TSUKUBA J. MATH.
Vol. 15 No. 1 (1991), 31—40

ON THE BRUN-TITCHMARSH THEOREM

By

Hiroshi MIKAWA

1. Introduction.

Let $\pi(x;q,a)$ denote the number of primes not exceeding x and being congruent to a modulo q. In 1936 P. Turán [6] showed that, under the extended Riemann hypothesis,

$$\pi(x;q,a)\sim\frac{x}{\varphi(q)\log x}\quad\text{as}\quad x\rightarrow\infty$$

for all $q\leqq x(\log x)^{-2-\varepsilon}(\varepsilon>0)$ and almost-all reduced residue classes a modulo q. The terminology "almost-all" means that the number of exceptional reduced classes is $o(\varphi(q))$ as $q\rightarrow\infty$.

In 1972 C. Hooley [1] demonstrated that there holds the inequality

$$\pi(x;q,a)\leqq\frac{(4+\varepsilon)x}{\varphi(q)\log(x^{2}/q)}\qquad(\varepsilon>0,\ x>x_{0}(\varepsilon))$$

for all $q\leqq x^{2/3}$ and almost-all a. Later Y. Motohashi [4] proved that the same is valid for $x^{2/3}<q\leqq x^{1-\varepsilon}$ as well. The purpose of this paper is to make an improvement upon this upper bound to large moduli.

THEOREM. *Let ε be a small positive constant and assume $x>x_{0}(\varepsilon)$. If q be given and $x^{6/7}\leqq q\leqq x(\log x)^{A}$ with $A>5$, then we have*

$$\pi(x;q,a)\leqq\frac{(18+\varepsilon)x}{\varphi(q)\log(x^{6}/q)}$$

for almost-all reduced classes a modulo q.

REMARK. It is of some interest to note that, using the argument of H. Iwaniec [3, section 2], one may easily show that

$$\pi(x;q,a)\leqq\begin{cases}\dfrac{(2+\varepsilon)x}{\varphi(q)\log(xq^{-3/8})} & \text{if}\quad q\leqq x^{5/6-\delta}\\ \dfrac{(1/2+\varepsilon)x}{\varphi(q)\log(x/q)} & \text{if}\quad x^{5/6-\delta}\leqq q\leqq x^{6/7-\delta}\quad(0<\delta<1/200)\end{cases}$$

for almost-all a.

Received June 6, 1990

We use the standatd notation in number theory. Especially, $\bar{r}$, used in either $\bar{r}/s$ or congruence (mod s), means $\bar{r}r\equiv 1(\bmod s)$. ε denotes a small positive constant and the constants implied in the symbols $\ll$ and O may depend only on ε. For convenience, we write $n\sim N$ when $N\leqq N_1<n\leqq N_2\leqq 2N$ for some N_1 and N_2.

I would like to thank Professor S. Uchiyama for encouragement and careful reading of the original manuscript. I would also thank the referee for making the paper easier to read.

2. Lemmas.

We first state the inequality of Rosser-Iwaniec sieve [2, 5] in a simplified form that is sufficient for our present aim.

LEMMA 1. *We have for any* $\varepsilon>0$ *and all* $x>x_0(\varepsilon)$

$$\pi(x\,;\,q,\,a)\leqq\frac{(2+\varepsilon)x}{\varphi(q)\log D}+\sum_{(d,q)=1}\lambda_d(D)r_d(x\,;\,q,\,a)$$

where $D\geqq 1$ *is an arbitrary parameter;*

$$r_d(x\,;\,q,\,a)=|\{n\,:\,n\leqq x,\,n\equiv a(\bmod q),\,d\mid n\}|-\frac{x}{qd}\,;$$

the sieving weights $(\lambda_d)=(\lambda_d(D))$ *have the following properties:*

$$\lambda_d=0 \quad \text{if} \quad d\geqq D,$$

$$|\lambda_d|\leqq\mu^2(d),$$

aud for any $M,\,N\geqq 1,\,MN=D,$

$$\lambda_d=\sum_{l\leq\log D}\sum_{\substack{m\leq M\\ d=mn}}\sum_{n\leq N}a_m(l,\,M,\,N)b_n(l,\,M,\,N)$$

with certain sequences (a) *and* (b), $|a_m|,\,|b_n|\leqq 1$.

