^n)$ such that the ratios ρ/d and d/ρ are positive and uniformly bounded on $\boldsymbol{R}^n\setminus\partial\Omega$. If also there are positive constants c_1 and c_2 such that $0<|\rho|\leqq c_1$ implies $|D\rho|\geqq c_2$, we say that ρ is *proper*. For a proper regularized distance ρ, we write $C_1(\rho)$ for the set of all such c_1. If we set

$$d^*(x)=\begin{cases} \min\{1, d(x)\} & \text{if } x\in\Omega \\ \max\{-1, d(x)\} & \text{if } x\notin\Omega \end{cases}$$

and replace d by d^* in the definition of regularized distance, we call the resulting ρ a *regularized bounded distance.*

To examine C^1 domains, we need some notation. We denote by MOC the set of all bounded, concave, continuous, increasing functions on $[0, \infty)$ which vanish at zero, and we denote by $DMOC$ the set of all elements of δ of MOC for which

$$\int_0^1 \frac{\delta(t)}{t}\,dt<\infty .$$

For $\zeta\in MOC$ we define Z by $Z(t)=\log_t\zeta(t)$. If u is defined on some open set S, we define

$$|u|_{0;S}=\sup_S|u|,$$

$$[u]_{Z;S}=\sup\{|u(x)-u(y)|/\zeta(|x-y|):\ x\neq y \text{ in } S\},$$

$$[u]'_{Z;S}=\sup\{|u(x)-u(y)|/\zeta(|x-y|):\ x\in S,\ y\in\partial S\},$$

$$|u|_{Z;S}=|u|_{0;S}+[u]_{Z;S},\qquad |u|'_{Z;S}=|u|_{0;S}/\zeta(1)+[u]'_{Z;S}$$

$$|u|_{1+Z;S}=|u|_{0;S}+|Du|_{Z;S},$$

$$H'_Z(S)=\{u:\ |u|'_{Z;S}<\infty\},$$

$$H_{1+Z}(S)=\{u:\ |u|_{1+Z;S}<\infty\}.$$

We say that $\partial\Omega\in H_{1+Z}$ if there is a function $g\in H_{1+Z}(\boldsymbol{R}^n)$ such that the ratios g/d and d/g are uniformly bounded on $\boldsymbol{R}^n\setminus\partial\Omega$. If $\zeta(t)=t^\alpha$ for some $\alpha\in(0, 1)$, then $Z(t)\equiv\alpha$ and our definition of $\partial\Omega\in H_{1+\alpha}$ is equivalent to the usual defintion of $\partial\Omega\in C^{1,\alpha}$(with appropriate uniformity for unbounded Ω); see [19, Theorem 2.3] for details. According to [19, Theorem 1.3] every H_{1+Z} domain has a proper regularized distance ρ such that

$$[D\rho]_Z\leqq 8[Dg]_Z,\qquad |D^2\rho|\leqq C(n,\ |Dg|_Z)\zeta(|\rho|)/|\rho|.$$

Because of its later importance, we denote by RD the set of all proper regularized bounded distance functions ρ with

$$|D^2\rho|\leqq C\zeta(|\rho|)/|\rho|.$$

For our energy estimate in Section 3, a more specific element of RD will be used. To construct it, we consider $\delta\in DMOC$ and $\partial\Omega\in H_{1+\Delta}$. For fixed constants $C_0\geqq 0$ and $\gamma\geqq 1$, we set

$$\mathcal{L}=\{L:\ Lu=a^{ij}D_{ij}u+b^iD_iu:\ I\leqq(a^{ij})\leqq\gamma I,\qquad |b|\leqq C_0\delta(\rho)/\rho\},$$

and we define the operator M on $C^2(\Omega)$ by

$$Mu(x)=\sup_{L\in\mathcal{L}} Lu(x).$$

As in the proof of [19, Theorem 4.1], we can find $\varepsilon>0$ and $k\in C^1[0, \infty)\cap C^2(0, \infty)$ with $k''\leqq 0$, $k'(0)=1$, $k'(\varepsilon)>0$ such that $\rho_0=k(\rho)$ satisfies $M\rho_0\leqq-\delta(\rho)/\rho$ in $\{0<\rho<\varepsilon\}$. If we now set

$$k_1(r)=\begin{cases} k(r) & \text{if } r\leqq\varepsilon, \\ k(\varepsilon)+\varepsilon k'(\varepsilon)(1-e^{1-r/\varepsilon})+\dfrac{1}{2}(\varepsilon^2k''(\varepsilon)+\varepsilon k'(\varepsilon))(1-e^{1-r/\varepsilon})^2 & \text{if } r>\varepsilon, \end{cases}$$

the function $\rho_1=k_1(\rho)$ is in RD and satisfies the estimates

$$M\rho_1\leqq-\frac{\delta(\rho)}{\rho}+c_3. \tag{1.1}$$

Now let λ be a positive constant, and for $R>0$, set $\Omega_R=\{x\in\Omega: |x|<R\}$ and write v_R for the unique solution of the problem

