TSUKUBA J. MATH.
Vol. 14 No. 2 (1990), 505—512

ON THE ANTI-SELF-DUALITY OF THE YANG-MILLS CONNECTION OVER HIGHER DIMENSIONAL KAEHLERIAN MANIFOLD

By

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1. Introduction.

Let M be a Kaehler manifold of complex dimension $n\geqq 2$, with a Kaehler form Φ, where Φ is locally expressed by $\Phi=\sqrt{-1}\sum g_{\alpha\bar{\beta}}dz^{\alpha}\wedge d\bar{z}^{\beta}$ and a Kaehler metric $g=\sum g_{\alpha\bar{\beta}}dz^{\alpha}\otimes d\bar{z}^{\beta}$. A connection A on a principal fibre bundle $\boldsymbol{P}$ over M with the structure group G is said to be *Yang-Mills* when it gives a critical point of the Yang-Mills functional. It satisfies the Yang-Mills equation $d_A*F_A=0$ for the curvature F_A. Thus with the Bianchi identity $d_AF_A=0$ Yang-Mills connection is a connection whose curvature is harmonic with respect to the covariant derivative d_A.

When M has complex dimension 2, i.e., Kaehler surface, the Hodge $*$ operator determines a decomposition

$$\Lambda^2T^*M=\Lambda^2_+\oplus\Lambda^2_-$$

of the space of 2-forms, where $\Lambda^2_\pm$ denotes the eigenspace subbundle of $*$ of eigenvalue ± 1. Thus $*^2=id$ implies that the adjoint bundle $\mathfrak{g}_P=\boldsymbol{P}\times_{Ad}\mathfrak{g}$ valued 2-form $F_A=dA+(1/2)[A\wedge A]$ splits into $F^+=(1/2)(F_A+*F_A)$ and $F^-=(1/2)(F_A-*F_A)$, which are called the *self-dual* part and the *anti-self-dual* part of F_A respectively, where $\mathfrak{g}$ denotes the Lie algebra of G. Thus a connection A on a principal fibre bundle $\boldsymbol{P}$ over a Kaehler surface M being *Yang-Mills* is equivalent to $d_AF^+=0$ or $d_AF^-=0$.

But for a higher dimensional Kaehler manifold these formulae give us no meaning. Thus instead of using Hodge $*$ operator let us introduce another operator #, which is defined in section 2 such as $\#=*^{-1}\circ L^{(n-2)}/(n-2)!$, where L means the multiplication by Φ. Then a connection A on a principal fibre bundle $\boldsymbol{P}$ over higher dimensional Kaehler manifold M being *Yang-Mills* is equivalent to $d_A\#F_A=0$ (cf. Proposition 3.1 (ii)).

Also let us define an operator $\tilde{\#}$ such that $\tilde{\#}$ is equal to # on $F^{2,0}+F^{0,2}+$

Received June 6, 1989. Revised January 26, 1990.

$F_0^{1,1}$, and $\tilde{\#}=\#/(n-1)$ on $F^0\otimes\Phi$, where $F^{p,q}$ is the (p, q)-component and $F_0^{1,1}$ means the primitive (1, 1) form and F^0 is 0-form. Then we can consider the *self-duality* and *anti-self-duality* of F_A in the sense of $\tilde{\#}F^+=F^+$ and $\tilde{\#}F^-=-F^-$, where the self-dual part is $F^+=F^{2,0}+F^{0,2}+F^0\otimes\Phi$ and the anti-self-dual part F^- is a form of type (1, 1) orthogonal to Kaehler form Φ, that is, $F_0^{1,1}$.

Then our anti-self-dual connection minimizes the Yang-Mills functional, and then is a Yang-Mills connection (cf. Theorem 4.2).

Now we can state main theorems which give the curvature form conditions for a Yang-Mills connection to be anti-self-dual, and which generalize some results of M. Itoh for Kaehler surfaces [3].

THEOREM A. *Let M be a complex n-dimensional compact Kaehler manifold with the sum of any two distinct eigenvalues of the Ricci tensor is positive. Let A be an irreducible Yang-Mills connection. If $[F^{2,0}\wedge F^{0,2}]=0$, then A is anti-self-dual.*

REMARK. M. Itoh [3] obtained the above result for a compact Kaehler surface with positive scalar curvature.

With another commutative curvature condition we also have the following.

THEOREM B. *Let M be a compact Kaehler manifold with the same condition as in Theorem A. If $[F^{2,0}\wedge F^{1,1}]=0$ and $[F^0\wedge F^{2,0}]=0$, then A is anti-self-dual.*

The author would like to express his hearty thanks to Professor Mitsuhiro Itoh for his valuable suggestion and encouragement to develop this paper and to the referee for his kind comments.