LEMMA 2. *Let* $\psi(t)=[t]-t+1/2$. *For* $H>2$ *we have*

$$\psi(t)=\sum_{0<|h|\leq H}\frac{e(ht)}{2\pi ih}+O\left(\min\left(1,\,\frac{1}{H\|t\|}\right)\right)$$

where $e(x)=e^{2\pi ix}$ *and* $\|x\|=\min_{n\in Z}|x-n|$. *Moreover,*

$$\min\left(1,\,\frac{1}{H\|t\|}\right)=\sum_{h\in Z}C_h e(ht)$$

with

$$C_h \ll \min\left(\frac{\log H}{H}, \frac{H}{h^2}\right).$$

LEMMA 3. *For any* $\varepsilon>0$, *we have*

$$\sum_{\substack{n\sim N\\(n,cd)=1}} e\left(b\frac{\bar{n}}{d}\right) \ll \tau(c)(b, d)^{1/2} d^{1/2+\varepsilon}\left(1+\frac{N}{d}\right).$$

Lemma 2 is well known. Lemma 3 is the Hooley's version of bounds for incomplete Kloosterman sums [1].

3. Proof of Theorem.

Maintaining the notation introduced in Lemma 1, we put

$$E_a = \sum_{(d,q)=1} \lambda_d r_d(x\,;\,q,\,a).$$

We use the following lemma:

LEMMA 4. *If* $M=x^{4/3-4\varepsilon}q^{-8/9}$ *and* $N=q^{7/9}x^{-2/3}$, *then we have*

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} |E_a|^2 \ll x(\log x)^3 + \frac{x^{2-\varepsilon}}{q}$$

uniformly for $x^{6/7} \leqq q < x$.

We postpone the proof of Lemma 4 until the final section. By Lemma 1, on choosing M and N as in Lemma 4, we have

$$\pi(x\,;\,q,\,a) \leqq \frac{(18+99\varepsilon)x}{\varphi(q)\log(x^6/q)} + E_a. \tag{1}$$

We denote by $\mathcal{E}$ the exceptional set of reduced classes modulo q, i.e.

$$\mathcal{E} = \left\{a : 1 \leqq a \leqq q,\ (a,\,q)=1,\ \pi(x\,;\,q,\,a) > \frac{(18+99\varepsilon)x}{\varphi(q)\log(x^6/q)}\right\}.$$

We shall show that $|\mathcal{E}| = o(\varphi(q))$, from which Theorem follows.

By (1) we see that $a \notin \mathcal{E}$ unless

$$E_a > \frac{\varepsilon x}{\varphi(q)\log(x^6/q)}.$$

We therefore get, by Lemma 4, that uniformly for $x^{6/7} \leqq q \leqq x(\log x)^{-A}$ with $A>5$

$$|\mathcal{E}|\left(\frac{\varepsilon x}{\varphi(q)\log(x^6/q)}\right)^2 < \sum_{a\in\mathcal{E}} |E_a|^2 \leqq \sum_{\substack{a=1\\(a,q)=1}}^{q} |E_a|^2 \ll x(\log x)^3 + \frac{x^2}{q}(\log x)^{-3}$$

or

$$|\mathcal{E}| \ll \varphi(q)\left\{\frac{q(\log x)^5}{x}+(\log x)^{-1}\right\}$$

$$\ll \varphi(q)\{(\log x^{5-A}+(\log x)^{-1}\}$$

as required.

4. Proof of Lemma 4, preliminaries.

In this section we reduce the proof of Lemma 4 to the estimation of R defined by (5) below. Since

$$E_a=\sum_{\substack{n\leq x\\ n\equiv a(q)}}\Big(\sum_{\substack{d|n\\ (d,q)=1}}\lambda_d\Big)-\Big(\sum_{(d,q)=1}\frac{\lambda_d}{d}\Big)\frac{x}{q},$$

we have

$$\sum_{\substack{a=1\\ (a,q)=1}}^{q}|E_a|^2\leqq\sum_{a=1}^{q}|E_a|^2=W-2V+U \tag{2}$$

where

$$U=\frac{x^2}{q^2}\Big(\sum_{(d,q)=1}\frac{\lambda_d}{d}\Big)^2$$

$$V=\sum_{n\leq x}\Big(\sum_{\substack{d_1|n\\ (d_1,q)=1}}\lambda_d\Big)\Big(\sum_{(d_2,q)=1}\frac{\lambda_{d_2}}{d_2}\Big)\frac{x}{q}$$

$$W=\sum_{\substack{n_1,n_2\leq x\\ n_1\equiv n_2(q)}}\Big(\sum_{\substack{d_1|n_1\\ (d_1,q)=1}}\lambda_{d_1}\Big)\Big(\sum_{\substack{d_2|n_2\\ (d_2,q)=1}}\lambda_{d_2}\Big).$$

