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DIAGONAL TRANSFORMATIONS OF TRIANGULATION ON SURFACES

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By

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1. Introduction

There is introduced in [3] an interesting theorem on maximal planar graphs, due to Wagner [6], as follows:

THEOREM 1. (K. Wagner) *Any two maximal planar graphs with the same number of vertices are equivalent under diagnal transformations.*

A *maximal planar graph* G is a simple graph embedded in the plane such that one can add no new edge to it in the plane, that is, such a one that each region or face is three-edged. The *diagonal transformation* is to switch the diagonal edge ac in the union of two adjacent triangular faces abc and acd, as shown in Figure 1. We however have to preserve the simpleness of graphs, that is, the diagonal transformation cannot be applied if vertices b and d are adjacent in G.

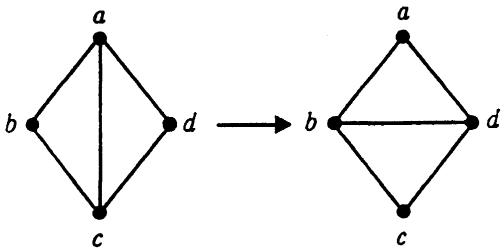

Figure 1. Diagonal transformation

In fact, it has been that every maximal planar graph can be transformed into the normal form given in Figure 2 by a finite sequence of diagonal transformations and hence any two maximal planar graphs are transferable via this normal form. The planarity of graphs ensures that the degree of an arbitrary vertex can be decreased to 3 by switching edges incident to it.

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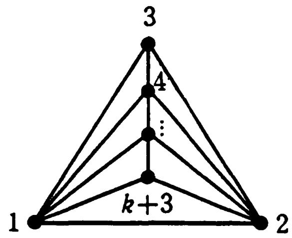

Figure 2. Normal form for maximal planar graphs with $k+3$ vertices

Theorem 1 can be translated naturally into the theorem that any two triangulations with n vertices on the sphere are equivalent under diagonal transformations. Dewdney [1] had already proved that any two triangulations with n vertices on the torus also are equivalent under diagonal transformations. Since the triangulation on the torus with fewest vertices is the unique embedding of the complete graph K_7 on seven vertices, we can take the triangulation given in Figure 3 as a normal form of toroidal triangulations.

THEOREM 2. (A.K. Dewdney) *Every triangulation of the torus can be transformed into the normal form in Figure* 3 *by diagonal transformations.*

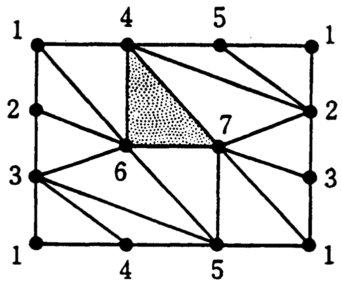

Figure 3. Normal form for toroidal triangulations

In this paper, we shall deal with triangulations of other closed surfaces and prove the following two theorems which imply that any two triangulations

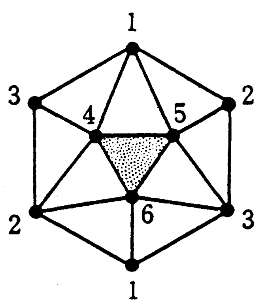

Figure 4. Normal form for projective-planar triangulations

with n vertices of the projective plane and Klein bottle are equivalent under diagonal transformations:

THEOREM 3. *Every triangulation of the projective plane can be transformed into the normal form in Figure* 4 *by diagonal transformations.*

THEOREM 4. *Every triangulation of the Klein bottle can be transformed into the normal form in Figure* 5 *by diagonal transformations.*

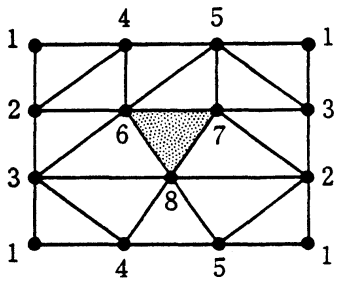

Figure 5. Normal form for Klein-bottlal triangulations

To get the actual triangulation of each normal form, we have to identify the parallel pairs of edges on the boundary of each polygonal disk so that the labels of vertices coincide, and have to add a suitable normal form of maximal planar graphs (Figure 2) to each shaded face so that the result has the same number of vertices as a given triangulation.

We shall use the terminology and notations in [2] for graph theory and quite standard ones for topology.

