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ON PRIME TWINS

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By

Hiroshi MIKAWA

1. Introduction.

It has long been conjectured that there exist infinitely many prime twins. There is even the hypothetical asymptotic formula for the number of prime pairs. Let

$$\Psi(y, 2k)=\sum_{2k<n\leq y}\Lambda(n)\Lambda(n-2k)$$

where Λ is the von Mangoldt function, then it is expected that

$$(*)\qquad \Psi(y, 2k)\sim\mathfrak{S}(2k)(y-2k)\qquad \text{as}\quad y\rightarrow\infty$$

with

$$\mathfrak{S}(2k)=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid k\\ p>2}}\left(\frac{p-1}{p-2}\right).$$

No proof of these has ever been given.

But it is well known that the above $(*)$ is valid for almost all $k\leqq y/2$. Recently, D. Wolke [4] has refined this classical result. He showed that in the range

$$2x\leqq y\leqq x^{8/5-\varepsilon},\qquad \varepsilon>0,$$

the formula $(*)$ holds true for almost all $k\leqq x$. Moreover he remarked that, on assuming the density hypothesis for L-series, the exponent $8/5$ may be replaced by 2.

In the present paper we shall improve this exponent beyond 2.

THEOREM. *Let ε, A and $B>0$ be given and*

$$2x\leqq y\leqq x^{3-\varepsilon}.$$

Then, except possibly for $O(x(\log x)^{-A})$ integers $k\leqq x$, we have

$$\Psi(y, 2k)=\mathfrak{S}(2k)(y-2k)+O(y(\log y)^{-B})$$

where the implied O-constants depend only on ε, A and B.

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Within the frame work of Wolke [4], we use H.L. Montgomery and R.C. Vaughan's technique on Circle method. They applied P.X. Gallagher's lemma in Fourier analysis [1] to the major arc. As for the minor arc, we also appeal to Gallagher's lemma. Then we utilize C. Hooley's devices for estimating a mean square of the trigonometric sums over primes in short intervals.

We use the standard notation in number theory. Especially, $\bar{m}$, used in either $\bar{m}/n$ or congruence modulo n, means that $\bar{m}m\equiv 1 \pmod{n}$. For a real number t, we write $\psi(t)=[t]-t+1/2$, $e(t)=e^{2\pi it}$ and $\|t\|=\min_{n\in\mathbf{Z}}|t-n|$. The convention $n\sim N$ means that $N<n\leqq N'\leqq 2N$ for some N'. The symbol F denotes a positive numerical constant, which is not the same at each occurrence.

2. Lemmas.

LEMMA 1. *Let $2<\Delta<N/2$. For arbitrary complex numbers a_n, we have*

$$\int_{|\beta|\leq 1/\Delta}\Big|\sum_{n\sim N}a_n e(\beta n)\Big|^2 d\beta \ll \Delta^{-2}\int_N^{2N}\Big|\sum_{t<n\leq t+\Delta/2}a_n\Big|^2 dt+\Delta(\sup_{n\sim N}|a_n|)^2.$$

with an absolute $\ll$-constant.

LEMMA 2. *Define*

$$\mathcal{J}(q,\Delta)=\sum_{\chi(\mathrm{mod}\, q)}\int_N^{2N}\Big|\sideset{}{^\#}\sum_{t<n\leq t+q\Delta}\chi(n)\Lambda(n)\Big|^2 dt$$

where $\#$ means that if χ is principal then $\chi(n)\Lambda(n)$ is replaced by $\Lambda(n)-1$. Let ε, A and $B>0$ be given. If $q\leqq(\log N)^B$ and $N^{1/6+\varepsilon}\leqq\Delta\leqq N^{1-\varepsilon}$, then we have

$$\mathcal{J}(q,\Delta)\ll(q\Delta)^2 N(\log N)^{-A},$$

where the implied constant depends only on ε, A and B.

Lemma 1 is a minor modification of [1, Lemma 1]. Lemma 2 is an analogous estimation on primes in almost all short intervals, and easily verified by using the same tools as that used by Wolke [4, p. 531].