$$Mv_R-\lambda v_R=-\frac{\delta(\rho)}{\rho} \quad \text{in } \Omega_R, \qquad v_R=\rho_1 \quad \text{on } \partial\Omega_R$$

given by [28] and [20]. The maximum principle and (1.1) imply that $0\leqq v_R\leqq c_3/\lambda+\rho_1$, and then the results in [20] show that $|v_R|_{1+Z;\Omega_{R/2}}$ is bounded independently of R for some $\zeta\in MOC$. Taking the limit as R tends to infinity then gives a bounded function $v\in H_{1+Z}$ such that

$$Mv-\lambda v=-\frac{\delta(\rho)}{\rho} \quad \text{in } \Omega, \qquad v=0 \quad \text{on } \partial\Omega.$$

Hence for θ a small positive constant, there is a constant c_4 such that $\bar{v}=2v+\theta\rho$ satisfies

$$M\bar{v}-\lambda\bar{v}\leqq-\frac{\delta(\rho)}{\rho}, \quad 0\leqq\bar{v}\leqq c_4, \quad \bar{v}\leqq c_4\rho, \quad |D\bar{v}|\leqq c_4 \quad \text{in } \Omega,$$

$$\bar{v}=0, \quad \frac{\partial\bar{v}}{\partial\nu}\geqq\theta c_2>0 \quad \text{on } \partial\Omega.$$

Now for $L\in\mathcal{L}$, define L^* by

$$L^*u=Lu+C_0\delta(\rho)^2|Du|^2/u-\lambda u.$$

If $\bar{\rho}=h(\bar{v})$ for some $h\in C^2[0,\infty)$ with $h''\leqq0$, $h'\geqq0$, and $h(0)=0$, then

$$\begin{aligned} L^*\bar{\rho}&\leqq h'(\bar{v})L^*\bar{v}+h''a^{ij}D_i\bar{v}D_j\bar{v} \\ &\leqq h'(\bar{v})\frac{\delta(\rho)}{\rho}\left[-1+\frac{C_0}{\theta}\delta(\rho)|D\bar{v}|^2\right]+h''|D\bar{v}|^2. \end{aligned}$$

Hence, for $\rho_0=\sup\{\rho: \delta(\rho)\leqq\theta/(2C_0c_4^2)\}$ and $\delta_0=C_0(\sup\delta)^2/\theta\rho_0$, we obtain

$$L^*\bar{\rho}\leqq-\frac{1}{2}h'(\bar{v})\frac{\delta(\rho)}{\rho}+(\delta_0h'+h'')|D\bar{v}|^2\leqq-c_5\frac{\delta(\rho)}{\rho} \quad \text{in } \Omega \tag{1.2a}$$

for $h(r)=1-\exp(-\delta_0r)$ and $c_5=(1/2)\delta_0\exp(-\delta_0c_4)$. Moreover

$$\frac{\partial\bar{\rho}}{\partial\nu}\geqq c_2\delta_0\theta \qquad \text{on } \partial\Omega, \tag{1.2b}$$

and $D\bar{\rho}$ is parallel to ν on $\partial\Omega$.

The preceding construction works in bounded or unbounded domains. In bounded domains, we can do better because λ need not be positive. To see this, we take v to be the solution of

$$Mv+\eta v=-\frac{\delta(\rho)}{\rho} \quad \text{in} \quad \Omega, \qquad v=0 \quad \text{on} \quad \partial\Omega$$

for some positive η. If η is small enough, the barrier constructed by Michael in [21] is a positive supersolution so the generalized maximum principle [27, Theorem 2.10] implies a bound on v(and hence its existence). With $\bar{v}$ constructed as before, $\bar{\rho}$ satisfies

$$h'\eta\bar{v}+L\bar{\rho}+C_0\delta(\rho)^2\frac{|D\bar{\rho}|^2}{\bar{\rho}}\leqq 0$$

and hence for $\bar{\eta}$ a sufficiently small positive constant and L^* given by

$$L^*u=Lu+C_0\delta(\rho)^2\frac{|Du|^2}{u}+\bar{\eta}u\,,$$

(1.2) holds.

When $\partial\Omega\in C^{1,\lambda}$ for some $\lambda\in(0, 1)$ and Ω is bounded, Petrushko [26] uses as ρ the solution of

$$\Delta\rho=-1 \quad \text{in} \quad \Omega, \qquad \rho=0 \quad \text{on} \quad \partial\Omega,$$

which can be seen to be in RD. In the next two sections we shall see the advantages of having a large choice of regularized distances available.

If $f\in H'_\Delta$, a useful variant of f can be defined via mollification.

