

ON ALMOST-PRIMES IN ARITHMETIC PROGRESSIONS

By

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1. Introduction.

Let P_r denote integers with at most r prime factors counted according to multiplicity. In 1975, Y. Motohashi [9] showed that there exists a P_3 such that

$$P_3 \equiv a \pmod{q}, \quad P_3 \ll q(\log q)^{70}$$

for any fixed non-zero integer a and almost-all moduli q with $(q, a) = 1$. His argument based upon the weighted linear sieve and the Brun-Titchmarsh theorem on average, which are due to H. E. Richert [10] and C. Hooley [4], respectively. H. Iwaniec's fundamental works [6, 7], therefore, suggest possibilities of an improvement upon the above result. In this paper we present an estimation for P_2 . We shall prove the following

THEOREM. *Let Q be a large parameter and a be any fixed integer, $0 < |a| \leq Q$. Then, except possibly for $O(Q/\log Q)$ moduli q with $(q, a) = 1$ and $Q < q \leq 2Q$, there exists a P_2 such that*

$$P_2 \equiv a \pmod{q}, \quad P_2 \leq \tau(a)q(\log q)^7$$

where the implied O -constant is absolute and τ denotes the divisor function.

Our proof of Theorem is performed by a simple modification of the argument in our previous paper [8], in which the dual problem is considered. In fact, the numerical work in the main term from sieve estimate is identical. Succeeding to Hooley's investigation [4] we treat the remainder terms with the same manner as in [8]. Our main lemma (see Lemma 1 below) is weaker than E. Fouvry's works [1-3] in its scope; however, it will be found to be suitable for an application to the weighted sieve.

We use the standard notation in number theory. Especially, \bar{r} , used in either r/s or congruence $(\text{mod } s)$, means that $\bar{r}r \equiv 1 \pmod{s}$. $\sum_{x=1}^y *$ stands for the summation with restriction $(x, y) = 1$. $n \sim N$ means $N \leq N_1 < n \leq N_2 \leq 2N$ for some N_1 and N_2 . ε denotes a small positive constant and the constants implied in the

symbols \ll and O may depend only on ε .

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2. Proof of Theorem.

Firstly we state the inequality for P_2 . This follows from Richert-Chen-Iwaniec's work, see [6]. For $q \leq x$ and $(q, a)=1$, put

$$\begin{aligned} \mathcal{A} &= \{n : n \leq x, n \equiv a \pmod{q}, (n, a)=1\}, \\ \mathcal{A}_d &= \{n : n \in \mathcal{A}, n \equiv 0 \pmod{d}\}, \\ \mathcal{P} &= \{p : p \nmid q\}, \quad \omega(d) = \begin{cases} 1, & (d, a)=1 \\ 0, & (d, a)>1, \end{cases} \end{aligned}$$

and

$$r(\mathcal{A}, d) = |\mathcal{A}_d| - \frac{\phi(a)}{a} \frac{x}{q} \frac{\omega(d)}{d}. \quad (2.1)$$

Let α, u, v be the parameters such that

$$\frac{1}{\alpha} < u < v, \quad \frac{2}{\alpha} \leq v \leq \frac{4}{u}, \quad u < 3. \quad (2.2)$$

Write

$$D = x^\alpha, \quad y = x^{1/u}, \quad z = x^{1/v}.$$

Then the following inequality is valid.

$$\begin{aligned} \Pi_2(x : q, a) &= |\{P_2 : P_2 \leq x, P_2 \equiv a \pmod{q}\}| \\ &> \prod_{\substack{p \leq z \\ p \nmid q}} \left(1 - \frac{\omega(p)}{p}\right) \frac{x}{q} \{C(\alpha, u, v) - E\} + \sum_{(d, q)=1} \lambda_d r(\mathcal{A}, d) - \sum_{z \leq p < y} \sum_{\substack{n \in \mathcal{A} \\ p^2 \mid n}} 1 \\ &= \Pi + E_2(q) + E_1(q), \quad \text{say,} \end{aligned} \quad (2.3)$$

where $C(\alpha, u, v)$ is some constant, E is a very small quantity and the sequence $(\lambda_d) = (\lambda_d(D))$ has the properties :

$$\lambda_d = 0 \quad \text{if } d \geq D.$$

$$|\lambda_d| \leq \mu^2(d).$$

and for any $M > y, N > 1, MN = D$,

$$\lambda_d = \sum_{l \leq (\log D)^2} \sum_{\substack{m \leq M \\ mn=d}} \sum_{\substack{n \leq N \\ mn=d}} a_m(l, M, N) b_n(l, M, N) \quad (2.4)$$

with $|a_m|, |b_n| \leq 1$.

We next choose $\alpha = 11/20 - 6\varepsilon, 1/u = 1/2 - 8\varepsilon, 1/v = \alpha/4 + 11/80 - 2\varepsilon$, then the

condition (2.2) is satisfied and, for sufficiently large x and small ε ,

$$ve^{-\tau}\{C(\alpha, u, v) - E\} > \frac{1}{200},$$

which is calculated in [8]. Hence

$$\Pi > \frac{1}{200} \frac{x}{q \log x}. \tag{2.5}$$

We turn to $E_2(q)$. We use the following lemma:

LEMMA 1. *Let $2Q < x$, $Q^{1/4}x^{-1/5} > x^\varepsilon$. If $M \leq Q^{1/2}x^{-4\varepsilon}$ and $N \leq Q^{1/4}x^{-1/5}$, then*

$$\sum_{\substack{Q < q \leq 2Q \\ (q, a) = 1}} \left| \sum_{(d, Q) = 1} \lambda_d r(\mathcal{A}, d) \right|^2 \ll \tau(a)x(\log x)^4 + x(a)^2Q + \frac{x^{2-\varepsilon/2}}{Q}.$$

We postpone the proof of lemma 1 until the final section. Now we set $x = \tau(a)Q(\log Q)^7$, $M = x^{1/2-5\varepsilon} > y = x^{1/u} = x^{1/2-8\varepsilon}$ and $N = x^{11/20-\varepsilon}$, then $M \leq Q^{1/2}x^{-4\varepsilon}$ and $N \leq Q^{1/4}x^{-1/5}$. Lemma 1 yields

$$\begin{aligned} \sum_{\substack{Q < q \leq 2Q \\ (q, a) = 1}} |E_2(q)|^2 &\ll \tau(a)x(\log x)^4 + \tau(a)^2Q + \frac{x^{2-\varepsilon/2}}{Q} \\ &\ll \left(\frac{x}{q}\right)^2 Q \frac{\tau(a)Q(\log Q)^4}{x} + \left(\frac{x}{q}\right)^2 Q^{1-\varepsilon/2} \\ &\ll \left(\frac{x}{q}\right)^2 Q(\log Q)^{-3}. \end{aligned} \tag{2.6}$$

Moreover we have

$$\begin{aligned} \sum_{\substack{Q < q \leq 2Q \\ (q, a) = 1}} |E_1(q)| &\leq \sum_q \sum_{x^{2\varepsilon} \leq p \leq x^{1/2}} \sum_{\substack{a < n \leq x \\ n \equiv a \pmod{q} \\ p^2 | n}} 1 \\ &\leq \sum_{x^{2\varepsilon} \leq p \leq x^{1/2}} \sum_{\substack{a < n \leq x \\ p^2 | n}} \tau(n-a) \\ &\ll x^\varepsilon \sum_{p \geq x^{2\varepsilon}} \frac{x}{p^2} \\ &\ll x^{1-\varepsilon} \ll \left(\frac{x}{Q}\right) Q(\log Q)^{-2}. \end{aligned} \tag{2.7}$$

In conjunction with (2.3), (2.5), (2.6) and (2.7) we obtain

$$\Pi_2(\tau(a)Q(\log Q)^7 : q, a) > \frac{3Cx}{q \log x} + E_1(q) + E_2(q) \tag{2.8}$$

where C is a positive absolute constant and $E_j(q)$ ($j=1, 2$) satisfy (2.7) and (2.6), respectively.