LEMMA 3. *For any $\varepsilon>0$, we have*

$$\sum_{\substack{n\sim N\\(n,d)=1}}e\Big(k\frac{\bar{n}}{d}\Big)\ll(k,d)^{1/2}d^{1/2+\varepsilon}\Big(1+\frac{N}{d}\Big)$$

where the implied constant depends only on ε.

LEMMA 4. *Let k be a positive integer. If $n\leqq X$, then*

$$\Lambda(n)=\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\sum_{\substack{n_1\cdots n_j n_{j+1}\cdots n_{2j}=n\\ n_{j+1},\cdots,n_{2j}\leq X^{1/k}}}(\log n_1)\mu(n_{j+1})\cdots\mu(n_{2j}).$$

Lemma 3 is the Hooley's version of bounds for incomplete Kloosterman sums [3, Chapter 2]. Lemma 4 is the combinatrial identity of D. R. Heath-Brown [2, Lemma 1].

3. Proof of Theorem.

Let ε, D and $E>0$ be given and x be a large parameter. Define

$$x^{1+\varepsilon}<N<N'\leqq 2N\leqq x^{3-\varepsilon},\qquad k\leqq x,$$

$$S(\alpha)=\sum_{N<n\leq N'}\Lambda(n)e(\alpha n),$$

$$Q_1=(\log x)^{2D},\qquad Q=N^{1/4},$$

$$\boldsymbol{M}=\bigcup_{q\leq Q_1}\bigcup_{\substack{1\leq a\leq q\\(a,q)=1}}I_{q,a},\qquad I_{q,a}=\left[\frac{a}{q}-\frac{1}{qQ},\ \frac{a}{q}+\frac{1}{qQ}\right],$$

$$\boldsymbol{m}=[Q^{-1},\ 1+Q^{-1}]\setminus\boldsymbol{M}.$$

Furtheremore, we write

$$\int_{Q^{-1}}^{1+Q^{-1}}|S(\alpha)|^2e(-2k\alpha)d\alpha=\int_{\boldsymbol{M}}+\int_{\boldsymbol{m}}.$$

We shall show that, for any positive constant E,

$$\int_{\boldsymbol{M}}=\sum_{q\leq Q_1}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-2k)(N'-N)+O(N(\log N)^{D-E}),\tag{3.1}$$

and

$$\sum_{k\leq x}\left|\int_{\boldsymbol{m}}\right|^2\ll xN^2(\log N)^{F-D}\tag{3.2}$$

where the implied constants in the symbols O and $\ll$ depend only on ε, D and E. Wolke [4] obtained essentially the same inequalities in the range $2x\leqq N\leqq x^{8/5-\varepsilon}$. Hence, following the argument of [4], we may derive Theorem from (3.1) and (3.2).

First we consider the major arc $\boldsymbol{M}$. For $\alpha\in I_{q,a}$, write $\alpha=a/q+\beta$. We then have, with the convention in Lemma 2, that

$$\begin{aligned}S(\alpha)&=\frac{1}{\varphi(q)}\sum_{\chi(q)}\tau(\bar{\chi})\chi(a)\sum_{n\sim N}\chi(n)\Lambda(n)e(\beta n)+O((\log N)^2)\\&=\frac{\mu(q)}{\varphi(q)}\sum_{n\sim N}e(\beta n)+\frac{\mu(q)}{\varphi(q)}\sum_{n\sim N}(\Lambda(n)-1)e(\beta n)\\&\quad+\frac{1}{\varphi(q)}\sum_{\substack{\chi(q)\\ \chi\neq\chi_0}}\tau(\bar{\chi})\chi(a)\sum_{n\sim N}\chi(n)\Lambda(n)e(\beta n)+O((\log N)^2)\end{aligned}$$

$$=\frac{\mu(q)}{\varphi(q)}\sum_{n\sim N}e(\beta n)+\frac{1}{\varphi(q)}\sum_{\chi(q)}\tau(\bar{\chi})\chi(a)\sum_{n\sim N}^{\#}\chi(n)\Lambda(n)e(\beta n)+O((\log N)^2),$$
$$=a+b+O((\log N)^2),\ \text{say.}$$

We write $\int_M |a|^2 d\alpha=A^2$ and $\int_M |b|^2 d\alpha=B^2$. By Cauchy's inequality, we have

$$\int_M=\int_M |a|^2 e(-2k\alpha)d\alpha+O(A(B+(\log N)^2)+B^2+(\log N)^4). \tag{3.3}$$