We first consider W. We interprete the congruence $n_1\equiv n_2(\bmod q)$ as $n_2=n_1+ql$. Changing the order of smmation we have

$$W=2\sum_{0<l\leq x/q}\sum_{\substack{d_1\\ (d_1d_2,q)=1}}\sum_{d_2}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{n\leq x-ql\\ n\equiv 0(d_1)\\ n+ql\equiv 0(d_2)}}1+\sum_{n\leq x}\Big(\sum_{\substack{d|n\\ (d,q)=1}}\lambda_d\Big)^2.$$

The simultaneous congruences $n\equiv 0(\bmod d_1)$, $n+ql\equiv 0(\bmod d_2)$ are soluble if and only if $(d_1, d_2)|l$, and, in case of $(d_1, d_2)|l$, reduce to the single congruence $n\equiv b(\bmod [d_1, d_2])$ where

$$\begin{cases} b\equiv 0 & (\bmod d_1)\\ b\equiv -ql & (\bmod d_2^*)\end{cases} \tag{3}$$

with $d_j^*=d_j/(d_1, d_2)$, $j=1, 2$. Thus,

$$W=2\sum_{0<l\leq x/q}\sum_{\substack{d_1\\ (d_1d_2,q)=1}}\sum_{d_1}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{n\leq x-ql\\ n\equiv b([d_1,d_2])}}1+O\Big(\sum_{n\leq x}\tau(n)^2\Big)$$

$$=W_1+2R+O(x(\log x)^3) \tag{4}$$

where

$$W_1=2\sum_{0<l\leq x/q}\sum_{\substack{d_1 \\ (d_1d_2,q)=1 \\ (d_1,d_2)\mid l}}\sum_{d_2}\lambda_{d_1}\lambda_{d_2}\frac{x-ql}{[d_1,d_2]}$$

and

(5) $$R=\sum_{0<l\leq x/q}\sum_{\substack{d_1 \\ (d_1d_2,q)=1 \\ (d_1,d_2)\mid l}}\sum_{d_2}\lambda_{d_1}\lambda_{d_2}\Big(\sum_{\substack{n\leq x-ql \\ n\equiv b([d_1,d_2])}}1-\frac{x-ql}{[d_1,d_2]}\Big).$$

Leaving the estimation of R to the next section, we here carry out the summation over l in W_1.

$$W_1=\sum_{\substack{d_1 \\ (d_1d_2,q)=1}}\sum_{d_2}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}\sum_{\substack{l\leq x/q \\ (d_1,d_2)\mid l}}2(x-ql).$$

We may assume $(d_1, d_2)\leqq x/q$, otherwise the sum over l is empty. By an elementaty argument we see that the inner sum is equal to

$$\frac{x^2}{q(d_1,d_2)}+O(x).$$

Hence,

$$W_1=\sum_{\substack{d_1 \\ (d_1d_2,q)=1 \\ (d_1,d_2)\leq x/q}}\sum_{d_2}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}\frac{x^2}{q(d_1,d_2)}+O\Big(\sum_{\substack{d_1 \\ (d_1,d_2)\leq x/q}}\sum_{d_2}\frac{x}{[d_1,d_2]}\Big)$$

$$=\frac{x^2}{q}\Big(\sum_{(d,q)=1}\frac{\lambda_d}{d}\Big)^2+O\Big(\frac{x^2}{q}\sum_{\substack{d_1 \\ (d_1,d_2)>x/q}}\sum_{d_2}\frac{1}{d_1d_2}\Big)+O(x(\log x)^3)$$