2. Self-duality and anti-self-duality.

Let M be an n-dimensional compact complex manifold with a Kaehler metric g. Let Φ be its Kaehler form. When M is a compact Kaehler surface, the Hodge $*$ operator is involutive. Thus naturally we can consider self-dual 2 form (or anti-self-dual 2 form). But in a higher dimensional manifold it gives us no meaning. However H. J. Kim [4] defined the involutive operator # as follows.

We denote by $A'=\sum A^p$ the exterior algebra of all smooth real valued forms on M. Now let us define the Lipschitz operator L by $L\phi=\phi\wedge\Phi$, $\phi\in A'$ and the operator $\Lambda: A'\to A'$ which is the adjoint of L. Then it is well known that $*$, L, and Λ satisfy the following relations

(2.1) $\Lambda=L^*=*^{-1}\circ L\circ *, \qquad (\Lambda L-L\Lambda)|_{A^k}=n-k, \qquad \Lambda(\Phi)=n\,.$

(2.2) $*^2|_{A^k}=(-1)^{k(n-k)}\,.$

(2.3) $*(\Phi^k/k!)=\Phi^{n-k}/(n-k)!\,, \qquad k=0, 1, \cdots n\,.$

We denote also by $A^{p,q}$ the space of C^∞-(p, q) forms on M and by $A_0^{p,q}$ the space of primitive (p, q) forms, that is,

$$A_0^{p,q}=\{\alpha\in A^{p,q}|\,\Lambda\alpha=0\}.$$

Then

LEMMA 2.1 (R. O. Well [7]). *Let* $k=p+q$.

(i) *if* $k\geqq n$, *then* $A_0^{p,q}=0$.

(ii) *if* $k\geqq n$, *then* $A_0^{p,q}=\{\alpha\in A^{p,q}|\,L^{n-k+1}\alpha=0\}$
$=\{\alpha\in A^{p,q}|\,C_{p,q}*L^{(n-k)}\alpha/(n-k)!=\alpha\}$,

where $C_{p,q}=(-1)^{pq}(\sqrt{-1})^{p^2-q^2}$.

The space A^2 of 2-forms is decomposed as

$$A^2=A^{2,0}+A^{0,2}+A_0^{1,1}+A_\Phi^{1,1}$$

where $A_\Phi^{1,1}$ denotes the space of (1, 1) type propotional to Φ. And let us now consider the operator # which is defined by H. J. Kim:

$$\#: A^2 \xrightarrow{L^{(n-2)}/(n-2)!} A^{2(n-1)} \xrightarrow{*^{-1}=*} A^2, \quad \text{i.e.,}\ \#=*^{-1}\circ L^{(n-2)}/(n-2)!$$

Then we have the following from the definition of # and Lemma 2.1.

LEMMA 2.2. (i) $A_0^{1,1}=\{\alpha\in A^2|\,\#\alpha=-\alpha\}$,

(ii) $A^{2,0}+A^{0,2}=\{\alpha\in A^2|\,\#\alpha=\alpha\}$,

(iii) $A_\Phi^{1,1}=\{\alpha\in A^2|\,\#\alpha=(n-1)\alpha\}$.

Now we define an operator

$$\tilde{\#}=\begin{cases} \# & \text{on } A^{2,0}+A^{0,2}+A_0^{1,1}, \\ \#/(n-1) & \text{on } A_\Phi^{1,1}. \end{cases}$$

Then we get $\tilde{\#}^2=id$ which implies that A^2 is decomposed into the self-dual part $A_+^2=A^{2,0}+A^{0,2}+A_\Phi^{1,1}$ and the anti-self-dual part $A_0^{1,1}$. Hence the curvature form F_A also can be splitted into the self-dual part $F^+=F^{2,0}+F^{0,2}+F^0\otimes\Phi$ and the anti-self-dual part $F^-=F_0^{1,1}$, i.e., $\tilde{\#}F^+=F^+$, and $\tilde{\#}F^-=-F^-$.

3. Anti-self-duality of Yang-Mills connection.

Let $\boldsymbol{P}$ be a principal fibre bundle over a compact Kaehler manifold M with a compact semi-simple Lie group G. Let A be a connection on $\boldsymbol{P}$. Then we get:

PROPOSITION 3.1. *The following conditions are equivalent.*

(i) *A is Yang-Mills i.e., $d_A * F_A=0$,*

(ii) $d_A \# F_A=0$,

(iii) $2\bar{\partial}_A F^{2,0}+n\partial_A(F^0\otimes\Phi)=0$,

(iv) $\partial_A^* F^{2,0}=-ni\partial_A F^0/2(n-1)$.