LEMMA 1. *Let $f\in H'_Z$ for some $\zeta\in MOC$. Then there is $F\in C^0(\bar{\Omega})\cap C^1(\Omega)$ such that*

(1.3) $$|f-F|\leqq 4|f|'_Z\zeta(d^*),$$

(1.4) $$|DF|\leqq C(n)[f]'_Z\zeta(d^*)/d^*.$$

If Ω is bounded, then

(1.3)′ $$|f-F|\leqq 4[f]'_Z\zeta(d),$$

(1.4)′ $$|DF|\leqq C(\Omega)[f]'_Z\zeta(d)/d.$$

Moreover if f is a matrix satisfying the inequality $f_1I\leqq f\leqq f_2I$ for some constants f_1 and f_2, then F satisfies the same inequality.

PROOF. Let ρ be any regularized bounded distance for Ω with $d^*/2\leqq\rho\leqq 2d^*$, let $\varphi\in C^{0,1}(\boldsymbol{R}^n)$ be nonnegative with support in the unit ball and $\int_{\boldsymbol{R}^n}\varphi(y)dy=1$, and set

$$F(x)=\int_{R^n} f(x+\rho(x)y/2)\varphi(y)dy. \tag{1.5}$$

To prove (1.3), we first note that

$$F(x)-f(x)=\int_{R^n}[f(x+\rho(x)y/2)-f(x)]\varphi(y)dy. \tag{1.6}$$

We estimate this integral by choosing $z\in\partial\Omega$ so that $|z-x|=d(x)$ and noting that $|x+\rho(x)y/2-z|\leqq 2d(x)$ if $|y|\leqq 1$, so that

$$\begin{aligned}|f(x+\rho(x)y/2)-f(x)|&\leqq f(x+\rho(x)y/2))-f(z)|+|f(x)-f(z)|\\&\leqq\{\zeta(2d(x))+\zeta(d(x))\}[f]_Z'\leqq 2\zeta(2d(x))[f]_Z'\leqq 4\zeta(d(x))[f]_Z'\end{aligned}$$

since ζ is increasing and concave. Using this estimate in (1.6) easily gives (1.3) when $d^*(x)=d(x)$ and (1.3)' when Ω is bounded. When $d^*=1$, we obtain (1.3) by noting that $|f-F|\leqq 2|f|_0$ and recalling the definition of $|f|_Z$.

To prove (1.4), we make the change of variable $y'=x+\rho(x)y/2$ in the integral for F, differentiate with respect to x and then change back to the original (y) variable of integration to find that

$$\begin{aligned}DF(x)=&\frac{-2}{\rho(x)}\int_{R^n} f(x+\rho(x)y/2)D\varphi(y)dy\\&-\frac{2}{\rho(x)}D\rho(x)\int_{R^n} f(x+\rho(x)y/2)\operatorname{div}(y\varphi(y))dy.\end{aligned} \tag{1.7}$$

Since $\int_{R^n} D\varphi(y)dy=0$ and $\int_{R^n}\operatorname{div}(y\varphi(y))=0$, it follows that

$$\begin{aligned}DF(x)=&\frac{-2}{\rho(x)}\int_{R^n}[f(x+\rho(x)y/2)-f(x)]D\varphi(y)dy\\&-\frac{2}{\rho(x)}D\rho(x)\int_{R^n}[f(x+\rho(x)y/2)-f(z)]\operatorname{div}(y\varphi(y))dy\end{aligned}$$

for the same z as before because we can take φ so that $|D\varphi|\leqq C(n)$. These integrals are estimated as before to infer (1.4) when $d^*=d$ and (1.4)' when Ω is bounded. When $d^*=1$, we proceed from (1.7).

Finally the preservation of linear inequalities is clear from (1.6) because φ is nonnegative. ■

2. Traces of $\widetilde{W}^{1,2}$ functions.

By using the class RD, we can prove that solutions of certain elliptic equations in the class $\widetilde{W}^{1,2}=W^{1,2}_{\mathrm{loc}}\cap L^2$ have traces. The first step in this program is to establish the equivalence of two conditions on $\widetilde{W}^{1,2}$ solutions.

LEMMA 2. *Let $\delta \in DMOC$, let $\partial\Omega \in H_{1+\Delta}$ and let a^{ij}, c, f be functions defined in Ω such that*

(2.1a) $$(a^{ij}) \geqq I,$$

(2.1b) $$a^{ij} \in H_\Delta, \qquad c \in L^\infty$$

(2.1c) $$|f|^2 (d^*)^3/\delta(d^*) \in L^1.$$

Then for any $\rho \in RD$, any $c_1 \in C_1(\rho)$, and any solution $u \in \widetilde{W}^{1,2}$ of

(2.2) $$-D_i(a^{ij} D_j u) + cu = f \qquad \text{in } \Omega,$$

the following conditions are equivalent:

(I) $\sup_{0<\sigma<c_1} \int_{\{\rho=\sigma\}} u^2 < \infty$,

(II) $\int_\Omega |Du|^2 d^* < \infty$.