(6) $$=U+O(x(\log x)^3).$$

We turn to V. Since

$$\sum_{n\leq x}\Big(\sum_{\substack{d_1\mid n \\ (d_1,q)=1}}\lambda_{d_1}\Big)=\sum_{(d_1,q)=1}\lambda_{d_1}\Big(\frac{x}{d_1}+O(1)\Big)$$

$$=\Big(\sum_{(d_1,q)=1}\frac{\lambda_{d_1}}{d_1}\Big)x+0(D),$$

we have

$$V=\Big\{\Big(\sum_{(d_1,q)=1}\frac{\lambda_{d_1}}{d_1}\Big)x+O(D)\Big\}\Big(\sum_{(d_2,q)=1}\frac{\lambda_{d_2}}{d_2}\Big)\frac{x}{q}$$

$$=U+O\Big(\frac{x}{q}D\log D\Big).$$

Combining this with (2), (4) and (6), we get

(7) $$\sum_{\substack{a=1 \\ (a,q)=1}}^{q}|E_a|^2\ll|R|+x(\log x)^3+\frac{x}{q}D\log D$$

where R is defined by (5).

5. Proof of Lemma 4.

In this section we estimate R by appealing to Lemmas 2 and 3. We shall show that $R\ll x^{2-\varepsilon}q^{-1}$, from which Lemma 4 follows by (7). We begin with expressing the innermost sum in (5) as

$$\psi\Big(\frac{x-ql}{[d_1, d_2]}-\frac{b}{[d_1, d_2]}\Big)-\psi\Big(-\frac{b}{[d_1, d_2]}\Big) \tag{8}$$

By the definition (3) of $b(\bmod [d_1, d_2])$ and the relation

$$\frac{\bar{m}}{n}+\frac{\bar{n}}{m}\equiv\frac{1}{mn}(\bmod 1) \qquad \text{for} \quad (m, n)=1,$$

we have

$$-\frac{b}{[d_1, d_2]}\equiv-\frac{b\bar{d}_2^*}{d_1}-\frac{b\bar{d}_1}{d_2^*}\equiv ql\frac{\bar{d}_1}{d_2^*}\equiv q\frac{l}{(d_1, d_2)}\frac{\bar{d}_1^*}{d_2^*} \quad (\bmod 1) \tag{9}$$

since $\mu^2(d_1)=\mu^2(d_2)=1$ and $(d_1, d_2)|l$. Furtheremore we decompose (λ_{d_2}) by Lemma 1, getting

$$\lambda_{d_2}=\sum_{c\leq\log MN}\ \sum_{rs=(d_1, d_2)}\ \sum_{mn=d_2^*}a_{rm}(c, M, N)b_{sn}(c, M, N). \tag{10}$$

In conjunction with (5), (8), (9) and (10) we may write

$$R=\sum_{\substack{\delta l\leq x/q\\(\delta, q)=1}}\ \sum_{(k,q)=1}\lambda_{\delta k}\sum_{c\leq\log MN}\ \sum_{rs=\delta}\sum_m\sum_n a_{rm}(c, M, N)b_{sn}(c, M, N).$$

$$\cdot\Big\{\psi\Big(\frac{x-q\delta l}{kmn}+ql\frac{\bar{k}}{mn}\Big)-\psi\Big(ql\frac{\bar{k}}{mn}\Big)\Big\}$$

$$\ll\sum_{\delta\leq x/q}\tau(\delta)\log x\sum_{K\leq M_0}\ \sum_{M\leq M_0}\ \sum_{N\leq N_0}\ \sup_{\alpha,\beta,\gamma}|R_1(\delta, K, M, N, \alpha, \beta, \gamma)| \tag{11}$$

with

$$R_1=R_1(\delta, K, M, N, \alpha, \beta, \gamma)$$

$$=\sum_{\substack{k\sim K\\(k, mn)=1\\(kmn, q)=1}}\ \sum_{l\leq L}\ \sum_{m\sim M}\ \sum_{n\sim N}\alpha(k)\beta(m)\gamma(n)\Big\{\psi\Big(\frac{x-q\delta l}{kmn}+ql\frac{\bar{k}}{mn}\Big)-\psi\Big(ql\frac{\bar{k}}{mn}\Big)\Big\}$$

where $M_0=x^{4/3-4\varepsilon}q^{-8/9}$, $N_0=q^{7/9}x^{-2/3}$; K, M, N's run through powers of 2; the supremum is taken over all sequences (α), (β), (γ) such that $|\alpha|$, $|\beta|$, $|\gamma|\leqq 1$; and $L=x/q\delta$. When $KMN\leqq x^{1-2\varepsilon}$, we trivially have