We proceed to the proof of Theorem. Put $x(t) = \tau(a)t(\log t)^7$ and $\mathcal{E} = \{q : Q < q \leq 2Q, (q, a) = 1, \Pi_2(x(q) : q, a) = 0\}$. We shall deduce $|\mathcal{E}| \ll Q(\log Q)^{-1}$,

from which our Theorem will follow. Now, by (2.8), we have

$$\Pi_2(x(q) : q, a) \geq \Pi_2(x(Q) : q, a) > \frac{3Cx}{q \log x} + E_1(q) + E_2(q).$$

For all $q \in \mathcal{E}$ we then see that

$$|E_1(q)| > \frac{Cx}{q \log x} \quad \text{or} \quad |E_2(q)| > \frac{Cx}{q \log x}.$$

Hence

$$\mathcal{E} \subset \mathcal{E}_1 \cup \mathcal{E}_2, \tag{2.9}$$

where $\mathcal{E}_j = \left\{ q : Q < q \leq 2Q, (q, a) = 1, |E_j(q)| > \frac{Cx}{q \log x} \right\}$. Furthermore,

$$|\mathcal{E}_j| \left(\frac{Cx}{2Q \log x} \right)^j < \sum_{q \in \mathcal{E}_j} |E_j(q)|^j \leq \sum_{\substack{Q < q \leq 2Q \\ (q, a) = 1}} |E_j(q)|^j.$$

Combining this with (2.6), (2.7) and (2.9) we get

$$\begin{aligned} |\mathcal{E}| &\leq \sum_{j=1,2} \left(\frac{Q \log Q}{x} \right)^j \cdot \left(\frac{x}{Q} \right)^j Q (\log Q)^{-j-1} \\ &\ll Q (\log Q)^{-1}, \end{aligned}$$

as required.

3. Proof of Lemma 1.

In this section we follow the argument of [4] with a minor modification. We use the following elementary lemma: *If $(c, d) = 1$, then*

$$\sum_{\substack{A < m \leq B \\ m \equiv a \pmod{d} \\ (m, c) = 1}} 1 = \frac{\phi(c)}{c} \frac{B-A}{d} + O(\tau(c)). \tag{3.1}$$

By the definition (2.1) of $r(\mathcal{A}, d)$ we have

$$\sum_{(d, q) = 1} \lambda_d r(\mathcal{A}, d) = \sum_{\substack{a < n \leq x \\ n \equiv a \pmod{q} \\ (n, a) = 1}} \left(\sum_{\substack{d | n \\ (d, q) = 1}} \lambda_d \right) - \frac{\phi(a)}{a} \frac{x-a}{q} \left(\sum_{(d, aq) = 1} \frac{\lambda_d}{d} \right).$$

By (3.1),

$$\begin{aligned} &\sum_{\substack{(q, a) = 1 \\ Q < q \leq 2Q}} \left| \sum_{(d, q) = 1} \lambda_d r(\mathcal{A}, d) \right|^2 \\ &\ll \sum_{\substack{(q, a) = 1 \\ Q < q \leq 2Q}} \left| \sum_{\substack{a < n \leq x \\ n \equiv a \pmod{q} \\ (n, a) = 1}} \left(\sum_{\substack{d | n \\ (d, q) = 1}} \lambda_d \right) - \left(\sum_{(d, aq) = 1} \frac{\lambda_d}{d} \right) \sum_{\substack{a < n \leq x \\ n \equiv a \pmod{q} \\ (n, a) = 1}} 1 \right|^2 + \tau(a)^2 Q \\ &= W - 2V + U + \tau(a)^2 Q, \quad \text{say.} \end{aligned} \tag{3.2}$$

Firstly we consider W .

$$\begin{aligned}
 W &= \sum_{\substack{(q, a)=1 \\ Q < q \leq 2Q}} \sum_{\substack{a < n_1 \leq x \\ n_1 \equiv a(q) \\ (n_1, a)=1}} \left(\sum_{\substack{d_1 | n_1 \\ (d_1, q)=1}} \lambda_{d_1} \right) \sum_{\substack{a < n_2 \leq x \\ n_2 \equiv a(q) \\ (n_2, a)=1}} \left(\sum_{\substack{d_2 | n_2 \\ (d_2, q)=1}} \lambda_{d_2} \right) \\
 &= \sum_{\substack{(q, a)=1 \\ Q < q \leq 2Q}} \sum_{(d_1 d_2, aq)=1} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{a < n_1 \leq x \\ n_1 \equiv a(q) \\ n_1 \equiv 0(d_1) \\ (n_1, a)=1}} 1 \sum_{\substack{a < n_2 \leq x \\ n_2 \equiv a(q) \\ n_2 \equiv 0(d_2) \\ (n_2, a)=1 \\ n_1 \neq n_2}} 1 + O\left(\sum_q \sum_{\substack{a < n \leq x \\ n \equiv a(q)}} \tau(n)^2 \right)
 \end{aligned}$$

We express the congruent condition $n_1 \equiv a(q), n_2 \equiv a(q)$ as $n_1 = a + ql_1, n_2 = a + ql_2$. Then, the condition for l_1 and l_2 is

$$l_1 \neq l_2 \leq \frac{x-a}{q}, \quad \begin{cases} a + ql_1 \equiv 0 \pmod{d_1} \\ a + ql_2 \equiv 0 \pmod{d_2} \end{cases}, \quad (l_1 l_2, a) = 1.$$

Since $(l_1 l_2, a) = 1, (l_1, d_1) = (l_2, d_2) = 1$. Changing the order of summation, we have

$$W = \sum_{(d_1 d_2, a)=1} \sum_{\substack{l_1 \neq l_2 \leq \frac{x-a}{Q} \\ (l_1 l_2, a)=1 \\ (l_1, d_1)=1 \\ (l_2, d_2)=1}} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{Q < q \leq \min(2Q, \frac{x-a}{l_1}, \frac{x-a}{l_2}) \\ q \equiv -al_1(d_1) \\ q \equiv -al_2(d_2) \\ (q, a d_1 d_2) = 1}} 1 + O\left(\sum_{a < n \leq x} \tau(n)^2 \tau(n-a) \right)$$

We write $L = (x-a)/Q$ and $Q' = Q'(l_1, l_2) = \min(2Q, (x-a)/l_1, (x-a)/l_2)$. The congruences are soluble iff $l_1 \equiv l_2 \pmod{(d_1, d_2)}$ and expressed as one congruence

$$q \equiv b \pmod{[d_1, d_2]}, \quad \begin{cases} b \equiv -al_1(d_1) \\ b \equiv -al_2(d_2^*), \quad d_2^* = d_2 / (d_1, d_2). \end{cases} \quad (3.3)$$

This congruence then absorbs the condition $(q, d_1 d_2) = 1$. Hence

$$W = \sum_{(d_1 d_2, a)=1} \sum_{\substack{l_1 \neq l_2 \leq L \\ l_1 \equiv l_2 \pmod{(d_1, d_2)} \\ (l_1 l_2, a)=1 \\ (l_1, d_1) = (l_2, d_2) = 1}} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{Q < q \leq Q' \\ q \equiv b \pmod{[d_1, d_2]} \\ (q, a) = 1}} 1 + O\left(\sum_{a < n \leq x} \tau_4(n) \tau(n-a) \right)$$

It is expected, by (3.1), that the innermost sum is approximately equal to

$$\frac{\phi(a)}{a} \frac{Q' - Q}{[d_1, d_2]}.$$

Thus,

$$W = W_0 + R + O(x(\log x)^4) \quad (3.4)$$

where

$$W_0 = \sum_{(d_1 d_2, a)=1} \sum_{\substack{l_1 \neq l_2 \leq L \\ l_1 \equiv l_2 \pmod{(d_1, d_2)} \\ (l_1, a d_1) = (l_2, a d_2) = 1}} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} \frac{\phi(a)}{a} \left(\min\left(2Q, \frac{x-a}{l_1}, \frac{x-a}{l_2}\right) - Q \right) \quad (3.5)$$

and

$$R = \sum_{(d_1 d_2, a)=1} \sum_{\substack{l_1 \neq l_2 \leq L \\ l_1 \equiv l_2 \pmod{(d_1, d_2)} \\ (l_1, a d_1) = (l_2, a d_2) = 1}} \lambda_{d_1} \lambda_{d_2} \left(\sum_{\substack{Q < q \leq Q' \\ q \equiv b \pmod{[d_1, d_2]} \\ (q, a) = 1}} 1 - \frac{\phi(a)}{a} \frac{Q' - Q}{[d_1, d_2]} \right). \quad (3.6)$$