By the familiar method, we have that

$$\int_M |a|^2 e(-2k\alpha)d\alpha=\sum_{q\le Q_1}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{|\beta|\le 1/qQ}\left|\frac{\mu(q)}{\varphi(q)}\sum_{n\sim N}e(\beta n)\right|^2 e\Big(-2k\Big(\frac{a}{q}+\beta\Big)\Big)d\beta$$
$$=\sum_{q\le Q_1}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-2k)\{(N'-N)+O(k)+O(qQ)\}$$
$$=\sum_{q\le Q_1}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-2k)(N'-N)+O(N(\log N)^{-E}), \tag{3.4}$$

since $Q<x$ and $x^{1+\varepsilon}<N$. Simply,

$$A^2\ll N\log\log N. \tag{3.5}$$

We proceed to estimate B. By Lemma 1, we have

$$B^2=\sum_{q\le Q_1}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{|\beta|\le 1/qQ}\left|\frac{1}{\varphi(q)}\sum_{\chi(q)}\tau(\bar{\chi})\chi(a)\sum_{n\sim N}^{\#}\chi(n)\Lambda(n)e(\beta n)\right|^2 d\beta$$
$$=\sum_{q\le Q_1}\frac{1}{\varphi^2(q)}\int_{|\beta|\le 1/qQ}\sum_{\chi(q)}|\tau(\bar{\chi})|^2\varphi(q)\left|\sum_{n\sim N}^{\#}\chi(n)\Lambda(n)e(\beta n)\right|^2 d\beta$$
$$\ll\sum_{q\le Q_1}\frac{q}{\varphi(q)}\sum_{\chi(q)}\left\{(qQ)^{-2}\int_N^{2N}\left|\sum_{t<n\le t+qQ/2}^{\#}\chi(n)\Lambda(n)\right|^2 dt+qQ(\log N)^2\right\}$$
$$\ll\sum_{q\le Q_1}\frac{q}{\varphi(q)}\cdot(qQ)^{-2}\mathcal{J}(q,Q/2)+Q_1^3Q(\log N)^2.$$

where $\mathcal{J}(q,\Delta)$ is defined in Lemma 2. Since $q\leqq(\log N)^{2D}$ and $Q=N^{1/4}$, we may apply Lemma 2 to $\mathcal{J}(q,Q/2)$. Thus, by Lemma 2, we have

$$B^2\ll\log\log N\cdot\sum_{q\le Q_1}(qQ)^{-2}\mathcal{J}(q,Q/2)+Q_1^3Q(\log N)^3$$
$$\ll N(\log N)^{2D-E'},\qquad E'=2E+1. \tag{3.6}$$

In conjunction with (3.3), (3.4), (3.5) and (3.6), we get the required estimation for (3.1).

Next we consider the minor arc $\boldsymbol{m}$.

$$I=\sum_{k\le x}\left|\int_m\right|^2\ll\int_m\int_m|S(\alpha_1)|^2|S(\alpha_2)|^2\min\Big(x,\frac{1}{\|\alpha_1-\alpha_2\|}\Big)d\alpha_1 d\alpha_2.$$

In case of $\|\alpha_1-\alpha_2\|>x^{-1}(\log N)^D=1/2\varDelta$, the corresponding integral is

$$\ll x(\log N)^{-D}\Big(\int_{Q^{-1}}^{1+Q^{-1}}|S(\alpha)|^2 d\alpha\Big)^2$$
$$\ll xN^2(\log N)^{2-D}.$$

In another case, we write $\alpha_1-\alpha_2=\beta$. Thus,

$$I\ll x\int_{\boldsymbol{m}}|S(\alpha)|^2\Big(\int_{\substack{\|\beta\|\leq 1/2\varDelta\\ \alpha+\beta\in\boldsymbol{m}}}|S(\alpha+\beta)|^2 d\beta\Big)d\alpha+xN^2(\log N)^{2-D}. \tag{3.7}$$

We note that $S(\alpha)$ has the period 1. By Lemma 1, the inner integral is

$$\ll\int_{|\beta|\leq 1/2\varDelta}|S(\alpha+\beta)|^2 d\beta$$
$$\ll\varDelta^{-2}\int_N^{2N}|\sum_{t<n\leq t+\varDelta}\varLambda(n)e(\alpha n)|^2 dt+\varDelta(\log N)^2. \tag{3.8}$$

Here we use the following lemma. We postpone the proof of Lemma 5 until the final section.

