TSUKUBA J. MATH.
Vol. 14 No. 2 (1990), 489–496

THE HOCHSCHILD COCYCLE CORRESPONDING TO A LONG EXACT SEQUENCE

Dedicated to Hiroyuki Tachikawa on his 60th birthday

By

Vlastimil DLAB and Claus Michael RINGEL

1. Let k be a field, and A an associative k-algebra with 1. Let M, N be right A-modules. We denote by $H^{\cdot}$ the Hochschild cohomology of A. It is well-known that there is a natural isomorphism

$$\eta_{MN}: \mathrm{Ext}_A^t(M, N) \longrightarrow H^t(A, \mathrm{Hom}_k(M, N))$$

see Cartan-Eilenberg [CE], Corollary IX. 4.4. For $t \geqq 1$, the elements of $\mathrm{Ext}_A^t(M, N)$ may be considered as equivalence classes of long exact sequences, see Mac Lane [M], chapter III. Let

$$E=(0 \longleftarrow M \overset{g_0}{\longleftarrow} Y_1 \overset{g_1}{\longleftarrow} Y_2 \longleftarrow \cdots \longleftarrow Y_t \overset{g_t}{\longleftarrow} N \longleftarrow 0)$$

be an exact sequence. We want to derive a recipe for obtaining a corresponding cocycle $A^{\otimes(t+2)} \rightarrow \mathrm{Hom}_k(M, N)$.

For $0 \leqq i \leqq t+1$, let Z_i be right A-modules, and for $0 \leqq i \leqq t$, let $\beta_i: Z_i \rightarrow Z_{i+1}$ be k-linear maps. With $\beta=(\beta_0, \cdots, \beta_t)$ we associate a map

$$\Omega_\beta: A^{\otimes(t+2)} \longrightarrow \mathrm{Hom}_k(Z_0, Z_{t+1})$$

defined by

$$(a_0, \cdots, a_{t+1})\Omega_\beta = \bar{a}_0\beta_0\bar{a}_1\beta_1 \cdots \bar{a}_t\beta_t\bar{a}_{t+1},$$

for $a_0, \cdots, a_{t+1} \in A$, where $\bar{a}_i$ denotes the scalar multiplication by a_i (on Z_i); note that all maps will be written on the right of the argument, thus the composition of $\beta_0: Z_0 \rightarrow Z_1$, and $\beta_1: Z_1 \rightarrow Z_2$ is denoted by $\beta_0\beta_1$.

Given the exact sequence E exhibited above, it clearly splits as a sequence of k-spaces, thus there are k-linear maps

$$M \overset{\gamma_0}{\longrightarrow} Y_1 \overset{\gamma_1}{\longrightarrow} Y_2 \longrightarrow \cdots \longrightarrow Y_t \overset{\gamma_t}{\longrightarrow} N$$

such that

$$\gamma_{i-1}\gamma_i=0, \quad g_{i-1}\gamma_{i-1}+\gamma_i g_i=1_{Y_i}, \qquad \text{for } 1 \leqq i \leqq t,$$

Received December 4, 1989.

and

$$\gamma_0 g_0 = 1_M, \qquad g_t \gamma_t = 1_N,$$

(see section 2).

THEOREM. *The map* $\Omega_\gamma : A^{\otimes(t+2)} \to \mathrm{Hom}_k(M, N)$ *is a cocycle, and the cohomology classes* $[\Omega_\gamma]$ *and* $\eta([E])$ *in* $H^t(A, \mathrm{Hom}_k(M, N))$ *are equal up to sign.*