In section 5 we shall estimate R . We proceed to consider W_0 . We perform the summation over l_1 and l_2 . Write

$$\sum_{l_1 \neq l_2} = \sum_{l_1 < l_2} + \sum_{l_1 > l_2}.$$

Then,

$$\sum_{l_1 < l_2} = \sum_{\substack{l_2 \leq L \\ (l_2, ad_2)=1}} \left(\min\left(2Q, \frac{x-a}{l_2}\right) - Q \right) \sum_{\substack{l_1 < l_2 \\ l_1 \equiv l_2 \pmod{(d_1, d_2)} \\ (l_1, ad_1)=1}} 1.$$

By (3.1), the inner sum is equal to

$$\begin{aligned} \sum_{\substack{l_1 < l_2 \\ l_1 \equiv l_2 \pmod{(d_1, d_2)} \\ (l_1, ad_1^*)=1}} 1 &= \frac{\phi(ad_1^*)}{ad_1^*} \frac{l_2}{(d_1, d_2)} + O(\tau(ad_1^*)) \\ &= \frac{1}{\phi((d_1, d_2))} \sum_{(l_1, ad_1)=1}^{l_1 < l_2} 1 + O(\tau(ad_1)), \end{aligned}$$

whence

$$\sum_{l_1 < l_2} = \frac{1}{\phi((d_1, d_2))} \sum_{\substack{l_2 \leq L \\ (l_2, ad_2)=1}} \left(\min\left(2Q, \frac{x-a}{l_1}\right) - Q \right) \sum_{\substack{l_1 < l_2 \\ (l_1, ad_1)=1}} 1 + O(x(ad_1)x) \quad (3.7)$$

Similarly,

$$\sum_{l_1 > l_2} = \frac{1}{\phi((d_1, d_2))} \sum_{\substack{l_1 \leq L \\ (l_1, ad_1)=1}} \left(\min\left(2Q, \frac{x-a}{l_1}\right) - Q \right) \sum_{\substack{l_2 < l_1 \\ (l_2, ad_2)=1}} 1 + O(\tau(ad_2)x). \quad (3.8)$$

In conjunction with (3.5), (3.7) and (3.8) we have

$$\begin{aligned} W_0 &= \sum_{(d_1 d_2, a)=1} \sum_{\substack{\lambda_{d_1} \lambda_{d_2} \phi(a) \\ d_1 d_2 a \phi((d_1, d_2))}} \sum_{(l_1, ad_1)=1} \sum_{\substack{l_1 \neq l_2 \leq L \\ (l_2, ad_2)=1}} \left(\min\left(2Q, \frac{x-a}{l_1}, \frac{x-a}{l_2}\right) - Q \right) \\ &\quad + O(\tau(a)x(\log x)^3) \\ &= W_1 + O(\tau(a)x(\log x)^3), \quad \text{say.} \end{aligned}$$

Combining this with (3.4) we get

$$W = W_1 + R + O(\tau(a)x(\log x)^4). \quad (3.9)$$

We turn to V .

$$\begin{aligned} V &= \sum_{\substack{(q, a)=1 \\ Q < q \leq 2Q}} \sum_{\substack{a < n_1 \leq x \\ n_1 \equiv a \pmod{q} \\ (n_1, a)=1}} \left(\sum_{\substack{d_1 | n_1 \\ (d_1, q)=1}} \lambda_{d_1} \right) \left(\sum_{(d_2, aq)=1} \frac{\lambda_{d_2}}{d_2} \right) \sum_{\substack{a < n_2 \leq x \\ n_2 \equiv a \pmod{q} \\ (n_2, a)=1}} 1 \\ &= \sum_{\substack{(q, a)=1 \\ Q < q \leq 2Q}} \sum_{(d_1 d_2, aq)=1} \lambda_{d_1} \frac{\lambda_{d_2}}{d_2} \sum_{\substack{a < n_1 \leq x \\ n_1 \equiv a \pmod{q} \\ n_1 \equiv 0 \pmod{d_1} \\ (n_1, a)=1}} 1 \sum_{\substack{a < n_2 \leq x \\ n_2 \equiv a \pmod{q} \\ (n_2, a)=1 \\ n_1 \neq n_2}} 1 + O((\log x) \sum_{a < n \leq x} \tau(n)\tau(n-a)). \end{aligned}$$

As before we write $n_1 = a + ql_1$ and $n_2 = a + ql_2$. Then the condition for l_1 and l_2 is

$$l_1 \neq l_2 \leq \frac{x-a}{q}, \quad a + ql_1 \equiv 0(d_1), \quad \begin{cases} (l_1 l_2, a) = 1 \\ (l_1, d_1) = 1 \end{cases}$$

Changing the order of summation we have

$$V = \sum_{(d_1 d_2, a)=1} \sum_{d_2} \lambda_{d_1} \frac{\lambda_{d_2}}{d_2} \sum_{\substack{l_1 \neq l_2 \leq L \\ (l_1 l_2, a)=1 \\ (l_1, d_1)=1}} \sum_{\substack{Q < q \leq Q' \\ q \equiv -a l_1 (d_1) \\ (q, a d_1 d_2)=1}} 1 + O(x(\log x)^3).$$

By (3.1) the innermost sum is equal to

$$\sum_{\substack{Q < q \leq Q' \\ q \equiv -a l_1 (d_1) \\ (q, a d_2^*)=1}} 1 = \frac{\phi(a d_2^*) (Q' - Q)}{a d_2^* d_1} + O(\tau(a d_2^*)),$$

whence

$$V = \sum_{(d_1 d_2, a)=1} \sum_{d_2} \frac{\lambda_{d_1} \lambda_{d_2} \phi(a d_2^*)}{d_1 d_2 a d_2^*} \sum_{\substack{l_1 \neq l_2 \leq L \\ (l_1 l_2, a)=1 \\ (l_1, d_1)=1}} (Q' - Q) + O\left(\tau(a) \left(\frac{x}{Q}\right)^2 D(\log x)^2 + x(\log x)^3\right) \tag{3.10}$$

Next we carry out the summation over l_1 and l_2 .

$$\begin{aligned} & \frac{\phi(d_2^*)}{d_2^*} \sum_{\substack{l_1 \neq l_2 \leq L \\ (l_1 l_2, a)=1 \\ (l_1, d_1)=1}} \left(\min\left(2Q, \frac{x-a}{l_1}, \frac{x-a}{l_2}\right) - Q \right) \\ &= \sum_{\substack{l_2 \leq L \\ (l_2, a)=1}} \left(\min\left(2Q, \frac{x-a}{l_2}\right) - Q \right) \frac{\phi(d_2^*)}{d_2^*} \sum_{\substack{l_1 < l_2 \\ (l_1, a d_1)=1}} 1 \\ & \quad + \sum_{\substack{l_1 \leq L \\ (l_1, a d_1)=1}} \left(\min\left(2Q, \frac{x-a}{l_1}\right) - Q \right) \frac{\phi(d_2^*)}{d_2^*} \sum_{\substack{l_2 < l_1 \\ (l_1, a)=1}} 1. \end{aligned} \tag{3.11}$$

(3.1) yields

$$\frac{\phi(d_2^*)}{d_2^*} \sum_{\substack{l_1 < l_2 \\ (l_1, a d_1)=1}} 1 = \frac{\phi([d_1, d_2])}{[d_1, d_2]} \sum_{\substack{l_1 < l_2 \\ (l_1, a)=1}} 1 + O(\tau(a d_1))$$

and

$$\frac{\phi(d_2^*)}{d_2^*} \sum_{\substack{l_2 < l_1 \\ (l_2, a)=1}} 1 = \frac{(d_1, d_2)}{\phi((d_1, d_2))} \sum_{\substack{l_2 < l_1 \\ (l_2, a d_1)=1}} 1 + O(\tau(a d_1) \log x).$$