PROOF. For $\tau>0$, set $g(t)=(t^{1/2}-\tau^{1/2})_+^2$, and for η a nonnegative $C_c^1(\boldsymbol{R}^n)$ function with $|\eta|+|D\eta| \leqq 2$, use $v=ug(\rho)\eta^2$ as test function in the weak form of (2.1):

$$\int_\Omega a^{ij} D_i u D_j u \eta^2 g + \int_\Omega a^{ij} D_j u D_i \rho u \eta^2 g'$$

$$+2\int_\Omega a^{ij} D_j u D_i \eta \eta u g + \int_\Omega c u^2 \eta^2 g = \int_\Omega f u \eta^2 g.$$

From Lemma 1 we obtain functions A^{ij} such that

$$|A^{ij}-a^{ij}| \leqq C\delta(d^*), \qquad |D_j(A^{ij})| \leqq C\frac{\delta(d^*)}{d^*}, \qquad I \leqq (A^{ij}) \leqq \gamma I,$$

and therefore, after an integration by parts,

$$\int_\Omega a^{ij} D_j u D_i \rho u \eta^2 g' = \int_\Omega (a^{ij}-A^{ij}) D_j u D_i \rho u \eta^2 g'$$

$$-\frac{1}{2}\int_\Omega \{g'(A^{ij} D_{ij}\rho + D_j(A^{ij}) D_i \rho) + g'' A^{ij} D_i \rho D_j \rho\} \eta^2 u^2$$

$$-\int_\Omega A^{ij} D_i \rho D_j \eta \eta u^2 g' - \frac{1}{2}\int_{\{\rho=\tau\}} A^{ij} \nu_j D_i \rho u^2 \eta^2 g'(\tau).$$

Now we use the special form of g and Cauchy's inequality to infer that

$$\int_\Omega |Du|^2 \eta^2 g$$

$$\leqq C\left[\int_{\{\rho>\tau\}} u^2 \eta^2 \left(\frac{\delta(\rho)}{\rho} + \tau^{1/2} \rho^{-3/2}\right) + \int_\Omega u^2 (|D\eta|^2 + \eta^2) + \int_\Omega f^2 \eta^2 \frac{(d^*)^3}{\delta(d^*)}\right]$$

Now, similarly to [6, (2.4)], we have

$$\int_{\{\rho>\tau\}} u^2\eta^2\left(\frac{\delta(\rho)}{\rho}+\tau^{1/2}\rho^{-3/2}\right)$$
$$\leqq\left[\frac{\delta(c_1)}{c_1}+\tau^{1/2}c_1^{-3/2}\right]\int_{\{\rho>\tau\}} u^2\eta^2+\left[2+\int_0^{c_1}\frac{\delta(t)}{t}dt\right]\sup_{\tau<t<c_1}\int_{\{\rho=t\}} u^2\eta^2 .$$

Hence

$$\int_\Omega |Du|^2\eta^2 g\leqq C\left[\sup_{0<\tau<c_1}\int_{\{\rho=\tau\}} u^2\eta^2+\int_\Omega u^2(\eta^2+|D\eta|^2)+\int_\Omega f^2\eta^2\frac{(d^*)^3}{\delta(d^*)}\right].$$

Because of (2.3) and our choice of η, the right hand side of this inequality is bounded independently of η and τ. Therefore (I) implies (II).

For the reverse implication, we set $g(t)=(t-\tau)_+$ and then $v=ug(\rho)\eta^2$ as before to obtain

$$\frac{1}{2}\int_{\{\rho=\tau\}}(A^{ij}\nu_j D_i\rho)u^2\eta^2=\int_\Omega a^{ij}D_iuD_ju\eta^2 g+2\int_\Omega a^{ij}D_juD_i\eta\eta ug$$
$$+\int_\Omega(a^{ij}-A^{ij})D_juD_i\rho u\eta^2 g'+\int_\Omega A^{ij}D_i\rho D_j\eta u^2\eta g'$$
$$-\frac{1}{2}\int_\Omega(A^{ij}D_{ij}\rho+D_j(A^{ij})D_i\rho)g'u^2\eta^2+\int_\Omega cu^2\eta^2 g-\int fu\eta^2 g .$$

Because ρ is proper there is a constant σ_0 such that $A^{ij}\nu_j D_i\rho>\sigma_0$ for $\rho<c_1$. Hence for $\tau\leqq c_1$,

$$\int_{\{\rho=\tau\}} u^2\eta^2\leqq C\int_\Omega\left[|Du|^2 d^*+u^2+f^2\frac{(d^*)^3}{\delta(d^*)}\right]+C\int_{\{\rho>\tau\}}\frac{\delta(\rho)}{\rho}u^2\eta^2 .$$

Just as in [6, Theorem 1] (see p. 632 there), we conclude from this estimate that

$$\sup_{0<\sigma<\sigma_1}\int_{\{\rho=\sigma\}} u^2<\infty$$

for some sufficiently small positive σ_1. A simple variant of [6, Lemma 1] completes the proof. ■

In this theorem, we did not need the specialized regularized distance $\bar{\rho}$ of Section 1. It will appear in Section 3 when we prove an energy inequality.