One reason for our interest in this problem is the following: Consider the case $t=2$. Given any bimodule ${}_AT_A$, the elements of $H^2(A, T)$ index the various "Hochschild extensions" $\tilde{A}$ of A by T (here, $\tilde{A}$ is a k-algebra with a square zero ideal I such that $\tilde{A}/I = A$, and such that I, as an $A-A$-bimodule, is isomorphic to T; note that the multiplication of $\tilde{A}$ can be recovered from A and T using the corresponding 2-cocycle, see [H] or [CE], XIV. 2). There is a recursive construction for quasi-hereditary algebras due to Parshall and Scott ([PS], Theorem 4.6) which uses Hochschild extensions of quasi-hereditary algebras A by bimodules of the form $\mathrm{Hom}_k(M, N)$, so we have to deal with 2-cocycles $A^{\otimes(4)} \to \mathrm{Hom}_k(M, N)$. Our presentation of such 2-cocycles using long exact sequences should help to understand these algebras. Also, we remark that the Hochschild cohomology groups with values in $\mathrm{Hom}_k(DA, A)$, where $DA = \mathrm{Hom}_k(A, k)$, play a prominent role in Tachikawa's discussion of the Nakayama conjecture [T].

2. *The splitting for E over k.* In order to work with the sequence E, it will be convenient to use the notation: $Y_{-1}=0$, $Y_0=M$, $Y_{t+1}=N$, $Y_{t+2}=0$, and to deal also with the zero maps $g_{-1}: Y_0 \to Y_{-1}$, $\gamma_{-1}: Y_{-1} \to Y_0$, $g_{t+1}: Y_{t+2} \to Y_{t+1}$, $\gamma_{t+1}: Y_{t+1} \to Y_{t+2}$; so that the conditions mentioned above can be rewritten in the form

$$\gamma_{i-1}\gamma_i = 0, \quad g_{i-1}\gamma_{i-1} + \gamma_i g_i = 1_{Y_i}, \qquad \text{for } 0 \leqq i \leqq t+1.$$

Let X_i be the image of g_i, thus we have short exact sequences

$$0 \longleftarrow X_{i-1} \overset{h_{i-1}}{\longleftarrow} Y_i \overset{f_i}{\longleftarrow} X_i \longleftarrow 0$$

for $1 \leqq i \leqq t$, with $g_0 = h_0$, $g_i = h_i f_i$ for $1 \leqq i \leqq t-1$, and $g_t = f_t$. These sequences split over k, thus we obtain k-linear maps $\varphi_i : Y_i \to X_i$, $\eta_{i-1}: X_{i-1} \to Y_i$ such that $\eta_{i-1}\varphi_i = 0$, $f_i\varphi_i = 1_{X_i}$, $\eta_{i-1}h_{i-1} = 1_{X_{i-1}}$ and $h_{i-1}\eta_{i-1} + \varphi_i f_i = 1_{Y_i}$ for all i. Now, take $\gamma_i = \varphi_i\eta_i : Y_i \to Y_{i+1}$, in this way we obtain a splitting of E over k.

3. *Preparation for the proof.* Let $A^e = A^{op} \underset{k}{\otimes} A$ be the enveloping algebra of A, where A^{op} is the opposite algebra of A. The $A-A$-bimodules are just the (right) A^e-modules, in particular, A itself is in a canonical way an A^e-module. For $n\geqq 0$, let $S_n=A^{\otimes(n+2)}$, and let $\nabla_n: S_{n+1}\to S_n$ be defined by

$$(a_0\otimes a_1\otimes\cdots\otimes a_{n+2})\nabla_n=\sum_{i=0}^{n+1}(-1)^i a_0\otimes\cdots\otimes(a_i a_{i+1})\otimes\cdots\otimes a_{n+2}.$$