$$R_1\ll\frac{x^2}{q\delta}x^{-2\varepsilon}. \tag{12}$$

From now on we assume

$$KMN>x^{1-2\varepsilon}. \tag{13}$$

We apply Lemma 2 to ψ-function in R_1, getting

$$R_1 = R_2 + R_3 \tag{14}$$

where

$$R_2 = \sum_{\substack{k\sim K \\ (k,mn)=1 \\ (kmn,q)=1}} \sum_{l\le L} \sum_{m\sim M} \sum_{n\sim N} \frac{\alpha(k)\beta(m)\gamma(n)}{\delta kmn} \sum_{0<|h|\le H} e\Big(hql\frac{\bar{k}}{mn}\Big)\int_0^{x-q\delta l} e\Big(\frac{ht}{\delta kmn}\Big)dt$$

$$R_3 \ll \sum_{j=1,2} \sum_{\substack{k\sim K \\ (k,mn)=(mn,q)=1}} \sum_{l\le L} \sum_{m\sim M} \sum_{n\sim N} \min\Big(1, \frac{1}{H\|(x_j/\delta kmn)+ql(\bar{k}/mn)\|}\Big)$$

with $x_1=0$ and $x_2=x-q\delta l$.

First we treat R_3. By Lemma 2,

$$R_3 \ll \sum_{j,1,2} \sum_{h\in Z} |C_h||S_h| \tag{15}$$

where

$$S_h = \sum_{\substack{k\sim K \\ (k\ mn)=(mn,q)=1}} \sum_{l\le L} \sum_{m\sim M} \sum_{n\sim N} e\Big(\frac{hx_j}{\delta kmn}\Big) e\Big(hql\frac{\bar{k}}{mn}\Big).$$

We preceed to the estimation of S_h. Trivially,

$$S_h \ll KLMN. \tag{16}$$

For $h\ne 0$ we have, by partial summation and Lemma 3,

$$S_h \ll \sum_{\substack{l}}\sum_{\substack{m \\ (mn,q)=1}}\sum_{n}\Big|\sum_{\substack{k\sim K \\ (k,mn)=1}} e\Big(hql\frac{\bar{k}}{mn}\Big)\Big|\Big(1+\frac{hx}{\delta Kmn}\Big)$$

$$\ll \Big(1+\frac{hx}{\delta KMN}\Big)\sum_{l}\sum_{\substack{m \\ (mn,q)=1}}\sum_{n}(hql,\ mn)^{1/2}(mn)^{1/2+\varepsilon}\Big(1+\frac{K}{mn}\Big)$$

$$\ll x_\varepsilon\Big(1+\frac{hx}{KMN}\Big)\sum_{l}\Big(\sum_{m}\sum_{n}\frac{(hl,\ mn)}{mn}\Big)^{1/2}\{(\sum_{m}\sum_{n}(mn)^2)^{1/2}+K(\sum_{m}\sum_{n}1)^{1/2}\}$$

$$\ll x^\varepsilon\Big(1+\frac{hx}{KMN}\Big)\sum_{l}\tau(hl)\{(MN)^{3/2}+K(MN)^{1/2}\}$$

$$\ll x^\varepsilon\Big(1+\frac{hx}{KMN}\Big)\tau(h)L(\log x)(M_0N_0)^{3/2}$$

$$\ll Lx^{1-5\varepsilon}(\log x)\Big(1+\frac{hx}{KMN}\Big)\tau(h), \tag{17}$$

since $M_0N_0 \leqq x^{2/3-4\varepsilon}$. Now we choose

$$H = \frac{KMN}{x^{1-3\varepsilon}};$$

then $H>2$ by (13). Thus, by (15), (16), (17) and Lemma 2, we have

$$R_3 \ll \Big(|C_0| + \sum_{|h|>H^2} |C_h|\Big) KLMN + \sum_{0<|h|\le H^2} |C_h|\, L x^{1-5\varepsilon}(\log x)\Big(1+\frac{hx}{KMN}\Big)\tau(h)$$