LEMMA 5. *For a real number α, define*

$$J=J(\alpha,\varDelta)=\int_N^{2N}|\sum_{t<n\leq t+\varDelta}\varLambda(n)e(\alpha n)|^2 dt.$$

Suppose that $|\alpha-a/q|\leqq q^{-2}$ with $(a,q)=1$. Then, for any small $\varepsilon>0$, we have

$$J\ll(\log N)^F\{\varDelta N(N^{1/3}+\varDelta q^{-1/2}+(\varDelta q)^{1/2})+\varDelta^2N^{1-\varepsilon}+\varDelta^3\}$$

where the implied constant depends only on ε.

Now, for any $\alpha\in\boldsymbol{m}$, there exist a and q such that

$$\Big|\alpha-\frac{a}{q}\Big|\leqq q^{-2},\quad (a,q)=1\quad\text{and}\quad Q_1<q\leqq Q.$$

Since

$$Q_1<q\leqq Q=N^{1/4}\ll x(\log N)^{-3D}\ll\varDelta/Q_1,$$
$$N\ll x^{3-\varepsilon}\ll x^3(\log N)^{-9D}\ll(\varDelta/Q_1)^3,$$

we have, by Lemma 5, that

$$J(\alpha,\varDelta)\ll\varDelta^2N(\log N)^FQ_1^{-1/2}$$

uniformly for $\alpha\in\boldsymbol{m}$. Combining this with (3.7) and (3.8), we get

$$I\ll x\int_{\boldsymbol{m}}|S(\alpha)|^2 d\alpha\cdot\varDelta^{-2}\sup_{\alpha\in\boldsymbol{m}}J(\alpha,\varDelta)+xN^2(\log N)^{3-D}$$
$$\ll xN^2(\log N)^{F-D}.$$

This gives (3.2) and, apart from the verification of Lemma 5, completes our proof of Theorem.

4. Proof of Lemma 5, preliminaries.

In this section we provide for the proof of Lemma 5. Throughout this section we assume that

$$\left|\alpha-\frac{a}{q}\right|\leqq q^{-2} \quad \text{with} \quad (a, q)=1, \quad \text{and} \quad q<\Delta<N/2. \tag{4.1}$$

Let f and g be arbitrary sequences such that $|f(n)|\leqq\log n$ and $|g(n)|\leqq\tau_5(n)\cdot\log n$. Moreover, let U and V be parameters and define

$$J\,\mathrm{I}_U=\int_N^{2N}\Big|\sum_{\substack{t<mn\leqq t+\Delta \\ m\geqq U}} g(n)e(\alpha mn)\Big|^2 dt,$$

$$J\,\mathrm{II}_{U,V}=\int_N^{2N}\Big|\sum_{t<dl\leqq t+\Delta}\Big(\sum_{\substack{mn=d \\ m\leqq U,\, n\leqq V}} g(n)\Big)e(\alpha dl)\Big|^2 dt,$$

and

$$J\,\mathrm{III}_U=\int_N^{2N}\Big|\sum_{\substack{t<mn\leqq t+\Delta \\ m\sim U}} f(m)g(n)e(\alpha mn)\Big|^2 dt.$$

In order to estimate the above integrals we use the elementary lemma; If $1<X\leqq Y$, then

$$\sum_{m\leqq X}\min\Big(\frac{Y}{m}, \frac{1}{\|\alpha m\|}\Big)\ll\Big(\frac{Y}{q}+X+q\Big)\log qX. \tag{4.2}$$

LEMMA 6.

$$J\,\mathrm{I}_U\ll(\log N)^F\{\Delta N(\Delta q^{-1/2}+(q\Delta)^{1/2})+\Delta^2(N/U)^2+\Delta^3\}$$