Also, let $\nabla_{-1}: S_0\to A$ be defined by

$$(a_0\otimes a_1)\nabla_{-1}=a_0a_1.$$

The S_n are $A-A$-bimodules, or, equivalently A^e-modules, the scalar multiplication of $a_0\otimes a_1\otimes\cdots a_{n+1}\in S_n$ by $a\otimes a'\in A^{op}\otimes_k A=A^e$ yields $(aa_0)\otimes a_1\otimes\cdots\otimes(a_{n+1}a')$. Note that for all $n\geqq -1$, the maps ∇_i are A^e-linear, in fact

$$A\xleftarrow{\nabla_{-1}}S_0\xleftarrow{\nabla_0}S_1\longleftarrow\cdots$$

is a projective resolution of A as a right A^e-module, it is called the *standard resolution* of A, see [CE], IX. 6. We can use this resolution in order to calculate $H^t(A, \mathrm{Hom}_k(M, N))=\mathrm{Ext}^t_{A^e}(A, \mathrm{Hom}_k(M, N))$.

4. Besides $\gamma=(\gamma_0, \cdots, \gamma_t)$, we also will need for $0\leqq r\leqq t$, the sequences $\gamma(r)=(\gamma_0, \cdots, \gamma_r)$, so that $\gamma(0)=(\gamma_0)$, $\gamma(t)=\gamma$. According to section 1, there is defined $\Omega_{\gamma(r)}: S_r\to\mathrm{Hom}_k(Y_0, Y_{r+1})$. In addition, by abuse of language, we also define $\Omega_{\gamma(-1)}: A\to\mathrm{Hom}_k(Y_0, Y_0)$ by $a\Omega_{\gamma(-1)}=\bar{a}$, for $a\in A$.

LEMMA. *For $0\leqq r\leqq t$, we have $\nabla_{r-1}\Omega_{\gamma(r-1)}=(-1)^r\Omega_{\gamma(r)}\mathrm{Hom}(1, g_r)$.*

PROOF. We introduce the following notation: let $\sigma_i=\gamma_i\bar{a}_i$, $\tau_i=\bar{a}_i\gamma_i: Y_i\to Y_{i+1}$ for $0\leqq i\leqq t-1$, and let $\sigma_{ij}=\sigma_i\sigma_{i+1}\cdots\sigma_j$, $\tau_{ij}=\tau_i\tau_{i+1}\cdots\tau_j$ for $0\leqq i\leqq j\leqq t-1$; by abuse of language, let $\sigma_{i+1,i}=1_{Y_i}$, and $\tau_{i+1,i}=1_{Y_{i+1}}$. Recall that

$$(a_{-1}\otimes\cdots\otimes a_r)\nabla_{r-1}=\sum_{i=0}^{r}(-1)^i a_{-1}\otimes\cdots\otimes(a_{i-1}a_i)\otimes\cdots\otimes a_r,$$

thus

$$\begin{aligned}(a_{-1}\otimes\cdots\otimes a_r)\nabla_{r-1}\Omega_{\gamma(r-1)}&=\sum_{i=0}^{r}(-1)^i\bar{a}_{-1}\sigma_{0,i-1}\tau_{i,r-1}\bar{a}_r\\&=\sum_{i=0}^{r}(-1)^i\bar{a}_{-1}\sigma_{0,i-1}(g_{i-1}\gamma_{i-1}+\gamma_i g_i)\tau_{i,r-1}\bar{a}_r,\end{aligned}$$

where we have inserted $1_{Y_i}=g_{i-1}\gamma_{i-1}+\gamma_i g_i$. Note that for $0\leqq i\leqq r-1$, we have

$$\begin{aligned}\sigma_{0,i-1}\gamma_i g_i\tau_{i,r-1}&=\sigma_{0,i-1}\gamma_i g_i\bar{a}_i\gamma_i\tau_{i+1,r-1}\\&=\sigma_{0,i-1}\gamma_i\bar{a}_i g_i\gamma_i\tau_{i+1,r-1}\\&=\sigma_{0,i}g_i\gamma_i\tau_{i+1,r-1},\end{aligned}$$

since g_i is A-linear. As a consequence, the last term of the summand with index i and the first term of the summand with index $i+1$ are equal up to sign, so they cancel. In addition, the first term of the summand with index $i=0$ involves $g_{-1}=0$, thus vanishes. It remains