We now examine the boundary condition $u=\varphi$ on $\partial\Omega$ for $\widetilde{W}^{1,2}$ functions. As in [7], etc., we interpret this boundary condition as saying that traces of u on certain interior hypersurfaces should converge to φ as these hypersurfaces approach the boundary in a reasonable way. When $\partial\Omega\in C^2$ the choice of hypersurfaces is made quite naturally because of a simple, smooth correspondence between points on $\partial\Omega$ and points on the hypersurface $d=$constant (when the constant is small enough); however, in our case there is no such simple correspondence. Instead, motivated by the definition for C^2 boundaries and by the desire for a definition which is invariant under smooth changes of independent variable, we present the following definition.

Let $\partial\Omega\in C^1$, $\varphi\in L^2(\partial\Omega)$, $u\in\widetilde{W}^{1,2}(\Omega)$. We say that $u=\varphi$ on $\partial\Omega$ if, for every $x_0\in\partial\Omega$, there is a neighborhood N of x_0 and an invertible C^1 map G such that

$$G(x_0)=0, \qquad G(N\cap\Omega)=B^+(0,1), \qquad G(N\cap\partial\Omega)=B^0(0,1) \tag{2.4a}$$

$$\lim_{\delta\to 0} u(G^{-1}(\cdot,\delta))=\varphi \quad \text{in} \quad L^2(N\cap\partial\Omega). \tag{2.4b}$$

We now show that all choices of G and N in this definition are equivalent and we find additional conditions which imply the existence of a trace.

LEMMA 3. *Let $\partial\Omega\in C^1$, $u\in\widetilde{W}^{1,2}$, and suppose $|Du|(d^*)^{1/2}\in L^2$ If there are two function φ_1 and φ_2 in $L^2(\partial\Omega)$ such that $u=\varphi_1$ and $u=\varphi_2$ on $\partial\Omega$, then $\varphi_1=\varphi_2$ a.e. on $\partial\Omega$.*

PROOF. Clearly it suffices to show that, if the hypotheses of the lemma are satisfied with Ω and $\partial\Omega$ replaced by $B^+(0,1)$ and $B^0(0,1)$, then $\varphi_1=\varphi_2$ a.e. on $B^0(0,1/2)$.

To this end, let $\psi=(\psi',\psi^n)$ be an invertible C^1 map of $\overline{B^+(0,1)}$ into itself which restricts to the identity on $B^0(0,1)$ and for $\delta>0$, consider the integral

$$\begin{aligned}
&\int_{B^0(0,1/2)}|u(x',\delta)-u(\psi(x',\delta))|^2dx' \\
&\quad\leqq 2\int_{B^0(0,1/2)}|u(x',\delta)-u(x',\psi^n(x',\delta))|^2dx' \\
&\qquad +2\int_{B^0(0,1/2)}|u(x',\psi^n(x',\delta))-u(\psi'(x',\delta),\psi^n(x',\delta))|^2dx'.
\end{aligned}$$

We now use I_1 and I_2 to denote the integrals on the right hand side of this inequality and observe that we only need to show that I_1 and I_2 tend to zero as $\delta\to 0$.

To estimate I_1, we write $\Sigma=B^0(0,1/2)$, so that

$$\begin{aligned}
I_1&=\int_\Sigma\left|\int_0^1 D_nu(x',t\delta+(1-t)\psi^n(x',\delta))\{\delta-\psi^n(x',\delta)\}dt\right|^2dx' \\
&\leqq\int_\Sigma\int_0^1|D_nu(x',t\delta+(1-t)\psi^n)|^2|\psi^n-\delta|^2dtdx'
\end{aligned}$$

by Jensen's inequality. Because $\psi^n\in(\varepsilon\delta,\delta/\varepsilon)$ for some $\varepsilon\in(0,1)$, we can use the change of variable $x^n=t\delta+(1-t)\psi^n$ in the inner integral to obtain

$$I_1 \leqq \int_\Sigma \int_{\min(\delta, \phi_n)}^{\max(\delta, \phi^n)} |D_n u(x', x^n)|^2 |\phi^n - \delta| \, dx^n dx'$$

$$\leqq C \int_{\Sigma \times (\varepsilon\delta, \delta/\varepsilon)} |D_n u(x)|^2 x^n dx,$$

which goes to zero with δ because $D_n u(x^n)^{1/2} \in L^2$.