Thus (3.11) is equal to

$$\frac{\phi([d_1, d_2])}{[d_1, d_2]} \sum_{\substack{l_1 < l_2 \leq L \\ (l_1 l_2, a)=1}} (Q' - Q) + \frac{(d_1, d_2)}{\phi((d_1, d_2))} \sum_{\substack{l_2 < l_1 \leq L \\ (l_1, a d_1)=1 \\ (l_2, a d_2)=1}} (Q' - Q) + O(\tau(a d_1) \log x)$$

Combining this with (3.10) we have

$$\begin{aligned}
 V = & \sum_{(d_1 d_2, a)=1} \frac{\lambda_{d_1} \lambda_{d_2}}{d_1 d_2} \frac{\phi(a)}{a} \cdot \left(\frac{\phi([d_1, d_2])}{[d_1, d_2]} \sum_{\substack{l_1 < l_2 \leq L \\ (l_1 l_2, a)=1}} (Q' - Q) \right) \\
 & + \frac{(d_1, d_2)}{\phi((d_1, d_2))} \sum_{\substack{l_2 < l_1 \leq L \\ (l_1, a d_1)=1 \\ (l_2, a d_2)=1}} (Q' - Q) \\
 & + O\left(\tau(a)x(\log x)^4 + \tau(a)\left(\frac{x}{Q}\right)^2 D(\log x)^2\right) \tag{3.12}
 \end{aligned}$$

Interchanging the role of (d_1, l_1) with that of (d_2, l_2) , we may obtain the corresponding expression to (3.12). Hence

$$2V = U_1 + W_1 + O\left(\tau(a)x(\log x)^4 + \tau(a)\left(\frac{x}{Q}\right)^2 D(\log x)^2\right) \tag{3.13}$$

where

$$U_1 = \sum_{(d_1 d_2, a)=1} \sum_{\substack{l_1 \neq l_2 \leq L \\ (l_1 l_2, a)=1}} \frac{\lambda_{d_1} \lambda_{d_2}}{d_1 d_2} \frac{\phi(a[d_1, d_2])}{a[d_1, d_2]} (Q' - Q).$$

Finally we consider U . By the same argument as above, we have

$$\begin{aligned}
 U = & \sum_{\substack{(q, a)=1 \\ Q < q \leq 2Q}} \left(\sum_{(d_1, aq)=1} \frac{\lambda_{d_1}}{d_1} \right) \left(\sum_{(d_2, aq)=1} \frac{\lambda_{d_2}}{d_2} \right) \sum_{\substack{a < n_1 \leq x \\ n_1 \equiv a \pmod{q} \\ (n_1, a)=1}} 1 \sum_{\substack{a < n_2 \leq x \\ n_2 \equiv a \pmod{q} \\ (n_2, a)=1}} 1 \\
 = & \sum_{\substack{(q, a)=1 \\ Q < q \leq 2Q}} \left(\sum_{(d_1, aq)=1} \frac{\lambda_{d_1}}{d_1} \right) \left(\sum_{(d_2, aq)=1} \frac{\lambda_{d_2}}{d_2} \right) \sum_{\substack{l_1 \neq l_2 \leq (x-a)/q \\ (l_1 l_2, a)=1}} 1 + O((\log x)^2 \sum_{a < n \leq x} \tau(n-a)) \\
 = & \sum_{(d_1 d_2, a)=1} \sum_{\substack{l_1 \neq l_2 \leq L \\ (l_1 l_2, a)=1}} \frac{\lambda_{d_1} \lambda_{d_2}}{d_1 d_2} \sum_{\substack{Q < q \leq Q' \\ (q, a[d_1, d_2])=1}} 1 + O(x(\log x)^3) \\
 = & \sum_{(d_1 d_2, a)=1} \sum_{\substack{l_1 \neq l_2 \leq L \\ (l_1 l_2, a)=1}} \frac{\lambda_{d_1} \lambda_{d_2}}{d_1 d_2} \left(\frac{\phi(a[d_1, d_2])}{a[d_1, d_2]} (Q' - Q) + O(\tau(a[d_1, d_2])) \right) \\
 & + O(x(\log x)^3) \\
 = & U_1 + O\left(\tau(a)\left(\frac{x}{Q}\right)^2 (\log x)^4 + x(\log x)^3\right) \tag{3.14}
 \end{aligned}$$

In conjunction with (3.2), (3.9), (3.13) and (3.14) we get

$$\begin{aligned}
 \sum_{\substack{Q < q \leq 2Q \\ (q, a)=1}} \left| \sum_{(d, q)=1} \lambda_d r(\mathcal{A}, d) \right|^2 & \ll W_1 + R - (U_1 + W_1) + U_1 + \tau(a)x(\log x)^4 \\
 & + \tau(a)\left(\frac{x}{Q}\right)^2 D(\log x)^2 + \tau(a)^2 Q \\
 & \leq R + \tau(a)x(\log x)^4 + \tau(a)^2 Q + \left(\frac{x}{Q}\right)^2 D x^\epsilon. \tag{3.15}
 \end{aligned}$$

4. Auxiliary results.

In the next section we use the following lemmas, see [5].

LEMMA 2. Let $\phi(t)=[t]-t+1/2$. For $H>2$, we have

$$\phi(t)=\sum_{0<|h|\leq H}\frac{e(ht)}{2\pi ih}+O\left(\min\left(1,\frac{1}{H\|t\|}\right)\right)$$

where $e(x)=e^{2\pi ix}$ and $\|x\|=\min_{n\in\mathbb{Z}}|x-n|$. Moreover,

$$\min\left(1,\frac{1}{H\|t\|}\right)=\sum_{n\in\mathbb{Z}}C_n e(ht),$$

with

$$|C_h|\ll\min\left(\frac{\log H}{H},\frac{H}{h^2}\right).$$

LEMMA 3. We have, for any $\varepsilon>0$,

$$\sum_{\substack{m\sim M \\ m\equiv x \pmod{y} \\ (m,cd)=1}}e\left(l\frac{\bar{m}}{d}\right)\ll\tau(c)(l,d)^{1/2}d^{1/2+\varepsilon}\left(1+\frac{M}{d}\right).$$

5. Proof of Lemma 1, continued.

In this section we shall estimate R by appealing to lemmas 3 and 4. We begin with treating the condition $(q, a)=1$ by Moebius function.

$$\begin{aligned} R &= \sum_{\substack{l_1\neq l_2\leq L \\ (l_1l_2, a)=1}} \sum_{\substack{(d_1, d_2)|l_1-l_2 \\ (d_1, al_1)=1 \\ (d_2, al_2)=1}} \lambda_{d_1}\lambda_{d_2} \sum_{e|a} \mu(e) \left(\sum_{\substack{Q<q\leq Q' \\ q\equiv b \pmod{[d_1, d_2]} \\ q=O(e)}} 1 - \frac{Q'-Q}{e[d_1, d_2]} \right) \\ &= \sum_{e|a} \mu(a) \sum_{\substack{l_1\neq l_2\leq L \\ (l_1l_2, a)=1}} \sum_{\substack{(d_1, d_2)|l_1-l_2 \\ (d_1, al_1)=1 \\ (d_2, al_2)=1}} \lambda_{d_1}\lambda_{d_2} \\ &\quad \times \left\{ \phi\left(\frac{Q'}{e[d_1, d_2]} - \frac{b\bar{e}}{[d_1, d_2]}\right) - \phi\left(\frac{Q}{e[d_1, d_2]} - \frac{b\bar{e}}{[d_1, d_2]}\right) \right\}. \end{aligned} \tag{5.1}$$