$$\ll \Big(\frac{\log H}{H} + \sum_{h>H^2}\frac{H}{h^2}\Big) KLMN$$

$$+ L x^{1-5\varepsilon}(\log x)\Big\{\sum_{0<h\le H}\tau(h)\Big(1+\frac{Hx}{KMN}\Big)\frac{\log H}{H} + \sum_{H<h\le H^2}\tau(h)\Big(1+\frac{hx}{KMN}\Big)\frac{H}{h^2}\Big\}$$

$$\ll L x^{1-2\varepsilon} + L x^{1-5\varepsilon}(\log x)\cdot x^{3\varepsilon}(\log x)^2$$

$$\ll \frac{x^2}{q\delta} x^{-2\varepsilon}(\log x)^3. \tag{18}$$

We turn to R_2. We have

$$R_2 \le 2\int_0^x \sum_{\substack{k\sim K \\ (k,m)=(m,q)=1}} \sum_{m\sim M} \frac{|\alpha(k)\beta(m)|}{\delta k m N}$$

$$\cdot\Big|\sum_{0<h\le H}\sum_{l\le (x-t)/q\delta}\sum_{\substack{n\sim N \\ (n,kq)=1}} \gamma(n)\frac{N}{n} e\Big(\frac{ht}{\delta kmn}\Big) e\Big(hql\frac{\bar{k}}{mn}\Big)\Big| dt$$

$$\ll \frac{x}{\delta KMN}\sup_{t,c}\sum_k\sum_m\Big|\sum_h\sum_l\sum_n c_n e\Big(\frac{ht}{\delta kmn}\Big) e\Big(hql\frac{\bar{k}}{mn}\Big)\Big|$$

where the supremum is taken over all sequences (c), $|c|\le 1$, and all $0\le t\le x$. Thus,

$$R_2 \ll \frac{x}{\delta KMN}\sup_{t,c}(KM)^{1/2}(S(t,c))^{1/2} \tag{19}$$

where

$$S=S(t,c)=\sum_{\substack{k\sim K \\ (kq,m)=1}}\sum_{m\sim M}\Big|\sum_{0<h\le H}\sum_{l\le (x-t)/q\delta}\sum_{\substack{n\sim N \\ (n,kq)=1}} c_n e\Big(\frac{ht}{\delta kmn}\Big) e\Big(hql\frac{\bar{k}}{mn}\Big)\Big|^2.$$

We proceed to the estimation of S. Expanding the square and changing the order of summation, we have

$$S=\sum_{h_1,h_2}\sum_{l_1,l_2}\sum_{n_1,n_2} c_{n_1}c_{n_2}\sum_k\sum_m e\Big(\Big(\frac{h_1}{n_1}-\frac{h_2}{n_2}\Big)\frac{t}{\delta km}\Big) e\Big(h_1 q l_1\frac{\bar{k}}{mn_1} - h_2 q l_2 \frac{\bar{k}}{mn_2}\Big)$$

$$\le \sum_{0<h_1,h_2\le H}\sum_{l_1,l_2\le L}\sum_{\substack{n_1,n_2\sim N \\ (mn_1n_2,q)=1}}\sum_{m\sim M}$$

$$\cdot\Big|\sum_{\substack{k\sim K \\ (k,mn_1n_2)=1}} e\Big(\frac{(h_1n_2-h_2n_1)t}{\delta kmn_1n_2}\Big) e\Big((h_1l_1n_2-h_2l_2n_1)q\frac{\bar{k}}{mn_1n_2}\Big)\Big|$$

Here, the contribution of the diagonal terms $h_1l_1n_2-h_2l_2n_1=0$ is at most

$$\sum_{h_1l_1n_2=h_2l_2n_1} KM \ll KM\sum_{r\le 2HLN}\tau_3(r)^2$$

$$\ll x^{\varepsilon} HKLMN$$

$$\ll x^{1-2\varepsilon}H^2 L. \tag{20}$$

By Lemma 3, the non-diagonal terms contribute to S at most

$$\sum_{h_1,h_2}\sum_{\substack{l_1,l_2\\ h_1l_1n_2\neq h_2l_2n_1\\ (mn_1n_2,q)=1}}\sum_{n_1,n_2}\sum_{m}\Big(1+\frac{HNx}{\delta KMN^2}\Big)\Big|\sum_{\substack{k\sim K\\ (k,mn_1n_2)=1}} e\Big((h_1l_1n_2-h_2l_2n_1)q\frac{\bar{k}}{mn_1n_2}\Big)\Big|$$