$$\begin{aligned}(a_{-1}\otimes\cdots\otimes a_r)\nabla_{r-1}\Omega_{\gamma(r-1)} &= (-1)^r\bar{a}_{-1}\sigma_{0,r-1}\gamma_r g_r\bar{a}_r \\ &= (-1)^r\bar{a}_{-1}\sigma_{0,r}g_r \\ &= (-1)^r(a_{-1}\otimes\cdots\otimes a_r)\Omega_{\gamma(r)}-g_r \\ &= (-1)^r(a_{-1}\otimes\cdots\otimes a_r)\Omega_{\gamma(r)}\mathrm{Hom}(1, g_r).\end{aligned}$$

This finishes the proof.

5. *An injective coresolution of the $A-A$-bimodule* $\mathrm{Hom}_k(M, N)$. We choose a projective resolution

$$0\longleftarrow M\overset{p_{-1}}{\longleftarrow}P_0\overset{p_0}{\longleftarrow}P_1\longleftarrow\cdots$$

of the A-module M, and an injective coresolution

$$0\longrightarrow N\overset{q^{-1}}{\longrightarrow}Q^0\overset{q^0}{\longrightarrow}Q^1\longrightarrow\cdots$$

of the A-module N. For $t\geqq 0$, let $L^t=\bigoplus_{i=0}^{t}\mathrm{Hom}_k(P_i, Q^{t-i})$, this is an $A-A$-bimodule, or, equivalently a right A^e-module. For $t\geqq 0$, define an A^e-linear map $\Delta^t: L^t\to L^{t+1}$ by

$$(\varphi_0, \cdots, \varphi_t)\Delta^t=(\varphi_0 q^t, (-1)^{t+1}p_0\varphi_0+\varphi_1 q^{t-1}, \cdots, (-1)^{t+1}p_{t-1}\varphi_{t-1}+\varphi_t q^0, (-1)^{t+1}p_t\varphi_t),$$

where $\varphi_i\in\mathrm{Hom}_k(P_i, Q^{t-i})$, and define $\Delta^{-1}: \mathrm{Hom}_k(M, N)\to L^0$ by $\Delta^{-1}=\mathrm{Hom}(p_{-1}, q^{-1})$. We obtain a sequence

$$0\longrightarrow\mathrm{Hom}_k(M, N)\overset{\Delta^{-1}}{\longrightarrow}L^0\overset{\Delta^0}{\longrightarrow}L^1\longrightarrow\cdots,$$

which is an injective coresolution, see [CE], IX, Cor. 2.7a.

In order to relate the given sequence E with the injective coresolution $Q^{\cdot}=(Q^{\cdot}, q^{\cdot})$, we define $u_{-1}=1_N$, and, inductively, we find $u_i: Y_{t-i}\to Q^i$ such that $g_{t-i}u_i=u_{i-1}q^{i-1}$, for $0\leqq i\leqq t$.

We are going to reformulate the previous lemma using the maps Δ^i and u_i. For $0\leqq r\leqq t-1$, let

$$\Omega'_r: S_r\longrightarrow L^{t-r-1}$$

be defined by

$$(a_0\otimes\cdots\otimes a_{r+1})\Omega'_r=(p_{-1}\cdot(a_0\otimes\cdots\otimes a_{r+1}(\Omega_{\gamma(r)}\cdot u_{t-r-1}, 0, \cdots, 0),$$

and similarly, let

$$\Omega'_{-1}: A \longrightarrow L^t$$

be defined by

$$(a)\Omega'_{-1}=(p_{-1}\bar{a}u_t, 0, \cdots, 0).$$

PROPOSITION. *For* $0\leqq r\leqq t-1$, *we have* $\nabla_{r-1}\Omega'_{r-1}=(-1)^r\Omega'_r\Delta^{t-r-1}$. *For* $r=t$, *we have* $\nabla_{t-1}\Omega'_{r-1}=(-1)^t\Omega_\gamma\Delta^{-1}$.