PROOF.

(i)⇔(ii) It is well known that a connection A being Yang-Mills if and only if the curvature satisfies Yang-Mills equation $d_A * F_A=0$. With $\delta_A \Phi^{n-2}=0$ the Yang-Mills equation $d_A * F_A=0$ implies

$$*d_A \# F_A=\delta_A(F_A\wedge\Phi^{n-2})/(n-2)!=0\,,$$

that is, $d_A \# F_A=0$, where δ_A means the formal adjoint of d_A such that $\delta_A=-*d_A*$.

Conversely $*d_A\# F_A=0$ gives $(\delta_A F_A)\wedge\Phi^{n-2}=0$ because $\delta_A\Phi^{n-2}=0$. Since the nondegeneracy of Φ^{n-2} is invariant by taking an orthonormal dual basis, we can assert that $(\delta_A F_A)\wedge\Phi^{n-2}=0$ implies $\delta_A F_A=0$, that is, $d_A * F_A=0$. From this fact a connection A being Yang-Mills is equivalent to $d_A\# F_A=0$.

(ii)⇔(iii) From Lemma 2.2 it follows that

$$\# F_A=F^{2,0}+F^{0,2}-F_0^{1,1}+(n-1)(F^0\otimes\Phi)\,.$$

Then by the assumption (ii) we have that

$$0=d_A\# F_A=(\partial_A+\bar{\partial}_A)(F^{2,0}+F^{0,2}-F_0^{1,1}+(n-1)(F^0\otimes\Phi))\,,$$

from which it follows that

$$\partial_A F^{0,2}-\bar{\partial}_A F_0^{1,1}+(n-1)\bar{\partial}_A(F^0\otimes\Phi)=0\,, \tag{3.1}$$

$$\bar{\partial}_A F^{2,0}-\partial_A F_0^{1,1}+(n-1)\partial_A(F^0\otimes\Phi)=0\,. \tag{3.2}$$

On the other hand, the Bianchi identity gives that

$$\partial_A F^{0,2}+\bar{\partial}_A(F^0\otimes\Phi)+\bar{\partial}_A F_0^{1,1}=0,\quad (\text{resp. } \bar{\partial}_A F^{2,0}+\partial_A(F^0\otimes\Phi)+\partial_A F_0^{1,1}=0). \tag{3.3}$$

Summing up (3.1) and (3.3), we obtain $2\partial_A F^{0,2}+n\bar{\partial}_A(F^0\otimes\Phi)=0$.

Conversely, it suffices to show that (3.1) holds since (3.1) and its conjugate part (3.2) is equivalent to $d_A \# F_A = 0$. Thus the left side of (3.1) becomes $-(\partial_A F^{0,2} + \partial_A F_0^{1,1} + \bar\partial_A(F^0 \otimes \Phi))$ because of the assumption (iii). Thus it vanishes from the Bianchi identity (3.3).

(iii)⇔(iv) The invariance of $F^{2,0}$ by # gives that

$$\frac{1}{(n-2)!}(\partial_A^* F^{2,0}) \wedge \Phi^{n-2} = -*\bar\partial_A F^{2,0} \cdot \tag{3.4}$$

Since $\#(F^0 \otimes \Phi) = (n-1)(F^0 \otimes \Phi)$, we have that

$$*\partial_A(F^0 \otimes \Phi) = \frac{1}{(n-1)!} *\partial_A *(F^0 \otimes \Phi^{n-1}) = -\frac{1}{(n-1)!}(\bar\partial_A^* F^0 \otimes \Phi) \wedge \Phi^{n-2}, \tag{3.5}$$

where we have used the definition of # and $\bar\partial_A^* = -*\partial_A *$.

Now we suppose the assumption (iii). Then (iii) implies $-*\bar\partial_A F^{2,0} = (n/2)*\partial_A(F^0 \otimes \Phi)$, from which, and using the invariance of the nondegeneracy of Φ^{n-2} to (3.4) and (3.5), it follows that

$$\partial_A^* F^{2,0} = -\frac{n}{2(n-1)} \bar\partial_A^*(F^0 \otimes \Phi) = -\frac{n}{2(n-1)} i \partial_A F^0 .$$

Conversely, the condition (iv) gives $-*\bar\partial_A F^{2,0} = (n/2)*\partial_A(F^0 \otimes \Phi)$ by virtue of (3.4) and (3.5). Thus the condition (iii) holds immediately.