2. General observations

Let F^2 be a closed surface, that is, a compact 2-manifold without boundary. A simple graph G embedded in F^2 is called a *triangulation* of F^2 if G divides F^2 into three-edged regions, called *faces* of G. Since G has no self-loop and no multiple edges, such a triangulation G induces a simplicial 2-complex structure of F^2 unless G is K_3 in the sphere. For each vertex v of a triangulation G, we define the *star neighborhood* $st(v, G)$ and the *link* $lk(v, G)$ of v as the union of triangular faces meeting v and its boundary cycle, respectively. Two triangulations G_1 and G_2 in F^2 are said to be *isomorphic* if there is a homeomorphism $h: F^2 \to F^2$ such that $h(G_1)=G_2$.

We define the diagonal trnsformation for triangulations in F^2 as the same local modification as is mentioned in introduction. Two triangulations of F^2 are said to be *equivalent* (under diagonal transformations) if one can be transformed into the other, up to isomorphism, by a finite sequence of diagonal transformations.

LEMMA 5. *Let G_1 and G_2 be two triangulations in F^2. If there are vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ of degree 3 such that $G_1 - v_1$ and $G_2 - v_2$ are isomophic, then G_1 and G_2 are equivalent.*

PROOF. Let G be a triangulation in F^2, isomorphic to $G_1 - v_1$ and hence to $G_2 - v_2$, and let v be an extra vertex of degree 3 added to a face abc of G. Figure 6 shows a transformation which carries v to a neighboring face acd. It should be noticed that vertices a and c are not adjacent in the second stage. By repeating this process, we can replace v in suitable faces of G to get G_1 and G_2. Thus, there is a sequence of diagonal transformations which transforms G_1 into G_2. ■

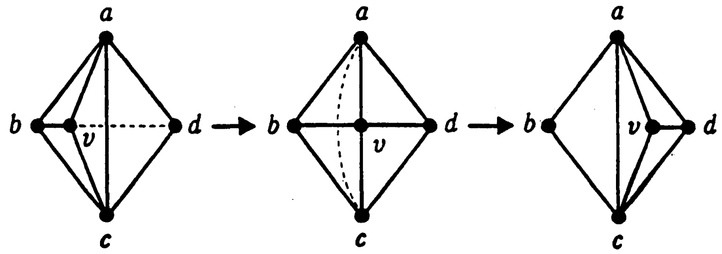

Fig. 6.

LEMMA 6. *Let G_1 and G_2 be two triangulations of a closed surface F^2 which have vertices v_1 and v_2 of degree 3, respectively. If $G_1 - v_1$ and $G_2 - v_2$ are equivalent, then G_1 and G_2 are equivalent.*

PROOF. We use induction on the length n of a sequence of triangulations $G_1 - v_1 = H_0, H_1, \cdots, H_n = G_2 - v_2$ such that H_{i-1} is transformed into H_i by a single diagonal transformation. If $n=0$, then G_1 and G_2 are equivalent by Lemma 5, which is the first step of our induction.

Let abc be the face of $G_1 - v_1$ which contains v_1 and suppose that the first diagonal transformation, applied to H_0, in a sequence of length $n>0$ switches an edge $a'c'$ to $b'd'$ in a rectangle $a'b'c'd'$.

If $a'c'$ is not an edge lying on the triangle abc, the transformation can be regarded as a diagonal transformation for G_1 directly. If $a'c'$ is one of edges on abc, say ac, then the transformation can be translated into two consecutive diagonal transformations for G_1 as shown in Figure 7, where a, b, c, d correspond to a', b', c', d' in order.

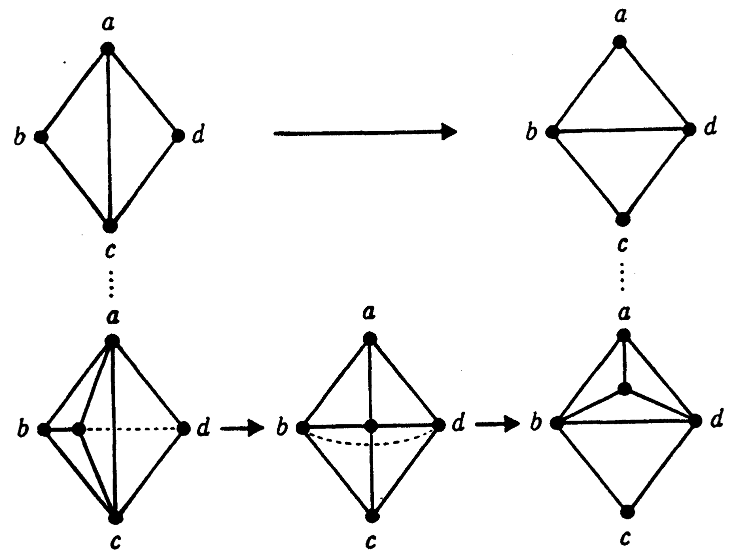

Figure 7.