PROOF. Since $N\leqq t<mn\leqq t+\Delta<3N$, we may attach the condition $N<mn\leqq 3N$. We widen the range of integral to $[0, 3N]$. Expanding the square, we interchange the order of summation and integration. Thus,

$$J\,\mathrm{I}\ll\sum_{\substack{N<m_1n_1,\, m_2n_2\leqq 3N \\ m_1,\, m_2\geqq U}}\sum g(n_1)g(n_2)e(\alpha(m_1n_1-m_2n_2))\cdot\text{meas.}\left\{t:\ \begin{matrix}0\leqq t\leqq 3N \\ m_in_i-\Delta\leqq t<m_in_i \\ i=1, 2.\end{matrix}\right\}$$

If $|m_1n_1-m_2n_2|>\Delta$, then meas.$\{\ \}=0$. Since $m_in_i-\Delta>N-\Delta>0$ and $m_in_i\leqq 3N$, the condition $0\leqq t\leqq 3N$ is weaker than $\max(m_1n_1, m_2n_2)-\Delta\leqq t<\min(m_1n_1, m_2n_2)$. Hence, we see

$$\text{meas.}\{\ \}=\max(0, \Delta-|m_1n_1-m_2n_2|).$$

The diagonal terms, $m_1n_1=m_2n_2$, contribute to $J\,\mathrm{I}$ at most

$$\Delta N(\log N)^F. \tag{4.3}$$

For the non-diagonal terms, say S, we write $|m_1n_1-m_2n_2|=r$. Then,

$$S=2\operatorname{Re}\sum_{0<r\leqq\Delta}(\Delta-r)e(\alpha r)\sum_{n_1,n_2}g(n_1)g(n_2)\sum_{\substack{m_1n_1-m_2n_2=r\\ N<m_1n_1,\, m_2n_2\leqq 3N\\ m_1,\, m_2\geqq U}}1$$

The condition on the innermost sum is equivalent to

$$N(r)=\max(n_1U,\ n_2U+r,\ N+r)<m_1n_1\leqq 3N$$

$$m_1n_1\equiv r \pmod{n_2}.$$

This congruence is soluble if and only if $(n_1, n_2)|r$. Write $n_i^*=n_i/(n_1, n_2)$ and $r^*=r/(n_1, n_2)$. Then the innermost sum is equal to

$$\begin{aligned}&\#\left\{m:\frac{N(r)}{n_1}<m\leqq\frac{3N}{n_1},\ m\equiv\overline{n_1^*}r^*\pmod{n_2^*}\right\}\\ &=\frac{3N-N(0)}{[n_1, n_2]}+O\left(\frac{|N(0)-N(r)|}{[n_1, n_2]}+1\right)\\ &=\Phi+\Phi',\ \text{say.}\end{aligned}$$

Here we note that $N(0)$ is independent of r. Changing the order of summation, Φ contributes to $J\,\mathrm{I}$

$$\begin{aligned}&\ll\sum\sum_{(n_1,n_2)\leqq\Delta}|g(n_1)g(n_2)|\frac{N}{[n_1, n_2]}\Big|\sum_{\substack{0<r\leqq\Delta\\ (n_1,n_2)|r}}(\Delta-r)e(\alpha r)\Big|\\ &\ll N(\log N)^F\sum_{n\leqq\Delta}\frac{\tau_5(n)^2}{n}\cdot\Delta\min\left(\frac{\Delta}{n},\ \frac{1}{\|\alpha n\|}\right)\\ &\ll\Delta N(\log N)^F\Delta^{1/2}\left(\sum_{n\leqq\Delta}\frac{1}{n}\min\left(\frac{\Delta}{n},\ \frac{1}{\|\alpha n\|}\right)\right)^{1/2}\\ &\ll\Delta N(\log N)^F(\Delta q^{-1/2}+(q\Delta)^{1/2}),\end{aligned} \tag{4.4}$$

by partial summation, Cauchy's inequality and (4.2). Since $|N(r)-N(0)|\leqq r\leqq\Delta$ and $n_i\leqq 3N/m_i\leqq 3N/U$, the contribution of Φ' is

$$\begin{aligned}&\ll\Delta\sum\sum_{(n_1,n_2)\leqq\Delta}|g(n_1)g(n_2)|\left\{\frac{\Delta}{[n_1, n_2]}+1\right\}\frac{\Delta}{(n_1, n_2)}\\ &\ll\Delta^3\left(\sum_n\frac{|g(n)|}{n}\right)^2+\Delta^2\left(\sum_n|g(n)|\right)^2\\ &\ll(\log N)^F(\Delta^3+\Delta^2(N/U)^2).\end{aligned}$$

Combining this with (4.3) and (4.4), we get the required bound for $J\,\mathrm{I}$.