PROOF. For $0\leqq r\leqq t$, and $a_0, \cdots, a_{r+1}\in A$, we have

$$\begin{aligned}(a_0\otimes\cdots\otimes a_{r+1})\nabla_{r-1}\Omega'_{r-1}&=(p_{-1}(a_0\otimes\cdots\otimes a_{r+1})\nabla_{r-1}\Omega_{\gamma(r-1)}u_{t-r}, 0, \cdots, 0)\\&=(-1)^r(p_{-1}(a_0\otimes\cdots\otimes a_{r+1})\Omega_{\gamma(r)}g_r u_{t-r}, 0', \cdots, 0)\\&=(-1)^r(p_{-1}(a_0\otimes\cdots\otimes a_{r+1})\Omega_{\gamma(r)}u_{t-r-1}q^{t-r-1}, 0, \cdots, 0),\end{aligned}$$

using the definition of Ω'_{r-1}, the lemma, and the defining condition for u_{t-r}. On the other hand, for $0\leqq r\leqq t-1$, we have

$$\begin{aligned}(a_0\otimes\cdots\otimes a_{r+1})\Omega'_r\Delta^{t-r-1}&=(p_{-1}(a_0\otimes\cdots\otimes a_{r+1})\Omega_r u_{t-r-1}, 0, \cdots, 0)\Delta^{t-r-1}\\&=(p_{-1}(a_0\otimes\cdots\otimes a_{r+1})\Omega_r u_{t-r-1}q^{t-r-1}, 0, \cdots, 0)\end{aligned}$$

using the definitions of Ω'_r, Δ^{t-r-1}, and the fact that $p_0p_{-1}=0$. Similarly, for $r=t$, we have

$$\begin{aligned}(a_0\otimes\cdots\otimes a_{t+1})\Omega_\gamma\Delta^{-1}&=p_{-1}(a_0\otimes\cdots\otimes a_{t+1})\Omega_\gamma q^{-1}\\&=p_{-1}(a_0\otimes\cdots\otimes a_{t+1})\Omega_{\gamma(t)}u_{-1}q^{-1},\end{aligned}$$

since $\Omega_\gamma=\Omega_{\gamma(t)}$ and $u_{-1}=1$.

6. *Some homological algebra.* We will need some basic result of homological algebra which we want to review. We have chosen already a projective resolution of M, and an injective coresolution of N. In order to calculate $\mathrm{Ext}^t(M, N)$ we may use one of these sequences, or else the double complex $\mathrm{Hom}_A(P_i, Q^j)$. So let $R^t=\bigoplus_{i=0}^{t}\mathrm{Hom}_A(P_i, Q^{t-i})$, this is a subset of $L^t=\bigoplus_{i=0}^{t}\mathrm{Hom}_k(P_i, Q^{t-i})$, and let $\delta^t: R^t\to R^{t+1}$ be the restriction of Δ^t to R^t, similarly, let $\delta^{-1}: \mathrm{Hom}_A(M, N)\to L^0$ be the restriction of $\Delta^{-1}=\mathrm{Hom}(p_{-1}, q^{-1})$ to $\mathrm{Hom}_A(M, N)$. So we obtain a complex

$$R^{\cdot}=(R^0\xrightarrow{\delta^0}R^1\xrightarrow{\delta^1}R^2\longrightarrow\cdots),$$

which we want to compare with the complexes

$$\mathrm{Hom}_A(P_{\cdot}, N)\quad\text{and}\quad\mathrm{Hom}_A(M, Q^{\cdot}).$$

Note that there are maps

$$\mathrm{Hom}(1, q^{-1}) : \mathrm{Hom}_A(P., N) \longrightarrow R^{\cdot},$$

$$\mathrm{Hom}(p_{-1}, 1) : \mathrm{Hom}_A(M, Q^{\cdot}) \longrightarrow R^{\cdot},$$

and they are quasi-isomorphisms: they induce isomorphisms when passing to the cohomology ([B], § 5.2).