We consider the argument in ϕ -function. By the definition (3.3) of $b \pmod{[d_1, d_2]}$ and the relation, for $(m, n)=1$,

$$\frac{\bar{m}}{n} + \frac{\bar{n}}{m} \equiv \frac{1}{mn} \pmod{1}, \tag{5.2}$$

we have

$$\begin{aligned}
-\frac{b\bar{e}}{[d_1, d_2]} &\equiv -\frac{b\overline{d_2^*e}}{d_1} - \frac{b\overline{d_1e}}{d_2^*} \\
&\equiv \frac{al_1\overline{d_2^*e}}{d_1} + \frac{al_2\overline{d_1e}}{d_2^*} \\
&\equiv \frac{a}{e} \left(\frac{\overline{d_2^*l_1}}{d_1} + \frac{\overline{d_1l_2}}{d_2^*} \right) \\
&\equiv \frac{a}{e} \left(\frac{1}{d_1d_2^*l_1} - \frac{\bar{d}_1}{d_2^*l_1} + \frac{\overline{d_1l_2}}{d_2^*} \right) \pmod{1}. \tag{5.3}
\end{aligned}$$

We write $l_1^* = l_1/(l_1, l_2)$, $l_2^* = l_2/(l_1, l_2)$. When

$$\begin{aligned}
(l_1, l_2) &= p_1^{a_1} \cdots p_r^{a_r} \\
l_1^* &= p_1^{b_1} \cdots p_r^{b_r} p_{r+1}^{a_{r+1}} \cdots p_s^{a_s} \\
p_i &\neq p_j \quad (i \neq j), \quad a_i \geq 1, \quad b_i \geq 0,
\end{aligned}$$

we define

$$\begin{aligned}
l_1^{**} &= p_{r+1}^{a_{r+1}} \cdots p_s^{a_s} \\
l_1^\# &= p_1^{a_1+b_1} \cdots p_r^{a_r+b_r}.
\end{aligned}$$

Then it follows $(l_1^{**}, l_1^\#) = 1$ and $(d_1d_2^*, l_1^\#) = 1$. By (5.2),

$$\frac{\bar{d}_1}{d_2^*l_1} \equiv \frac{\bar{d}_1}{d_2^*l_1^{**} \cdot l_1^\#} \equiv \frac{\overline{d_1l_1^\#}}{d_2^*l_1^{**}} + \frac{\overline{d_1d_2^*l_1^{**}}}{l_1^\#} \pmod{1}.$$

Moreover,

$$-\frac{\overline{d_1l_1^\#}}{d_2^*l_1^{**}} + \frac{\overline{d_1l_2}}{d_2^*} \equiv \frac{k}{d_2^*l_1^{**}} \pmod{1}$$

where $k = -\overline{d_1l_1^\#} + \widehat{d_1l_2l_1^{**}}$ with $\widehat{d_1l_2d_1l_2} \equiv 1 \pmod{d_2^*}$. We then have, with a certain integer n ,

$$\begin{aligned}
d_1l_1^\#l_2^*k &= -\overline{d_1l_1^\#}d_1l_1^\#l_2^* + \widehat{d_1l_2d_1l_2}l_1^\#l_1^{**} \\
&\equiv -l_2^* + \widehat{d_1l_2d_1l_2}l_1^* \\
&\equiv -l_2^* + (1 + nd_2^*)l_1^* \\
&\equiv l_1^* - l_2^* \pmod{d_2^*l_1^{**}}.
\end{aligned}$$

Since $(d_1l_1^\#l_2^*, d_2^*l_1^{**}) = 1$,

$$k \equiv (l_1^* - l_2^*)\overline{d_1l_1^\#l_2^*} \pmod{d_2^*l_1^{**}}.$$

The condition $(d_1, d_2) | l_1 - l_2$ implies $(d_1, d_2) | l_1^* - l_2^*$ since $((d_1, d_2), l_1l_2) = 1$, whence

$$k \equiv \frac{(l_1^* - l_2^*)}{(d_1, d_2)} \overline{d_1^\#l_1^\#l_2^*} \pmod{d_2^*l_1^{**}},$$

or

$$-\frac{\overline{d_1 l_1^*}}{d_2^* l_1^{**}} + \frac{\overline{d_1 l_2}}{d_2^*} \equiv \frac{l_1^* - l_2^*}{(d_1, d_2)} \frac{\overline{d_1^* l_1^* l_2^*}}{d_2^* l_1^{**}} \pmod{1}. \quad (5.5)$$

In conjunction with (5.3), (5.4) and (5.5) we finally obtain

$$\begin{aligned} -\frac{b\bar{e}}{[d_1, d_2]} &\equiv \frac{a}{e} \left(\frac{1}{d_1 d_2^* l_1} - \frac{\bar{d}_1}{d_2^* l_1} + \frac{\overline{d_1 l_2}}{d_2^*} \right) \\ &\equiv \frac{a}{e} \left(\frac{1}{d_1 d_2^* l_1} - \frac{\overline{d_1 l_1^*}}{d_2 l_1^{**}} - \frac{\overline{d_1 d_2^* l_1^{**}}}{l_1^*} + \frac{\overline{d_1 l_2}}{d_2^*} \right) \\ &\equiv \frac{a}{e} \left(\frac{1}{d_1 d_2^* l_1} - \frac{\overline{d_1 d_2^* l_1^{**}}}{l_1^*} + \frac{l_1^* - l_2^*}{(d_1, d_2)} \frac{\overline{d_1^* l_1^* l_2^*}}{d_2^* l_1^{**}} \right) \pmod{1} \\ &= \frac{a/e}{d_1 d_2^* l_1} + \theta\left(\frac{d_1^*}{d_2^*}\right), \quad \text{say.} \end{aligned} \quad (5.6)$$

We write

$$\begin{aligned} a = ef, \quad (d_1, d_2) = \delta, \quad d_j^* = \nu_j, \quad \frac{l_1^* - l_2^*}{\delta} = l \\ \frac{fQ'}{a\delta} + \frac{f}{\delta l_1} = \xi_2, \quad \frac{fQ}{a\delta} + \frac{f}{\delta l_1} = \xi_1. \end{aligned} \quad (5.7)$$

By (5.6) and (5.7), (5.1) is reduced to the following :

$$R = \sum_{f|a} \mu\left(\frac{a}{j}\right) \sum_{\substack{l_1^* l_2^* \leq L \\ (l_1^*, a) = 1}} \sum_{\substack{\delta | l_1^* - l_2^* \\ (\delta, a l_1^* l_2^*) = 1}} R_1(f, l_1, l_2, \delta)$$

where

$$\begin{aligned} R_1 &= R_1(f, l_1, l_2, \delta) \\ &= \sum_{\substack{(\nu_1, \nu_2) = 1 \\ (\nu_1, a l_1) = 1 \\ (\nu_2, a l_2) = 1}} \lambda_{\delta \nu_1} \lambda_{\delta \nu_2} \left\{ \psi\left(\frac{\xi_2}{\nu_1 \nu_2} + \theta\left(\frac{\nu_1}{\nu_2}\right)\right) - \psi\left(\frac{\xi_1}{\nu_1 \nu_2} + \theta\left(\frac{\nu_1}{\nu_2}\right)\right) \right\} \end{aligned} \quad (5.8)$$

with

$$\begin{aligned} \theta\left(\frac{\nu_1}{\nu_2}\right) &= \theta\left(\frac{\nu_1}{\nu_2} : f, l_1, l_2, \delta\right) \\ &= -f \frac{\overline{\delta \nu_1 \nu_2 l_1^{**}}}{l_1^*} + fl \frac{\overline{\nu_1 l_1^* l_2^*}}{\nu_2 l_1^{**}}. \end{aligned} \quad (5.9)$$

and ξ_j 's are defined by (5.7).