LEMMA 7.

$$J\,\mathrm{II}_{U,V}\ll(\log N)^F\{\Delta N(\Delta q^{-1/2}+(q\Delta)^{1/2})+\Delta^3\}+\Delta^2(N^{1-\varepsilon}+N^{7\varepsilon}U^{3/2}V^4).$$

PROOF. Put

$$a(d)=\sum_{\substack{mn=d\\ m\leq U,\, n\leq V}} g(n).$$

By the similar argument to that in Lemma 6, we have

$$J\,\mathrm{II} \ll (\log N)^F\{\Delta N(\Delta q^{-1/2}+(\Delta q)^{1/2})+\Delta^3\}+R \tag{4.5}$$

where

$$R=\Delta\sum_{0<r\leq\Delta}\Big|\sum_{(d_1,d_2)\mid r} a(d_1)a(d_2)\Psi(d_1{}^*, d_2{}^*, r^*)\Big|$$

with

$$\Psi(d_1{}^*, d_2{}^*, r^*)=\psi\Big(\frac{3N}{[d_1, d_2]}-r^*\frac{\overline{d_1{}^*}}{d_2{}^*}\Big)-\psi\Big(\frac{N+r}{[d_1, d_2]}-r^*\frac{\overline{d_1{}^*}}{d_2{}^*}\Big).$$

We proceed to estimate R. write

$$(d_1, d_2)=\delta, \qquad d_1=m_1n_1=\delta mn, \qquad d_2{}^*=d, \qquad r^*=k,$$

$$m_1=am, \qquad n_1=bn, \qquad \delta=ab.$$

Then,

$$(mn, d)=1,$$

$$a(d_1)=\sum_{abmn=d_1} g(bn), \qquad a(d_2)=a(\delta d),$$

$$\Psi(d_1{}^*, d_2{}^*, r^*)=\Psi(mn, d, k).$$

Next, we decompose the range of variables d and m into the sum of $[2^j, 2^{j+1}]$ type intervals. Let D, M run through powers of 2, and $D\leqq UV$, $M\leqq U$. We then obtain $O((\log N)^2)$ sums of the sum with $d\sim D$, $m\sim M$. If $DM\leqq N^{1-2\varepsilon}V^{-1}$, then we use the trivial estimation $|\Psi|\leqq 1$. Thus, we see

$$R\ll\Delta^2N^{1-\varepsilon}+\Delta^2N^{\varepsilon}V\sup_{DM>N^{1-2\varepsilon}V^{-1}}\sum_{\substack{d\sim D\\ (d,n)=1}}\Big|\sum_{\substack{m\sim M\\ (m,d)=1}}\psi\Big(\frac{T}{dmn}-k\frac{\overline{mn}}{d}\Big)\Big| \tag{4.6}$$

where the supremum is taken over D, M, T, r and n such that $D\leqq UV$, $M\leqq U$, $T\leqq 3N$, $r\leqq\Delta$ and $n\leqq V$.

Here we use the well known lemma; For arbitrary real numbers x_m and $H>2$, we have

$$\Big|\sum_{m\sim M}\psi(x_m)\Big|\ll\frac{M}{H}+\sum_{0<h\leq H}\frac{1}{h}\Big|\sum_{m\sim M}e(hx_m)\Big|.$$

Now, we choose $H=DMVN^{3\varepsilon-1}$. Then $H>2$, since $DMV>N^{1-2\varepsilon}$. Thus,

$$\begin{aligned}\sum_d\Big|\sum_m\psi\Big| &\ll\frac{DM}{H}+\sum_{0<h\leq H}\frac{1}{h}\sum_{\substack{d\sim D\\ (d,n)=1}}\Big|\sum_{\substack{m\sim M\\ (m,d)=1}}e\Big(\frac{hT}{dmn}\Big)e\Big(-hk\frac{\overline{mn}}{d}\Big)\Big| \\ &=N^{1-3\varepsilon}V^{-1}+\sum_h\frac{1}{h}S(h), \qquad \text{say}.\end{aligned} \tag{4.7}$$