Note. *M. Itoh obtained the above results for the case* $n=2$ *in the paper* [3].

DEFINITION. A connection A is said to be irreducible when it admits no nontrivial covariantly constant Lie algebra valued 0-form.

By using the above proposition we get the following.

COROLLARY 3.2. *Let A be an irreducible Yang-Mills connection and its curvature is* (1.1) *type, then it is anti-self-dual.*

PROOF. Anti-self-dual Yang-Mills connection is characterized by the self-dual part $F^+ = F^{2,0} + F^{0,2} + F^0 \otimes \Phi$ vanishes. Since F is of type (1, 1), $F^{2,0}$ and $F^{0,2}$ vanishes. By Proposition 3.1 (iii) $\partial_A(F^0 \otimes \Phi) = 0$ (or $\bar\partial_A(F^0 \otimes \Phi) = 0$), which implies $F^0 \otimes \Phi = 0$ by the irreducibility of A. Thus the self-dual part F^+ vanishes.

Using Proposition 3.1, we also obtain the following Lemma·

LEMMA 3.3. *Let A be a Yang-Mills connection. Then* $\square_A F^{2,0} = \frac{n}{2(n-1)} i[F^0 \wedge F^{2,0}]$, *where* $\square_A$ *means* $\partial_A \partial_A^* + \partial_A^* \partial_A$.

PROOF. By Proposition 3.1 (iv) we have $\square_A F^{2,0}=-\frac{n}{2(n-1)}i\partial_A\partial_A F^0$. From this and the formula $d_A d_A F^0=[F_A\wedge F^0]$ we obtain the above fact.

Applying Ricci formula for the $\mathfrak{g}_P^C$-valued (2, 0) form Ψ, then we obtain ([12])

$$(3.6)\qquad (\square_A\Psi)_{\mu\nu}=-\sum g^{\bar{\sigma}\tau}\nabla_{\bar{\sigma}}\nabla_\tau\Psi_{\mu\nu}-\sum g^{\bar{\sigma}\tau}[F_{\mu\bar{\sigma}},\Psi_{\tau\nu}]+\sum g^{\bar{\sigma}\tau}[F_{\nu\bar{\sigma}},\Psi_{\tau\mu}]+\sum(R_\mu{}^\varepsilon\Psi_{\varepsilon\nu}-R_\nu{}^\varepsilon\Psi_{\varepsilon\mu}).$$

With this formula and Lemma 3.3 we will show here Theorem A in the introduction.

PROOF OF THEOREM A. For the component $F^{2,0}$ of type (2, 0) the above formula (3.6) becomes

$$(3.7)\qquad (\square_A F^{2,0})_{\mu\nu}=(\nabla_A^*\nabla_A F^{2,0})_{\mu\nu}-\sum g^{\bar{\sigma}\tau}[F_{\mu\bar{\sigma}},F_{\tau\nu}]+\sum g^{\bar{\sigma}\tau}[F_{\nu\bar{\sigma}},F_{\tau\mu}]+(\lambda_\mu+\lambda_\nu)F_{\mu\nu},$$

where λ_μ means the eigenvalues of the Ricci operator R.

Computing the inner product of $\square_A F^{2,0}$ and $F^{2,0}$, then under the assumption $[F^{2,0}\wedge F^{0,2}]=0$ we obtain the following integral formula

$$\int_M(|\nabla_A F^{2,0}|^2+\sum_{\mu<\nu}(\lambda_\mu+\lambda_\nu)|F_{\mu\nu}^{2,0}|^2)dv=0\,.$$

Here we used Lemma 3.3 and the fact that $\langle\langle i[F^0\wedge F^{2,0}], F^{2,0}\rangle\rangle dv=\langle i[F^0\wedge F^{2,0}]\wedge *F^{0,2}\rangle=\langle F^0, i[F^{2,0}\wedge F^{0,2}]\rangle=0$. Thus $F^{2,0}$ vanishes because of $\sum_{\mu<\nu}(\lambda_\mu+\lambda_\nu)>0$. So is $F^{0,2}$. Hence Proposition 3.1 (iii) and the irreducibility of Yang-Mills connection implies $F^0\otimes\Phi=0$. This means $F=F_0^{1,1}$. That is, A is anti-self-dual.