Next we decompose $(\lambda_{\delta \nu})$. Since $\mu^2(\delta \nu) = 1$, we may write

$$\lambda_{\delta \nu} = \sum_{k \leq (\log D)^2} \sum_{\substack{bd = \delta \\ bm \leq M}} \sum_{\substack{mn = \nu \\ dn \leq N \\ \mu^2(n) = 1}} a_{bm}(k, M, N) b_{dn}(k, M, N).$$

Thus,

$$R_1 \ll (\log D)^4 \tau(\delta)^2 \sum_{\substack{M_1 \\ M_1, M_2 \leq M \\ M_1, N_2 \leq M}} \sum_{N_1} \sum_{M_2} \sum_{N_2} \sup |R_2|$$

where M_1, N_1, M_2, N_2 's run through powers of 2;

$$\begin{aligned} R_2 &= R_2(M_1, N_1, M_2, N_2; \alpha_1, \beta_1, \alpha_2, \beta_2) \\ &= \sum_{\substack{m_1 \sim M_1 \\ n_1 \sim N_1 \\ (m_1 n_1, m_2 n_2) = 1 \\ (m_1 n_1, a l_1) = (n_2 n_2, a l_2) = 1 \\ \mu^2(n_1 n_2) = 1}} \sum_{\substack{m_2 \sim M_2 \\ n_2 \sim N_2}} \alpha_1(m_1) \beta_1(n_1) \alpha_2(m_2) \beta_2(n_2) \left\{ \phi\left(\frac{\xi_2}{m_1 n_1 m_2 n_2} + \theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right. \\ &\quad \left. - \phi\left(\frac{\xi_1}{m_1 n_1 m_2 n_2} + \theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right\}; \quad (5.10) \end{aligned}$$

and the supremum is taken over all sequences $(\alpha_1), (\beta_1), (\alpha_2), (\beta_2)$ such that $|\alpha_1|, |\beta_1|, |\alpha_2|, |\beta_2| \leq 1$. Moreover

$$R_1 \ll (\log D)^8 \tau(\delta)^2 \left\{ \sup_{\substack{x_1, x_2, \beta_1, \beta_2 \\ M_1 N_1 M_2 N_2 > Q x^{-2\epsilon}}} |R_2| + Q x^{-2\epsilon} \right\}. \quad (5.11)$$

We shall show

$$\sup |R_2| \ll Q x^{-3\epsilon/2}$$

from which Lemma 1 follows. Actually, by (5.9), we then have

$$\begin{aligned} R &\ll \sum_{f|a} \sum_{l_1 \neq l_2 \leq L} \sum_{\delta | l_1 - l_2} (\log D)^8 \tau(\delta)^2 Q x^{-3\epsilon/2} \\ &\ll \left(\frac{x}{Q}\right)^2 x^\epsilon Q x^{-3\epsilon/2} \\ &\ll \frac{x^{2-3\epsilon/2}}{Q}. \end{aligned}$$

Combining this with (3.15) we get Lemma 1.

Now we estimate R_2 defined by (5.10). By Lemma 2 we have

$$R_2 = R_3 + R_4$$

where

$$R_3 = \sum_{\substack{m_1 \\ (m_1 n_1, m_2 n_2) = 1 \\ (m_1 n_1, a l_1) = (n_2 n_2, a l_2) = 1 \\ \mu^2(n_1 n_2) = 1}} \sum_{n_1} \sum_{m_2} \sum_{n_2} \frac{\alpha_1(m_1) \beta_1(n_1) \alpha_2(m_2) \beta_2(n_2)}{m_1 n_1 m_2 n_2} \sum_{0 < |h| \leq H} e\left(h \theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \int_{\xi_1}^{\xi_2} e\left(\frac{ht}{m_1 n_1 m_2 n_2}\right) dt, \quad (5.12)$$

$$R_4 \ll \sum_{j=1,2} \sum_{\substack{m_1 \\ (n_1 n_1, n_2 n_2) = 1 \\ (m_1 n_1, a l_1) = 1 \\ (m_2 n_2, a l_2) = 1}} \sum_{n_1} \sum_{m_2} \sum_{n_2} \min\left(1, \frac{1}{H \left\| \left(\frac{\xi_j}{m_1 n_1 m_2 n_2} + \theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right\|}\right).$$

Firstly we consider R_4 . By Lemma 2

$$\begin{aligned}
 R_4 &\ll x^\varepsilon \sum_{j=1,2} \sum_{\substack{k \sim M_1 N_1 \\ (k,r)=1}} \sum_{\substack{r \sim M_2 N_2 \\ (k,al_1)=(r,al_2)=1}} \min \left(1, \frac{1}{H \left\| \left(\frac{\xi_j}{kr} + \theta \left(\frac{k}{r} \right) \right) \right\|} \right) \\
 &\ll x^\varepsilon \sum_{j=1,2} \sum_{h \in \mathbb{Z}} |C_h| \left| \sum_k \sum_r e \left(\frac{h \xi_j}{kr} \right) e \left(h \theta \left(\frac{k}{r} \right) \right) \right| \\
 &= x^\varepsilon \sum_{j=1,2} \sum_{h \in \mathbb{Z}} |C_h| |S(h)|, \quad \text{say.} \tag{5.13}
 \end{aligned}$$

We proceed to estimate $S(h)$. Trivially,

$$S(h) \ll M_1 N_1 M_2 N_2. \tag{5.14}$$

For $h \neq 0$, we get, by partial summation and the definition (5.9) of θ ,

$$\begin{aligned}
 S(h) &\ll \sum_{\substack{r \sim M_2 N_2 \\ (r,al_2)=1}} \left(1 + \frac{h \xi_j}{M_1 N_1 r} \right) \left| \sum_{\substack{k \sim M_1 N_1 \\ (k,ral_1)=1}} e \left(h \theta \left(\frac{k}{r} \right) \right) \right| \\
 &\ll \left(1 + \frac{hQ}{\delta M_1 N_1 M_2 N_2} \right) \sum_{\substack{r \sim M_2 N_2 \\ (r,al_2)=1}} \left| \sum_{b=1}^{l_1^*} e \left(-h f \frac{\delta k r l_1^{**}}{l_1^*} \right) \sum_{\substack{k \sim M_1 N_1 \\ k=b(l_1^*) \\ (k,ral_1^{**})=1}} e \left(h f l_1 \frac{k l_1^* l_2^*}{r l_1^{**}} \right) \right|
 \end{aligned}$$

Lemma 3 yields

$$\begin{aligned}
 S(h) &\ll \left(1 + \frac{hQ}{M_1 N_1 M_2 N_2} \right) \sum_{\substack{r \sim M_2 N_2 \\ (r,al_2)=1}} \sum_{b=1}^{l_1^*} \left| \sum_{\substack{k \sim M_2 N_2 \\ k=b(l_1^*) \\ (k,ar l_1^{**})=1}} e \left(h f l_1 \frac{k l_1^* l_2^*}{r l_1^{**}} \right) \right| \\
 &\ll x^{\varepsilon/2} \left(1 + \frac{hQ}{M_1 N_1 M_2 N_2} \right) \sum_{(r,al_2)=1} l_1^* \tau(a) (h f l_1, r l_1^{**})^{1/2} (r l_1^{**})^{1/2} \left(1 + \frac{M_1 N_1}{r l_1^{**}} \right) \\
 &\ll \tau(a) x^{\varepsilon/2} \left(1 + \frac{hQ}{M_1 N_1 M_2 N_2} \right) l_1 \sum_r (h l_1, r)^{1/2} \left(r^{1/2} + \frac{M_1 N_1}{r^{1/2}} \right) \\
 &\ll \tau(a) x^{\varepsilon/2} \left(1 + \frac{hQ}{M_1 N_1 M_2 N_2} \right) l_1 \left(\sum_r \frac{(h l_1, r)}{r} \right)^{1/2} \left\{ \left(\sum_r r^2 \right)^{1/2} + M_1 N_1 \left(\sum_r 1 \right)^{1/2} \right\} \\
 &\ll \tau(a) x^\varepsilon \left(1 + \frac{hQ}{M_1 N_1 M_2 N_2} \right) l_1 \tau(h) \{ (M_2 N_2)^{3/2} + M_1 N_1 (M_2 N_2)^{1/2} \}, \tag{5.15}
 \end{aligned}$$

since

$$\sum_{m \sim M} \frac{(A, m)}{m} \ll \tau(A). \tag{5.16}$$

In conjunction with (5.13), (5.14), (5.15) and Lemma 2, we obtain

$$\begin{aligned}
 R_4 &\ll x^\varepsilon M_1 N_1 M_2 N_2 (|C_0| + \sum_{|h| > H M_2 N_2} |C_h|) \\
 &\quad + \tau(a) x^{2\varepsilon} l_1 \{ (M_2 N_2)^{3/2} + M_1 N_1 (M_2 N_2)^{1/2} \} \sum_{0 < |h| \leq H M_2 N_2} |C_h| \tau(h) \left(1 + \frac{hQ}{M_1 N_1 M_2 N_2} \right)
 \end{aligned}$$