$$\ll\Big(1+\frac{Hx}{KMN}\Big)\sum_{h_1,h_2}\sum_{\substack{l_1,l_2\\ h_1l_1n_2\neq h_2l_2n_1\\ (mn_1n_2,q)=1}}\sum_{n_1,n_2}\sum_{m} x^{\varepsilon}((h_1l_1n_2-h_2l_2n_1)q,\ mn_1n_2)^{1/2}$$

$$\cdot(mn_1n_2)^{1/2}\Big(1+\frac{K}{mn_1n_2}\Big)$$

$$\ll x^{4\varepsilon}\sum_{h_1,h_2}\sum_{l_1,l_2}\Big(\sum_{\substack{m,n_1,n_2\\ h_1l_1n_2\neq h_2l_2n_1}}\frac{(h_1l_1n_2-h_2l_2n_1,\ mn_1n_2)}{mn_1n_2}\Big)^{1/2}$$

$$\cdot\Big\{\Big(\sum_{m,n_1,n_2}(mn_1n_2)^2\Big)^{1/2}+K\Big(\sum_{m,n_1,n_2}1\Big)^{1/2}\Big\}.$$

Here we easily see

$$\sum_{\substack{m,n_1,n_2\\ h_1l_1n_2\neq h_2l_2n_1}}\frac{(h_1l_1n_2-h_2l_2n_1,\ mn_1n_2)}{mn_1n_2}\ll x^{\varepsilon}.$$

Therefore, the contribution of the non-diagonal terms is

$$\ll x^{5\varepsilon}(HL)^2\{(MN^2)^{3/2}+K(MN^2)^{1/2}\}$$

$$\ll x^{5\varepsilon}H^2L^2M_0^{3/2}N_0^{3}.$$

Combining this with (19) and (20), we have

$$R_2\ll\frac{x^{3\varepsilon}}{\delta H}\{M_0^2N_0(x^{1-2\varepsilon}H^2L+x^{5\varepsilon}H^2L^2M_0^{3/2}N_0^3)\}^{1/2}$$

$$\ll\frac{1}{\delta}\Big\{\frac{x^2}{q}x^{4\varepsilon}M_0^2N_0+\Big(\frac{x}{q}\Big)^2x^{11\varepsilon}M_0^{7/2}N_0^4\Big\}^{1/2}$$

$$\ll\frac{1}{\delta}\Big\{\frac{x^{2-4\varepsilon}}{q}\Big(\frac{x^{4/3}}{q^{8/9}}\Big)^2\Big(\frac{q^{7/9}}{x^{2/3}}\Big)+\Big(\frac{x}{q}\Big)^2x^{-3\varepsilon}\Big(\frac{x^{4/3}}{q^{8/9}}\Big)^{7/2}\Big(\frac{q^{7/9}}{x^{2/3}}\Big)^4\Big\}^{1/2}$$

$$\ll\frac{x^2}{q\delta}x^{-3\varepsilon/2}. \tag{21}$$

In conjunction with (11), (12), (14), (18) and (21) we get

$$R\ll\sum_{\delta\leq x/q}\tau(\delta)(\log x)^4\frac{x^2}{q\delta}x^{-3\varepsilon/2}\ll\frac{x^{2-\varepsilon}}{q},$$

as required.

This completes the proof of our Theorem.

References

[1] Hooley, C., On the Brun-Titchmarsh theorem., J. reine angew. Math. **225** (1972), 60-79.

[2] Iwaniec, H., A new form of the error term in the linear sieve., Acta Arith. **37** (1980), 307-320.

[3] Iwaniec, H., On the Brun-Titchmarsh theorem., J. Math. Soc. Japan **34** (1982), 95-123.

[4] Motohashi, Y., On some improvements of the Brun-Titchmarsh theorem., J. Math. Soc. Japan **26** (1974), 306-327.

[5] Motohashi, Y., Sieve methods and prime number theory., Tata Inst. Lect. Math. **72**, 1983.

[6] Turán, P., Über die Primzahlen der arithmetischen Progression., Acta Sci. Math. Szeged. **8** (1936/37), 226-235.

Institute of Mathematics
University of Tsukuba