In either case, G_1 is equivalent to a triangulation G_1' such that $G_1'-v_1'$ is isomorphic to H_1 for a vertex v_1' of degree 3. By the induction hypothesis, G_1' is equivalent to G_2 and hence so is G_1. ■

Let G be a triangulation in F^2 and v a vertex of G with neighbors u_1, u_2, $\cdots$, $u_n (n \geqq 4)$ lying cyclically on the link $lk(v, G)$ in this order. Suppose that no edges incident to v can be switched by a diagonal transformation, and hence that the degree of v cannot be decreased by only deformation within the star neighborhood $st(v, G)$ of v. Then there must exist n edges $u_i u_{i+2}$ $(i=1, 2, \cdots, n-2)$, $u_{n-1}u_1$ and $u_n u_2$ (or two edges $u_1 u_3$ and $u_2 u_4$ if $n=4$). We define the graph F_n as the union of the wheel $st(v, G)$ with center v and these n (or two) edges. In particular, F_4 and F_5 are isomorphic to the complete graph K_5 and K_6, respectively, but F_n is not complete if $n \geqq 6$.

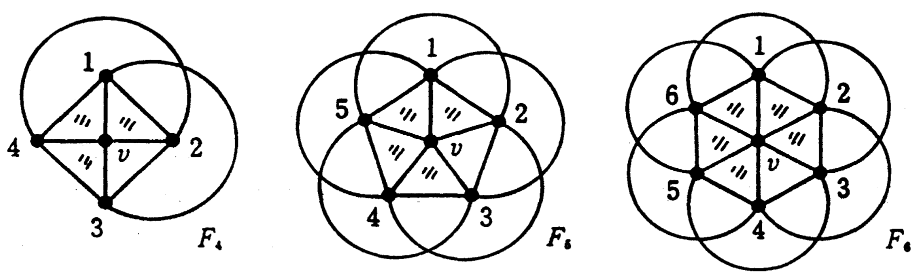

Figure 8.

LEMMA 7. *Let G be a triangulation in F^2 and C a cycle of G which bounds a 2-cell D^2 in F^2. Then $G \cap D^2$ can be deformed by a sequence of diagonal trans-*

formations and deletions of vertices of degree 3 *so that afterward the neighbors of each vertex in the interior of* D^2 *are contains in* C. *After such deformation, the interior of* D^2 *contains at most* $\lfloor (m-2)/2 \rfloor$ *vertices of* G *if* C *has length* m.

PROOF. Let v be a vertex of G in the interior of D^2 and apply diagonal transformations to edges incident to v as long as possible. If $deg(v)=3$, we remove v out of G and continue the following argument for another vertex.

Here we can assume that v lies at the center of the graph $F_n(n \geqq 4)$. If an edge $u_i u_{i+2}$ were contained in D^2, then the triangle $vu_i u_{i+2}$ would bound a 2-cell in D^2 which contains u_{i+1} but not u_{i+3}. In this case, no edge could join u_{i+1} to u_{i+3}, a contradiction. Thus, all of $u_i u_{i+2}$'s are placed in F^2-D^2. This implies that each u_i $(i=1, 2, \cdots, n)$ lies on C, the boundary of D^2.

Therefore, if C has length 3, then D^2 cannot contain any vertex after the deformation, which corresponds to that $\lfloor (m-2)/2 \rfloor = 0$ if $m=3$. When some vertices remain in the interior of D^2 with $m \geqq 4$, we estimate the number of them inductively as follows.

If D^2 contains two or more vertices, then it does not coincide with the star neighborhoods $st(v, G)$ of any vertex v and there is an edge on the link $lk(v, G)$ which divides D^2 into two 2-cells. Let m_1 and m_2 denote the length of their boundary cycles, respectively. Then we have $m=m_1+m_2-2$. By the induction hypothesis, we can assume that those 2-cells contains at most $(m_1-2)/2$ and $(m_2-2)/2$, and hence D^2 contains at most $(m-2)/2=(m_1-2)/2+(m_2-2)/2$ vertices. ∎

3. Proofs of theorems

We shall prove Theorems 3 and 4 through this section. Our proofs of these theorems will proceed in a common manner as follows.