Furthermore, by partial summation and Lemma 3, we see

$$S(h) \ll \left(1+\frac{hT}{DMN}\right)\sum_d (hk,\, d)^{1/2} d^{1/2+\varepsilon}\left(1+\frac{M}{d}\right)$$

$$\ll \left(1+\frac{HT}{DM}\right)N^{\varepsilon}\left(\sum_d \frac{(hk,\, d)}{d}\right)^{1/2}\{(\sum_d d^2)^{1/2}+M(\sum_d 1)^{1/2}\}$$

$$\ll VN^{5\varepsilon}\{D^{3/2}+MD^{1/2}\}$$

$$\ll N^{5\varepsilon}U^{3/2}V^{5/2}. \tag{4.8}$$

In conjunction with (4.5), (4.6), (4.7) and (4.8), we get the required bound for $J\mathrm{II}$.

LEMMA 8. *If* $U<\Delta$, *then we have*

$$J\mathrm{III}_U \ll \Delta N(\log N)^F\left(U+\frac{\Delta}{q}+\frac{\Delta}{U}+q\right).$$

PROOF. We may impose the restriction $N/2U<n<3N/U$. Then we extend the interval of integral to $[0, 6N]$. Moreover, by Cauchy's inequality, the in tegrand is

$$|\Sigma|^2 \ll \sum_{m'} |f(m')|^2 \cdot \sum_m |\sum_n|^2$$

$$= \sum_{m'\sim U} |f(m')|^2 \cdot \sum_{\substack{N/2U<n_1,\, n_2<3N/U\\ t<mn_1,\, mn_2\leq t+\Delta}} \sum_{m\sim U} \sum g(n_1)g(n_2)e(\alpha m(n_1-n_2)). \tag{4.9}$$

Now, we perform the integration. If $m|n_1-n_2|>\Delta$, then the integral vanishes. Since $mn_i<2U\cdot 3N/U=6N$ and $mn_i-\Delta>U\cdot N/2U-\Delta>0$, the end points of the integral have no effect on. Hence, the value of integral is exactly equal to

$$\max(0,\, \Delta-m|n_1-n_2|).$$

The diagonal terms, $n_1=n_2$, contribute at most

$$\sum_{m\sim U}\sum_{n\sim N/U} g(n)^2\cdot\Delta \ll \Delta N(\log N)^F. \tag{4.10}$$

As to the non-diagonal terms, S say, we write $|n_1-n_2|=r$, getting

$$S=2\,\mathrm{Re}\sum_{0<r\leq\Delta}\ \sum_{N/2U<n,\, n-r<3N/U} g(n)g(n-r) \sum_{\substack{m\sim U\\ 0<m\leq\Delta/r}} e(\alpha mr)(\Delta-mr).$$

Since $U<m\leqq\Delta/r$ in the innermost sum, we see $r\leqq\Delta/U$. Thus, by partial summation and (4.2), we have

$$S \ll \sum_{0<r\leq\Delta/U}\ \sum_{n\sim N/U} |g(n)g(n-r)|\cdot\Delta\min\left(\frac{\Delta}{r},\, \frac{1}{\|\alpha r\|}\right)$$

$$\ll \Delta\frac{N}{U}(\log N)^F\left(\frac{\Delta}{q}+\frac{\Delta}{U}+q\right).$$

Combining this with (4.9) and (4.10), we get Lemma 8.