Since $F^{1,1}=F^0\otimes\Phi+F_0^{1,1}$, where $F^0=(1/n)\langle F_\Phi^{1,1},\Phi\rangle$, Lemma 3.3 and the formula (3.7) give that

$$(\nabla_A^*\nabla_A F^{2,0})_{\mu\nu}-\frac{5n-4}{2(n-1)}i[F^0, F_{\mu\nu}]+(\lambda_\mu+\lambda_\nu)F_{\mu\nu}+\sum_\sigma([(F_0)_{\mu\bar{\sigma}}, F_{\sigma\mu}]-[(F_0)_{\mu\bar{\sigma}}, F_{\sigma\nu}])=0\,.$$

Applying this formula, by the similar way as in Theorem A we have Theorem B.

DEFINITION. A connection on a complex n-dimensional Kaehler manifold is said to be with harmonic curvature if $F^{2,0}$ is harmonic, i.e., $\partial_A^* F^{2,0}=0$.

Then a Yang-Mills connection with harmonic curvature by Proposition 3.1 (iv) satisfies that $F^0=0$ and $F^{1,1}=F_0^{1,1}$. From this fact and Theorem B we can also assert that

COROLLARY 3.4. *Let M be a compact Kaehler manifold with the same assumption as in Theorem A. Let A be an irreducible Yang-Mills connection with harmonic curvature. If* $[F_0^{1,1}\wedge F^{2,0}]=0$, *then A is anti-self-dual.*

4. Another characterization of anti-self-dual connection.

Let $\boldsymbol{P}$ be a principal fibre bundle over compact Kaehler manifold M with structure group $G=SU(r)$. And let A be a connection in $\boldsymbol{P}$. Then it is well known that Yang-Mills functional $\mathfrak{YM}(A)$ is given by

$$\mathfrak{YM}(A)=\frac{1}{2}\int_M(-Tr)(F\wedge *F)=\frac{1}{2}\int_M |F|^2\frac{\Phi^n}{n!}.$$

where $\Phi^n/n!$ is the volume form of compact Kaehler manifold M.

Now we assert the following formula.

LEMMA 4.1.

$$-Tr\,F\wedge *F=Tr\,F\wedge F\wedge\frac{\Phi^{n-2}}{(n-2)!}+2|F^{2,0}+F^{0,2}|^2 vol_\Phi+n|F^0\otimes\Phi|^2 vol_\Phi,$$

where $vol_\Phi=\Phi^n/n!$.

PROOF. Since the curvature is decomposed as $F=F^{2,0}+F^{0,2}+F^0\otimes\Phi+F_0^{1,1}$, Lemma 1.5 yields $\#F=*\left(F\wedge\frac{\Phi^{n-2}}{(n-2)!}\right)=F^{2,0}+F^{0,2}+(n-1)F^0\otimes\Phi-F_0^{1,1}$. Then we get

$$*(F^{2,0}+F^{0,2})=(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!},\quad (n-1)*(F^0\otimes\Phi)=(F^0\otimes\Phi)\wedge\frac{\Phi^{n-2}}{(n-2)!},$$

$$*F_0^{1,1}=-F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!}.$$

Thus it follows that

$$*F=(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!}+F^0\otimes\frac{\Phi^{n-1}}{(n-1)!}-F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!}.$$

Then by a direct calculation we have

$$\text{(4.1)}\qquad F\wedge *F=(F^{2,0}+F^{0,2})\wedge(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!}+\frac{1}{(n-1)!}F^0\otimes F^0\Phi^n -F_0^{1,1}\wedge F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!},$$

$$\text{(4.2)}\qquad Tr\,F\wedge F\wedge\frac{\Phi^{n-2}}{(n-2)!}=Tr\,(F^{2,0}+F^{0,2})\wedge(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!} +Tr\,F^0\otimes F^0\cdot\frac{\Phi^n}{(n-2)!}+Tr\,F_0^{1,1}\wedge F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!}.$$

Thus, combining (4.1) and (4.2), we obtain Lemma 4.1.

From the above Lemma 4.1 we obtain

THOREM 4.2. *Let M be a compact Kaehler manifold. Let A be a connection in the principal fibre bundle* $\boldsymbol{P}$ *over M with structure group* $G=SU(r)$. *Then* $\mathfrak{Y}\mathfrak{M}(A)\geqq \frac{1}{2}\int_M C(\boldsymbol{P})\wedge \frac{\Phi^{n-2}}{(n-2)!}$, *where* $C(\boldsymbol{P})=Tr\, F\wedge F=8\pi^2 c_2(E)$, $E=\boldsymbol{P}\times_{SU(r)}C^r$. *The equality holds if and only if A is anti-self-dual.*

REMARK. H. J. Kim showed that the Yang-Mills functional is bounded below by a topological constant and this minimum is achieved if and only if the curvature is Einstein ([4]).

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