To estimate I_2, we set $\tilde{u} = u \circ \phi$ and we define new coordinates (y', y^n) by the equations $\phi'(y', \delta) = x'$, $y^n = \phi^n(x', \delta)$. For δ small enough, the matrix $\partial\phi'/\partial y'$ is invertible and hence these equations determine y as a function of (x', δ). Similarly, $\partial\phi^n/\partial\delta$ does not vanish so (x', δ) is determined as a function of y. It follows that

$$I_2 \leqq 2 \int_{B^0(0, 1/2)} |u(x', \phi^n(x', \delta)) - u(x', \phi^n(y', \delta))|^2 dx'$$

$$+ C \int_{B^0(0, 1)} |\tilde{u}(y', \delta) - \tilde{u}(\phi', \delta)|^2 d\phi'$$

The first integral here is estimated as I_1 was. The second integral goes to zero by a standard argument: Fix $\varepsilon > 0$ and choose v uniformly continuous in $B^0(0, 2)$ so that $\|v - \varphi_2\|_{L_2(B^0(0,1))} \leqq \varepsilon/5$, then η so small that $|v(x) - v(y)| \leqq \varepsilon/5\omega_n$ if $|x - y| < \eta$. If δ is so small that $\|\tilde{u}(\cdot, \delta) - \varphi_2\|_{L^2} \leqq \varepsilon/5$ and $|y' - \phi'| < \eta$, then

$$\int_{B^0(0, 1)} |\tilde{u}(y', \delta) - \tilde{u}(\phi', \delta)|^2 d\phi' \leqq \varepsilon.$$

Hence $(u - u \circ \phi)(\cdot, \delta) \to 0$ in $L^2(B^0(0, 1))$. Since $u \circ \phi(\cdot, \delta) \to \varphi_2$, it follows that $u(\cdot, \delta) \to \varphi_2$ and hence $\varphi_1 = \varphi_2$. ■

Note that the proof of Lemma 3 implies a stronger result, namely, that if $u = \varphi$ on $\partial\Omega$ for a particular choice of G, then any choice of G gives a trace and that trace must be φ. Moreover the hypotheses are weaker than those of Lemma 2: we only need $\partial\Omega \in C^1$ and $u \in \widetilde{W}^{1,2}$ satisfying condition (II) of that lemma. To show that a trace exists, we assume the full hypotheses of Lemma 1. Because the hypotheses are invariant under $H_{1+\Delta}$ changes of variables for $\delta \in DMOC$, the arguments in [6, Section 3] and [2, Theorems 2 and 3] are easily modified to give the following result. We don't go into detail here since the result will not be used in our existence results for the Dirichlet problem.

LEMMA 4. *Suppose all the hypotheses of Lemma 2, including condition* (I) *or condition* (II), *are satisfied. Then there is a unique $\varphi \in L^2(\partial\Omega)$ for which $u = \varphi$ on $\partial\Omega$.*

In the next section we will use the easy observation that the definition of

$u=\varphi$ on $\partial\Omega$ for $\varphi\in L^2$ agrees with the definition that φ is the trace of u when $u\in W^{1,2}$.

3. Applications to the Dirichlet Problem.

We are now ready to attack the problem

$$-D_i(a^{ij}D_j u)+cu=f \quad \text{in } \Omega, \quad u=\varphi \text{ on } \partial\Omega \tag{3.1}$$

when $\varphi\in L^2(\partial\Omega)$. The main tool not yet given is the following energy estimate.

KEMMA 5. *Let $\lambda, \lambda_1, \lambda_0, \gamma$ be nonnegative constants with $\lambda\geqq\lambda_1>0$ and $\gamma\geqq 1$. Let $\delta\in DMOC$, let $\partial\Omega\in H_{1+\Delta}$, and suppose conditions* (2.1a-c) *are satisfied. If also*

$$|a^{ij}|\leqq\gamma, \qquad [a^{ij}]_\Delta\leqq\lambda_0, \tag{3.2a}$$

$$c\geqq\lambda, \tag{3.2b}$$

then for any $\rho\in RD$ and any $c_1\in C_1(\rho)$, there is a constant $C=C(\gamma, \delta, \lambda_0, \lambda_1, \rho, c_1)$ such that any solution $u\in\widetilde{W}^{1,2}$ of (3.1) *satisfies the estimate*

$$\int_\Omega|Du|^2d^*+\sup_{0<\sigma<c_1}\int_{(\rho=\sigma)}u^2+\lambda\int_\Omega u^2d^*\leqq C\left[\int_\Omega f^2\frac{(d^*)^3}{\delta(d^*)}+\int_{\partial\Omega}\varphi^2\right]. \tag{3.3}$$