5. Proof of Lemma 5.

Unless

$$N^{1/3}<\Delta/2 \quad \text{and} \quad q<\Delta<N/2, \tag{5.1}$$

then Lemma 5 is trivial. So we may assume (5.1). Since $n\leqq 3N$, we appeal to Lemma 4 with $X=8N$ and $k=3$. $\Lambda(n)$ is decomposed into a linear combination of $O(1)$ sums

$$\Lambda^*(n)=\sum_{\substack{n_1n_2n_3n_4n_5n_6=n\\ n_4,n_5,n_6\leq 2N^{1/3}}}(\log n_1)\mu(n_4)\mu(n_5)\mu(n_6).$$

It is sufficient to show Lemma 5 with Λ^* in place of Λ. Moreover, we may assume $\min(n_1, n_2, n_3)=n_3$, for the other cases are similarly treated. We then see that

$$n_i\leqq(3N)^{1/3} \qquad \text{for} \quad i=3, 4, 5, 6.$$

Put $n'=n_3n_4n_5n_6$. Let $v>2$ be a parameter, and $z=v^4$. We divide the integrand of J according to the following three cases.

(1) $n'\leqq z$ and $n_1>N^{1/2}v$,

(2) $n'\leqq z$ and $n_1\leqq N^{1/2}v$,

(3) $n'>z$.

Let Σ(i) denote the corresponding sum to case (i).

In case (1), we may write

$$\Lambda^*(n)=\sum_{\substack{n_1n'=n\\ n_1>N^{1/2}v}}(\log n_1)g(n')$$

with $|g(n)|\leqq\tau_5(n)$. By partial summation and Cauchy's inequality, we have

$$\int_N^{2N}|\Sigma(1)|^2dt\ll(\log N)^2\sup_{u\geq N^{1/2}v}J\,\mathrm{I}_u.$$

In case (2),

$$\begin{aligned}\Lambda^*(n)&=\sum_{n_1n_2n'=n}(\log n_1)g(n')\\ &=\sum_{n_2n''=n}\Big(\sum_{\substack{n_1n'=n''\\ n_1\leq N^{1/2}v,\ n'\leq z=v^4}}(\log n_1)g(n')\Big)\end{aligned}$$

with $|g(n)|\leqq\tau_4(n)$. Hence,

$$\int_N^{2N}|\Sigma(2)|^2dt\ll(\log N)^2\sup_{u\leq N^{1/2}v}J\,\mathrm{II}_{u,v^4}.$$

In case (3), since

$$v^4=z<n'=n_3n_4n_5n_6\leqq(\max_{i=3,4,5,6}n_i)^4,$$

there exists an index i such that

$$v<n_i\leqq(3N)^{1/3}.$$

So we may write

$$\Lambda^*(n)=\sum_{\substack{n_in''=n\\ v<n_i\leqq(3N)^{1/3}}} f(n_i)g(n'')$$

with $|f(n)|\leqq 1$, $|g(n)|\leqq\tau_5(n)\log n$. Decomposing this interval into the sum of $[2^j, 2^{j+1}]$ type intervals, we see

$$\int_N^{2N}|\Sigma(3)|^2dt\ll(\log N)^2\sup_{v<u<2N^{1/3}} J\mathrm{III}_u.$$

By the above argument, we have

$$J\ll\{\sup_{u>N^{1/2}v} J\,\mathrm{I}_u+\sup_{u\leqq N^{1/2}v} J\,\mathrm{II}_{u,v^4}+\sup_{v<u<2N^{1/3}} J\mathrm{III}_u\}(\log N)^2.$$

Because of (5.1), all of the assumptions in (4.1) and Lemma 8 are satisfied. We choose $v=N^{2\varepsilon}$ with any $0<\varepsilon<1/200$. Thus, by Lemmas 6, 7 and 8, we get

$$\begin{aligned}J&\ll(\log N)^F\{\Delta N(\Delta q^{-1/2}+(\Delta q)^{1/2})+\Delta^3\}\\&\quad+\Delta^2(\log N)^F(N/N^{1/2}v)^2+\Delta^2N^{1-\varepsilon}+\Delta^2N^{7\varepsilon}(N^{1/2}v)^{3/2}(v^4)^4\\&\quad+\Delta N(\log N)^F\left(N^{1/3}+\frac{\Delta}{q}+\frac{\Delta}{v}+q\right)\\&\ll(\log N)^F\{\Delta N(\Delta q^{-1/2}+(\Delta q)^{1/2}+N^{1/3})+\Delta^2N^{1-\varepsilon}+\Delta^3\},\end{aligned}$$

as required.

This completes our proof.

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Institute of Mathematics
University of Tsukuba