$$\begin{aligned}
&\ll x^\varepsilon M_1 N_1 M_2 N_2 \left(\frac{\log H}{H} + \sum_{h > H M_2 N_2} \frac{H}{h^2} \right) + \tau(a) x^{2\varepsilon} l_1 \{ (M_2 N_2)^{3/2} + M_1 N_1 (M_2 N_2)^{1/2} \} \\
&\quad \cdot \sum_{0 < h \leq H} \frac{\log H}{H} \tau(h) \left(1 + \frac{HQ}{M_1 N_1 M_2 N_2} \right) + \sum_{H < h \leq H M_2 N_2} H \tau(h) \left(\frac{1}{h^2} + \frac{Q}{h M_1 N_1 M_2 N_2} \right) \} \\
&\ll x^\varepsilon M_1 N_1 M_2 N_2 \left(\frac{\log H}{H} + \frac{1}{M_2 N_2} \right) \\
&\quad + \tau(a) x^{2\varepsilon} l_1 \{ (M_2 N_2)^{3/2} + M_1 N_1 (M_2 N_2)^{1/2} \} (\log H)^2 \left(1 + \frac{HQ}{M_1 N_1 M_2 N_2} \right).
\end{aligned}$$

Now, we choose

$$H = \frac{M_1 N_1 M_2 N_2 X^{4\varepsilon}}{Q},$$

then $H > 2$ since $M_1 N_1 M_2 N_2 > Q x^{-2\varepsilon}$ in (5.11). Thus,

$$R_4 \ll x^\varepsilon (Q x^{-3\varepsilon} + M_1 N_1) + \tau(a) x^{7\varepsilon} l_1 \{ (M_2 N_2)^{3/2} + M_1 N_1 (M_2 N_2)^{1/2} \}$$

or

$$\begin{aligned}
\sup |R_4| &\ll Q x^{-2\varepsilon} + \tau(a) x^{7\varepsilon} \left(\frac{x}{Q} \right) (MN)^{3/2} \\
&\ll Q x^{-2\varepsilon} + x^{8\varepsilon} \left(\frac{x}{Q} \right) (Q^{3/4} x^{-1/5-4\varepsilon})^{3/2} \\
&\ll Q x^{-2\varepsilon} + Q x^{2\varepsilon} \left(\frac{x^{1/5}}{Q^{1/4}} \right)^{7/2} \\
&\ll Q x^{-3\varepsilon/2}. \tag{5.17}
\end{aligned}$$

We turn to R_3 defined by (5.12).

$$\begin{aligned}
R_3 &= \frac{1}{N_1 N_2} \sum_{\substack{m_1 \\ (m_1, a l_1)=1 \\ (m_2, a l_2)=1}} \sum_{\substack{m_2 \\ (m_1, m_2)=1}} \alpha_1(m_1) \alpha_2(m_2) \int_{\xi_1/m_1 m_2}^{\xi_2/m_1 m_2} \sum_{0 < |h| \leq H} \sum_{\substack{n_1 \\ (m_1 n_1, m_2 n_2)=1 \\ (n_1, a l_1)=(n_2, a l_2)=1 \\ \mu^2(n_1 n_2)=1}} \sum_{n_2} \beta_1(n_1) \beta_2(n_2) \\
&\quad \cdot \frac{N_1 N_2}{n_1 n_2} e\left(\frac{ht}{n_1 n_2}\right) e\left(h\theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) dt \\
&\ll \frac{1}{N_1 N_2} \sum_{\substack{m_1 \\ (m_1, m_2)=1 \\ (m_1, l_1)=1 \\ (m_2, f l_2)=1}} \sum_{m_2} \int_0^{4Q/\delta M_1 M_2} \left| \sum_{0 < h \leq H} \sum_{\substack{n_1 \\ (m_1 n_1, m_2 n_2)=1 \\ (n_1, a l_1)=(n_2, a l_2)=1}} \sum_{n_2} \beta_1(n_1) \beta_2(n_2) \right. \\
&\quad \left. \cdot \frac{N_1 N_2}{n_1 n_2} e\left(\frac{ht}{n_1 n_2}\right) \cdot e\left(h\theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right| dt \\
&\ll \frac{Q}{\delta M_1 N_1 M_2 N_2} \sup \sum_{m_1} \sum_{m_2} \left| \sum_h \sum_{n_1} \sum_{n_2} c(h, n_1, n_2) e\left(h\theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right| \tag{5.18}
\end{aligned}$$

where the supremum is taken over all sequences $|c(h, n_1, n_2)| \leq 1$.

In the next section we shall prove the following :

LEMMA 4. For any sequence (c) with $|c| < 1$, we have

$$S = \sum_{\substack{m_1 \sim M_1 \\ (m_1, m_2)=1 \\ (m_1, l_1)=1 \\ (m_2, fl_2)=1}} \sum_{m_2 \sim M_2} \left| \sum_{0 < h \leq H} \sum_{\substack{n_1 \sim N_1 \\ (m_1 n_1, m_2 n_2)=1 \\ (n_1, al_1)=(n_2, al_2)=1 \\ \mu^2(n_1 n_2)=1}} \sum_{n_2 \sim N_2} c(h, n_1, n_2) e\left(h\theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right|^2 \\ \ll x^\varepsilon H M_1 N_1 M_2 N_2 + x^\varepsilon l_1 H^2 (M_1 + M_2) M_2^{1/2} N_1^2 N_2^3.$$

We apply Lemma 4 to (5.18).

$$R_3 \ll \frac{Q}{\delta M_1 N_1 M_2 N_2} \sup (M_1 M_2)^{1/2} (S)^{1/2} \\ \ll \frac{x^{4\varepsilon}}{H} \{x^\varepsilon H (M_1 M_2)^2 N_1 N_2 + x^\varepsilon l_1 H^2 (M_1 + M_2) M_1 M_2^{3/2} N_1^2 N_2^3\}^{1/2} \\ \ll x^{9\varepsilon/2} \{Q x^{-4\varepsilon} M_1 M_2 + l_1 (M_1 + M_2) M_1 M_2^{3/2} N_1^2 N_2^3\}^{1/2}$$

or

$$\sup |R_3| \ll x^{9\varepsilon/2} \{Q^{1/2} x^{-2\varepsilon} M + l_1^{1/2} M^{7/4} N^{5/2}\} \\ \ll x^{9\varepsilon/2} \left\{ Q x^{-6\varepsilon} + \left(\frac{x}{Q}\right)^{1/2} (Q^{1/2} x^{-4\varepsilon})^{7/4} \left(\frac{Q^{1/4}}{x^{1/5}}\right)^{5/2} \right\} \\ \ll Q x^{-3\varepsilon/2}.$$

Combining this with (5.17) we get

$$\sup |R_2| \leq \sup |R_3| + \sup |R_4| \ll Q x^{-3\varepsilon/2},$$

as required.