PROOF: For $\bar\rho$ a proper regularized bounded distance function to be chosen, use $v=u\bar\rho$ as test function in the weak form of the differential equation in (3.1). For A^{ij} as before, we

$$\begin{aligned}\int_\Omega\{a^{ij}D_iuD_ju\bar\rho+cu^2\bar\rho\}&=\int_\Omega(A^{ij}-a^{ij})D_i\bar\rho D_ju u+\frac{1}{2}\int_{\partial\Omega}A^{ij}D_i\bar\rho\nu_j\varphi^2\\&\quad+\int_\Omega fu\bar\rho+\frac{1}{2}\int_\Omega[A^{ij}D_{ij}\bar\rho+2D_j(A^{ij})D_i\bar\rho]u^2\\&\leqq C\int_{\partial\Omega}\varphi^2+\frac{1}{2}\int|Du|^2\bar\rho+\frac{\varepsilon}{2}\int u^2\frac{\delta(d^*)}{(d^*)^2}\bar\rho+C(\varepsilon)\int f^2\frac{(d^*)^2}{\delta(d^*)}\bar\rho\\&\quad+\frac{1}{2}\int_\Omega\left\{A^{ij}D_{ij}\bar\rho+2D_j(A^{ij})D_i\bar\rho+C\delta(d^*)^2\frac{|D\bar\rho|^2}{\rho}\right\}u^2\end{aligned}$$

for any $\varepsilon>0$. If we now subtract $\frac{1}{2}\int_\Omega\{|Du|^2+\lambda_1u^2\}\bar\rho$ from both sides of this inequality and set

$$\Phi=\int_{\partial\Omega}\varphi^2+\int_\Omega f^2\frac{(d^*)^2}{\delta(d^*)}\bar\rho,$$

we see that

$$\int_\Omega\{|Du|^2+cu^2\}\bar\rho\leqq\int_\Omega u^2\left(L^*\bar\rho+\varepsilon\frac{\delta(d^*)}{(d^*)^2}\bar\rho\right)+C(\varepsilon)\Phi$$

for some L^* as in Section 1 (with λ there replaced by λ_1). By taking $\bar{\rho}$ to satisfy (1.2) and then ε small enough, we obtain

$$\int_\Omega \{|Du|^2+cu^2\}\bar{\rho} \leqq C\Phi. \tag{3.4}$$

Now the proof of (II) implies (I) in Theorem 1 gives

$$\sup_{0<\sigma<c_1}\int_{(\rho=\sigma)} u^2 \leqq C\Big[\int_\Omega\{|Du|^2+cu^2\}d^*+\frac{\delta(c_1)}{c_1}\int_\Omega u^2 d^*+\Phi\Big].$$

Moreover $\dfrac{\delta(c_1)}{c_1} \leqq C(\lambda_1, c_1, \delta)\lambda_1 \leqq C_c$, so recalling (3.4), we infer

$$\sup_{0<\sigma<c_1}\int_{(\rho=\sigma)} u^2 \leqq C\Phi. \tag{3.5}$$

The estimate (3.3) follows immediately from (3.4) and (3.5). ■

The considerations of Section 2 show that (3.3) is valid for any proper C^1 regularized distance ρ.

For bounded domains, we can repeat the preceding argument with λ_1 a negative constant sufficiently close to zero. In this way we deduce the following analog of Lemma 4.

LEMMA 4′. *Let λ, λ_0, γ be nonnegative constants with $\gamma \geqq 1$. Let $\delta \in DMOC$, let $\partial\Omega \in H_{1+\Delta}$ and suppose Ω is bounded. If conditions* (2.1a–c) *and* (3.2) *are satisfied, then for any $\rho \in RD$ and any $c_1 \in C_1(\rho)$, there is a constant $C=C(\gamma, \delta, \lambda_0, \rho, c_1)$ such that any $\widetilde{W}^{1,2}$ solution of* (3.1) *obeys the estimate*

$$\int_\Omega |Du|^2 d+\sup_{0<\sigma<c_1}\int_{(\rho=\sigma)} u^2+\max\{1, \lambda\}\int_\Omega u^2 d \leqq C\Big[\int_\Omega f^2\frac{d^3}{\delta(d)}+\int_{\partial\Omega}\varphi^2\Big]. \tag{3.3)$'$ (}$$

These energy estimates lead to existence results via approximation and the corresponding results for $f \in L^2$ and $\varphi \in H^{1/2}(\partial\Omega)$. Verification that the limit function takes on the prescribed boundary values is the same as on [26, p. 573].

THEOREM. *Let $\delta \in DMOC$, let $\partial\Omega \in H_{1+\Delta}$, and suppose conditions* (2.1a, b) *hold. If* $\inf c>0$, *then for any f satisfying* (2.1c) *and any $\varphi \in L^2(\partial\Omega)$, there is a unique solution $u \in \widetilde{W}^{1,2}$ of* (3.1). *If λ, λ_0, λ_1, and γ are nonnegative constants with $\lambda \geqq \lambda_1 > 0$ and* (3.2) *holding, then for any $\rho \in RD$ and $c_1 \in C_1(\rho)$,* (3.3) *is valid. For bounded Ω,* (3.1) *has a unique solution if $c \geqq 0$, and then* (3.3)′ *holds.*

PROOF. If $f \in L^2$ and $\varphi \in H^{1/2}(\partial\Omega)$, the existence of a unique solution to the Dirichlet problem is well-known (see, for example, [13, Theorem 8.3]) and the solution satisfies (3.3). For the more general f and φ considered here, we approximate by sequences (f_m) in L^2 and (φ_m) in $H^{1/2}$ such that

$$\int_{\Omega}(f-f_m)^2\frac{(d^*)^3}{\delta(d^*)}\to 0, \qquad \int_{\partial\Omega}(\varphi-\varphi_m)^2\to 0$$

as $m\to\infty$, and denote by u_m the solution of (3.1) corresponding to f_m and φ_m.