A triangulation G of F^2 is said to be *pseudo-minimal* if G is equivalent to no triangulation which has a vertex of degree 3. By Lemma 6, we can conclude that any two triangulations with the same number of vertices are equivalent if any two pseudo-minimal triangulations are equivalent. So our goal is to show that any pseudo-minimal triangulation of the projective plane and Klein bottle are equivalent to the normal forms in Figures 4 and 5, respectively.

Let F^2 be one of the projective plane and Klein bottle and G a pseudo-minimal triangulation in F^2. Suppose that the minimum degree $\delta(G)$ of G is the smallest among the pseudo-minimal triangulations equivalent to G and let v be a vertex of G such that $n=deg(v)=\delta(G) \geqq 4$. Then v lies at the center of F_n. By Euler's formula, any triangulation of the projective plane (or of Klein bottle) has a vertex of degree at most 5 (or 6), so we have $n \leqq 5$ (or $n \leqq 6$) if F^2 is the projective plane (or the Klenin bottle).

We shall keep the situation in the previous paragraph hereafter and often use the fact that any m-gonal region contains no vertex if $m<n$, which follows from Lemma 7.

Case of the projective plane:

Assume that F^2 is the projective plane and $n=4$. Then the triangle vu_1u_3 is a non-trivial loop in F^2 and hence it is the center line of a Möbius band in F^2. Cut open F^2 along vu_1u_3, then we get the hexagonal 2-cell as shown in the left hand of Figure 9, where the vertex with label i corresponds to u_i.

If square 1243 (precisely $u_1u_2u_4u_3$) had a dianoal, then G would have multiple edges 14 or 23, contrary to the simpleness of G. Thus, there i s a unique vertex of degree 4 in square 1243 which is adjacent to 1, 2, 3, 4, by Lemma 7. Also square 1324 contains no diagonal but a unique vertex of degree 4. If we switch the three edges u_1u_2, u_2u_3 and u_2u_4, then the right hand of Figure 9 will be obtained. This contradicts that G is pseudo-minimal since the resulting triangulation has a vertex of degree 3.

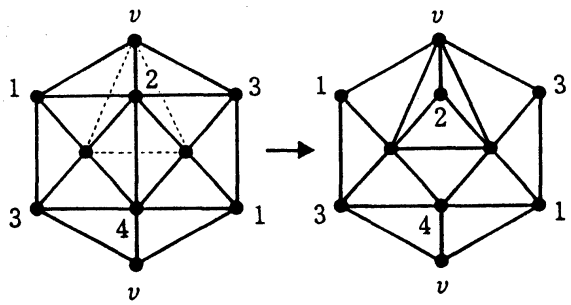

Figure 9.

Now suppose that $n=5$. The graph F_5 is isomorphic to K_6 and has the unique embedding obtained in Figure 4. Since such an embedding of F_5 is triangular, G has to coincide with F_5 by Lemma 7.

The above argument concludes that there is a unique pseudo-minimal triangulation of the projective plane, isomorphic to K_6 as a graph. By Lemm 6, any triangulation is equivalent to K_6 with extra vertices of degree 3 added in order. So we can choose Figure 4 as a normal form of projective-planar triangulations. Now Theorem 3 has been proved. ■

Case of the Klein bottle:

Assume that F^2 is the Klein bottle. Each triangle $vu_iu_{i+2}(i=1, 2, \cdots, n)$ is a non-trivial loop in F^2 and is the center line of an annulus or a Möbius band in turn. In the former case (or latter case), such a loop is said to be 2-*sided* (or 1-*sided*). Note that any 2-sided loop given as vu_iu_{i+2} cuts open the Klein bottle into an annulus.

Case 1. First, suppose that at least one of them, say vu_2u_4, is 2-sided. Then vu_1u_3 is 1-sided and F^2 can be cut open along the bouquet of vu_2u_4 and vu_1u_3 into a rectangle with v's at four corners; otherwise, F^2 would be a torus.