6. Proof of Lemma 4.

It remains to establish Lemma 4. By expanding the square and changing the order of summation, we have

$$S = \sum_{0 < h_1, h_2 \leq H} \sum_{\substack{n_j (j=1, 3, 2, 4) \\ (n_1, n_2)=(n_3, n_4)=1 \\ (n_1 n_3, al_1)=(n_2 n_4, al_2)=1 \\ \mu^2(n_1 n_2)=\mu^2(n_3 n_4)=1}} c(h_1, n_1, n_2) c(h_2, n_3, n_4) \sum_{\substack{m_1 \\ (m_1 n_1, n_2 n_2)=1 \\ (n_1 n_3, n_2 n_4)=1 \\ (m_1, l_1)=(m_2, fl_2)=1}} \sum_{m_2} e\left(h_1 \theta\left(\frac{m_1 n_1}{m_2 n_2}\right) - h_2 \theta\left(\frac{m_1 n_3}{m_2 n_4}\right)\right) \\ \leq \sum_{h_1} \sum_{h_2} \sum_{n_j} \sum_{m_2} \sum_{\substack{l_1^\# \\ b=1}} \left| \sum_{\substack{m_1 \sim M_1 \\ m_1 \equiv b(l_1^\#) \\ (m_1, m_2 n_2 n_4 l_1^{**})=1}} e\left(fl\left(h_1 \frac{m_1 n_1 l_1^\# l_2^*}{m_2 n_2 l_1^{**}} - h_2 \frac{m_1 n_3 l_1^\# l_2^*}{m_2 n_4 l_1^{**}}\right)\right) \right|$$

We proceed to treat the argument in the above exponential sums. We have

$$h_1 \frac{m_1 n_1 l_1^\# l_2^*}{m_2 n_2 l_1^{**}} - h_2 \frac{m_1 n_3 l_1^\# l_2^*}{m_2 n_4 l_1^{**}} = \frac{g}{m_2 \frac{n_2}{(n_2, n_3)} \frac{n_4}{(n_1, n_4)} l_1^{**}}$$

where

$$g = h_1 \frac{1}{(n_2, n_3)(n_1, n_4)} \frac{n_4}{(m_1 n_1 l_1^* l_2^*)'} - h_2 \frac{1}{(n_1, n_4)} \frac{n_2}{(n_2, n_3)} (m_2 n_3 l_1^* l_2^*)''$$

with $A'A \equiv 1 \pmod{m_2 n_2 l_1^{**}}$ and $A''A \equiv 1 \pmod{m_2 n_4 l_1^{**}}$. We then find, with certain integers k_1 and k_2 , that

$$\begin{aligned} & m_1 n_1 n_3 l_1^* l_2^* g \\ &= h_1 \frac{n_3}{(n_2, n_3)} \frac{n_4}{(n_1, n_4)} (1 + k_1 m_2 n_2 l_1^{**}) - h_2 \frac{n_1}{(n_1, n_4)} \frac{n_2}{(n_2, n_3)} (1 + k_2 m_2 n_4 l_1^{**}) \\ &\equiv h_1 \frac{n_3}{(n_2, n_3)} \frac{n_4}{(n_1, n_4)} - h_2 \frac{n_1}{(n_1, n_4)} \frac{n_2}{(n_2, n_3)} \pmod{m_2 \frac{n_2}{(n_2, n_3)} \frac{n_4}{(n_1, n_4)} l_1^{**}} \\ &= r, \quad \text{say.} \end{aligned}$$

Since $(m_1 n_1, m_2 n_2) = (m_1 n_3, m_2 n_4) = 1$ and $\mu^2(n_1 n_2) = \mu^2(n_3 n_4) = 1$, we have

$$\left(m_1 n_1 n_3 l_1^* l_2^*, m_2 \frac{n_2}{(n_2, n_3)} \frac{n_4}{(n_1, n_4)} l_1^{**} \right) = 1,$$

whence

$$g \equiv \overline{m_1 n_1 n_3 l_1^* l_2^* r} \pmod{m_2 \frac{n_2}{(n_2, n_3)} \frac{n_4}{(n_1, n_4)} l_1^{**}}.$$

Now Lemma 3 yields

$$\begin{aligned} S &\leq \sum_{h_1} \sum_{h_2} \sum_{n_j (j=1,2,3,4)} \sum_{\substack{m_2 \\ b}} \sum_b \left| \sum_{\substack{m_1 \sim M_1 \\ m_1 = b(l_1^*)}} e\left(flr \frac{m_1 n_1 n_3 l_1^* l_2^*}{d l_1^{**}} \right) \right| \\ &\ll \sum_{h_1} \sum_{h_2} \sum_{n_j} M_1 M_2 + x^{\varepsilon/2} \sum_{h_1} \sum_{h_2} \sum_{n_j} \sum_{\substack{m_2 \\ (d, f)=1}} l_1^* (flr, d)^{1/2} (d l_1^{**})^{1/2} \left(1 + \frac{M_1}{d} \right) \\ &\ll M_1 M_2 \sum_{s \leq 4H N_1 N_2} \tau_3(s)^2 + x^{\varepsilon/2} l_1 \sum_{h_1, h_2} \left(\sum_{n_j} \sum_{m_2} \frac{(kr, d)}{d} \right)^{1/2} \left\{ \left(\sum_{n_j} \sum_{m_2} d^2 \right)^{1/2} + M_1 \left(\sum_{n_j} \sum_{m_2} 1 \right)^{1/2} \right\} \\ &\ll x^\varepsilon H M_1 N_1 M_2 N_2 + x^{\varepsilon/2} l_1 \sum_{h_1, h_2} \left(\sum_{n_j} \sum_{m_2} \frac{(kr, d)}{d} \right)^{1/2} \\ &\quad \{ ((M_2 N_2^2)^3 N_1^2)^{1/2} + M_1 (M_2 N_2^2 N_1^2)^{1/2} \}. \end{aligned}$$

Here we easily see

$$\sum = \sum_{n_j} \sum_{m_2} \frac{(kr, d)}{d} \ll x^\varepsilon N_1^2.$$

In fact, if we write

$$\begin{aligned} \delta_1 &= (n_1, n_2), & \delta_2 &= (n_2, n_3) \\ n_1 &= \delta_1 n'_1, & n_2 &= \delta_2 n'_2, & n_3 &= \delta_2 n'_3, & n_4 &= \delta_1 n'_4, \end{aligned}$$

and

$$\gamma=(n'_2, n'_4), \quad n'_2=\gamma n_2, \quad n'_4=\gamma n_4,$$

then we have

$$\begin{aligned} \Sigma &= \sum_{\substack{n_1 \\ k(h_1 n'_3 n_4 - h_2 n'_1 n_2) \neq 0}} \sum_{n_3} \sum_{\substack{\delta_1 | n_1 \\ \delta_2 | n_3}} \sum_{\substack{\delta_2 | n_3 \\ (n_2, n_4)=1}} \sum_{m_2} \sum_{\gamma} \sum_{\substack{n_2 \\ (n_2, n_4)=1}} \sum_{n_4} \frac{(k(h_1 n'_3 n_4 - h_2 n'_1 n_2), m_2 \gamma n_2 n_4)}{m_2 \gamma n_2 n_4} \\ &\leq \sum_{n_1} \sum_{n_3} \sum_{\delta_1} \sum_{\delta_2} \sum_{n_2} \sum_{n_4} \sum_{m_2} \sum_{\gamma} \frac{(B, m_2)}{m_2} \frac{(B, \gamma)}{\gamma} \frac{(B, n_2)}{n_2} \frac{(B, n_4)}{n_4} \end{aligned}$$

where $B=k(h_1 n'_3 n_4 - h_2 n'_1 n_2) \neq 0$. Since $(n_2, n_4)=1$,

$$(B, n_2)=(k h_1 n'_3 n_4, n_2)=(k h_1 n'_3, n_2)$$

and

$$(B, n_4)=(k h_2 n'_1, n_4).$$

By (5.16),

$$\begin{aligned} \Sigma &\ll (x^{e/5})^2 \sum_{n_1} \sum_{n_3} \sum_{\delta_1 | n_1} \sum_{\delta_2 | n_3} \sum_{n_2} \frac{(k h_1 n'_3, n_2)}{n_2} \sum_{n_4} \frac{(k h_2 n'_1, n_4)}{n_4} \\ &\ll (x^{e/5})^4 \sum_{n_1} \sum_{n_3} \tau(n_1) \tau(n_3) \\ &\ll x^e N_1^2. \end{aligned}$$

Therefore we get

$$\begin{aligned} S &\ll x^e H M_1 N_1 M_2 N_2 + x^e l_1 H^2 \{M_2^{3/2} N_1^2 N_2^3 + M_1 M_2^{1/2} N_1^2 N_2\} \\ &\ll x^e H M_1 N_1 M_2 N_2 + x^e l_1 H^2 (M_1 + M_2) M_2^{1/2} N_1^2 N_2^3, \end{aligned}$$

as required.

This completes the proof of our Theorem.

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