From (3.3) applied to u_i-u_j for any i, j, we see that the sequence (u_m) is Cauchy in $\widetilde{W}^{1,2}$, and therefore it has a limit u, which satisfies the differential equation. To show that u also has trace φ on $\partial\Omega$, we note that for any $\varepsilon>0$, there is an integer N such that i, $j>N$ implies that

$$\int_{\partial\Omega}(\varphi-\varphi_i)^2<\varepsilon/3, \qquad \sup_{0<\sigma<c_1}\int_{\{\rho=\sigma\}}(u_i-u_j)^2<\varepsilon/3.$$

Moreover there is $\tau\in(0, c_1)$ such that

$$\sup_{0<\sigma<\tau}\int_{\{\rho=\sigma\}}(u_N-\varphi_N)^2<\varepsilon/3.$$

(Of course here, strictly speaking this last inequality can only be achieved when $\partial\Omega$ is a hyperplane, but using local coordinates, we can make sense of this inequality globally.) It then follows that

$$\sup_{0<\sigma<\tau}\int_{\{\rho=\sigma\}}(u-\varphi)^2<\varepsilon,$$

which means, because ε is arbitrary, that $u=\varphi$ on $\partial\Omega$. ■

Our results are easily extended in several ways. First, as in [6], we can consider the general linear equation

$$-D_i(a^{ij}D_ju+b^iu)+d^iD_iu+(c+\lambda)u=-D_ig^i+f$$

for the functions b, c, d, f, g in appropriate classes, which allow these coefficients to be unbounded near the boundary and only locally in an L^p space. Note, however, that the conditions on b, c, d for the existence of traces are different from those for the energy inequality or the solvability of the Dirichlet problem (cf. [5, Chapter 2]).

We can also consider quasilinear equations of the form

$$-D_i(a^{ij}(x, u)D_ju)+\lambda u=f(x, u, Du)$$

under the hypotheses detailed in [4] provided conditions (2.1a, b) hold uniformly for the functions $a^{ij}(\cdot, z)$. In this case the full strength of the construction of $\bar{\rho}$ in Section 1 is needed because now the coefficients of L^* depend on u in a way controlled through the operator M.

4. L^p Boundary Values.

When $\varphi\in L^p(\partial\Omega)$, our considerations need to be modified only slightly. In Lemma 2, the equivalent conditions are

$$(\mathrm{I})\quad \sup_{0<\sigma<c_1}\int_{\{\rho=\sigma\}} u^p<\infty$$

$$(\mathrm{II})_p\quad \int_\Omega |Du|^2|u|^{p-1}d^*<\infty$$

provided (2.1c) is modified to

$$(2.1\mathrm{c})_p\qquad |f|^p(d^*)^{2p-1}/\delta(d^*)^{p-1}\in L^1,$$

and $u\in\widetilde{W}^{1,p}=W^{1,p}_{\mathrm{loc}}\cap L^p\cap W^{1,2}_{\mathrm{loc}}$. The proof is easily modified by using $v=(u^2+\varepsilon)^{(p-2)/2}ug(\rho)\eta^2$ as test function and then sending ε to zero.

Similar modifications apply to Lemma 3 because condition $(\mathrm{II})_p$ implies that $\bar{u}=u^{p/2}$ sgn u satisfies the hypotheses of Lemma 3. Because $(1-\varepsilon)^p\leqq 1-\varepsilon^p$ for $p>1$ and $0<\varepsilon<1$, the definition of $u=\varphi$ in $L^p(\partial\Omega)$ is the same as $\bar{u}=\bar{\varphi}$ in $L^2(\partial\Omega)$.

Existence of a trace in L^p follows from the L^2 theory as in [7], [15], [16], etc.

The energy inequality

$$(3.3)_p\qquad \int_\Omega |Du|^2|u|^{p-1}d^*+\sup_{0<\sigma<c_1}\int_{\{\rho=\sigma\}}u^p+\lambda\int_\Omega u^pd^*$$
$$\leqq C\left[\int_\Omega |f|^p\frac{(d^*)^{2p-1}}{\delta(d^*)^{p-1}}+\int_{\partial\Omega}|\varphi|^p\right]$$

is proved by similar considerations and that the Dirichlet problem (3.1) has a unique $\widetilde{W}^{1,p}$ solution, if inf $c>0$, for any f satisfying $(2.1\mathrm{c})_p$ and any $\varphi\in L^p(\partial\Omega)$. The restriction $u\in W^{1,2}_{\mathrm{loc}}$ can be removed for $1<p<2$ by easy approximation arguments. (Analogous statements for bounded domains are easily verified.)

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