Consider now the given exact sequence E. Its equivalence class $[E]$ in $\mathrm{Ext}_A^t(M, N) = H^t(\mathrm{Hom}_A(P., N))$ is given by the cocylce $u_t : M \to Q_t$. Under the map $\mathrm{Hom}(p_{-1}, 1) : \mathrm{Hom}_A(M, Q^{\cdot}) \to R^{\cdot}$, the cocyle u_t is mapped onto the cocycle $(p_{-1}u_t, 0, \cdots, 0) \in \bigoplus_{i=0}^{t} \mathrm{Hom}_A(P_i, Q^{t-i}) = R^t$.

7. *Proof of the theorem.* We apply the previous considerations to the ring A^e (instead of A), and the A^e-modules A and $\mathrm{Hom}_k(M, N)$. For A, we use the standard resolution $S. = (S., \nabla.)$, for $\mathrm{Hom}_k(M, N)$, we use the injective coresolution $L^{\cdot} = (L^{\cdot}, \Delta^{\cdot})$. We form $C^t = \bigoplus_{i=0}^{t} \mathrm{Hom}_{A^e}(S_i, L^{t-i})$, with differential $D^t : C^t \to C^{t+1}$ given by

$$(\Phi_0, \cdots, \Phi_t)D^t = (\Phi_0\Delta^t, (-1)^{t+1}\nabla_0\Phi_0 + \Phi_1\Delta^{t-1}, \cdots,$$
$$(-1)^{t+1}\nabla_{t-1}\Phi_{t-1} + \Phi_t\Delta^0, (-1)^{t+1}\nabla_t\Phi_t),,$$

for $\Phi_i \in \mathrm{Hom}_{A^e}(S_i, L^{t-i})$. The maps

$$\mathrm{Hom}(1, \Delta^{-1}) : \mathrm{Hom}_{A^e}(S., \mathrm{Hom}_k(M, N)) \longrightarrow C^{\cdot}$$

and

$$\mathrm{Hom}(\nabla_{-1}, 1) : \mathrm{Hom}_{A^e}(A, L^{\cdot}) \longrightarrow C^{\cdot}$$

are quasi-isomorphisms. Clearly, we have an isomorphism

$$\rho : \mathrm{Hom}_{A^e}(A, L^{\cdot}) \longrightarrow R^{\cdot},$$

since for A-modules X, Y, the bimodule maps $\Sigma : A \to \mathrm{Hom}_k(X, Y)$ correspond bijectively to the elements of $\mathrm{Hom}_A(X, Y)$, with $(\Sigma)\rho = (1)\Sigma$.

It remains to chase elements via the various quasi-isomorphisms

$$\mathrm{Hom}_{A^e}(S., \mathrm{Hom}_k(M, N)) \xrightarrow{\mathrm{Hom}(1, \Delta^{-1})} C^{\cdot} \xleftarrow{\mathrm{Hom}(\nabla_{-1}, L)} \mathrm{Hom}_{A^e}(A, L^{\cdot}),$$

and

$$\mathrm{Hom}_A(M, Q^{\cdot}) \xrightarrow{\mathrm{Hom}(p_{-1}, 1)} R^{\cdot} \cong \mathrm{Hom}_{A^e}(A, L^{\cdot}).$$

The last map $\mathrm{Hom}(p_{-1}, 1)$ sends the cocycle u_t onto the element $(p_{-1}u_t, 0, \cdots, 0) \in R^t$, thus to Ω'_{-1} in $\mathrm{Hom}_{A^e}(A, L^t)$. So it remains to consider the elements

$$\Omega_\gamma\Delta^{-1}=(\Omega_\gamma)\,\mathrm{Hom}\,(1,\Delta^{-1}) \quad \text{and} \quad \nabla_{-1}\Omega'_{-1}=(\Omega'_{-1})\,\mathrm{Hom}\,(\nabla_{-1},1)$$

in C^t. Let $\varepsilon_{2i}=(-1)^i$, and $\varepsilon_{2i+1}=(-1)^{t+i}$, thus $\varepsilon_j=(-1)^{t+j+1}\varepsilon_{j-1}$, for all j. Let $\Phi_i=\varepsilon_i\Omega'_i$ for $0\leqq i\leqq t-1$, and $(\Psi_0,\cdots,\Psi_t):=(\Phi_0,\cdots,\Phi_{t-1})D^{t-1}$. Then

$$\Psi_0=\Phi_0\Delta^{t-1}=\varepsilon_0\Omega'_0\Delta^{t-1}=\nabla_{-1}\Omega'_{-1}$$

$$\Psi_t=\varepsilon_t\nabla_{t-1}\Phi_{t-1}=\varepsilon_t(-1)^t\Omega_\gamma\Delta^{-1},$$

whereas, for $1\leqq r\leqq t-1$,

$$\begin{aligned}\Psi_r&=(-1)^t\nabla_{r-1}\Phi_{r-1}+\Phi_r\Delta^{t-1-r}\\&=(-1)^t\varepsilon_{r-1}\nabla_{r-1}\Omega'_{r-1}+\varepsilon_r\Omega'_r\Delta^{t-1-r}\\&=(-1)^t\varepsilon_{r-1}(-1)^r\Omega'_t\Delta^{t-r-1}+(-1)^{t+r+1}\varepsilon_{r-1}\Omega'_r\Delta^{t-1-r}=0,\end{aligned}$$

always using the proposition. This shows that

$$(\nabla_{-1}\Omega'_{-1},0,\cdots,0,(-1)^t\varepsilon_t\Omega_\gamma\Delta^{-1})=(\Phi_0,\cdots,\Phi_{t-1})D^{t-1}$$

is a coboundary in $C^\cdot$, thus $\nabla_{-1}\Omega'_{-1}$ and $(-1)^{t+1}\varepsilon_t\Omega_\gamma\Delta^{-1}$ yield the same cohomology class in $H^t(C^\cdot)$.

Let us summerize: the composition of $H^t(\mathrm{Hom}(p_{-1},1))$, $H^t(p^{-1})$, $H^t(\mathrm{Hom}(\nabla_{-1,1}))$ and $H^t(\mathrm{Ham}\,(1,\Delta^{-1}))^{-1}$ yields a natural isomorphism

$$\eta_{MN}:\mathrm{Ext}^t_A(M,N)\longrightarrow H^t(A,\mathrm{Hom}_k(M,N))$$

and $\eta_{MN}([E])=(-1)^{t+1}\varepsilon_t[\Omega_\gamma]$, thus $\eta_{MN}([E])$ and $[\Omega_\gamma]$ are equal up to sign. This completes the proof.

REMARK. As the proof shows, the precise relation (under the given identification of $H^t(A,\mathrm{Hom}_k(M,N))$ and $\mathrm{Ext}^t_A(M,N)$) is

$$\eta_{MN}([E])=(-1)^{i+1}[\Omega_\gamma],$$

where i is the largest integer with $2i\leqq t$ (for $t=2i$, we have the sign $(-1)^{t+1}\varepsilon_{2i}=(-1)^{t+1+i}=(-1)^{i+1}$, for $t=2i+1$, we have $(-1)^{t+1}\varepsilon_{2i+1}=(-1)^{t+1}(-1)^{t+i}=(-1)^{i+1}$).

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V. Dlab
Department of Mathematics
Carleton University
Ottawa
Canada

C. M. Ringel
Fakultät für Mathematik
Universität
D-4800 Bielefeld 1
West-Germany