Assume that $n=4$ under the above condition. Let $24x$ and $24y(x\neq y)$ be the two triangles adjacent to edge 24 (precisely u_2u_4) and let $13s$ and $13t(s\neq t)$ be the such triangles for edge 13 (Figure 10(i)). If x and y were not adjacent in G, we could replace the diagonal 24 with xy in $2x4y$ and next $1v$ with 24 so that afterward v has degree 3, contrary to G being pseudo-minimal. Thus, x and y and also s and t are joined by an edge, respectively. Up to symmetry, we have the two possibilities shown in Figures 10(ii) and (iii), where $x=s$, $y=t$ and $x=s$, $y\neq t$, respectively.

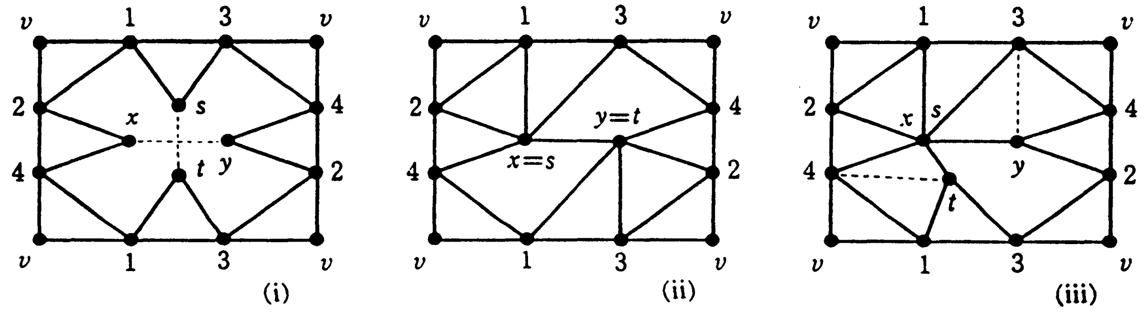

Figure 10.

In either figure, if there is a vertex z in rectangle $43xy$ or $14st$, then z has degree 4 and we can replace the diagonal xy or st in rectangle $xzy*$ or $szt*$ and carry out the same deformation as in the previous paragraph, a contradiction. Thus, both $43xy$ and $14st$ contain no vertex and are divided into two triangles by diagonals $3y$ and $4t$, respectivly. It is however impossible in case of Figure 10(ii).

By Lemma 7, pentagon $23txy$ in Figure 10(iii) contains at most one vertex. If there is no vertex in $23txy$, then we get the normal form in Figure 5 after adding diagonals $2t$, yt. If there is a vertex in $23txy$, then we have the three possibilities shown in Figures 11(i), (ii) and (iii). However, the diagonal transformations indicated by dashed lines transform them into one that have vertices of degree 3, contrary to G being pseudo-minimal. Therefore, we conclude that if v has degree 4, then the pseudo-minimal triangulation G is the normal form in the Klein bottle.

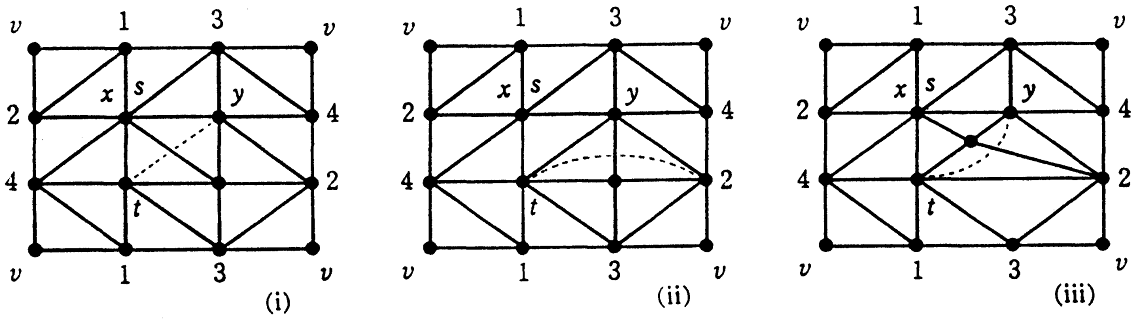

Figure 11.

Now suppose that v has degree 5. In this case, we have Figure 12(i) as the rectangle obtained from F^2 by cutting it open along the bouquet of vu_2u_4 and vu_1u_3, and consider triangles $24x$, $24y$, $13s$ and $13t$. As in the previous case, we may assume that there are edges xy and st in G. The simpleness of G implies that $x \neq 2, 3, 4$; $y \neq 1, 2, 3, 4, 5$; $s \neq 1, 2, 3, 5$; $t \neq 1, 2, 3, 4$. The vertex x might be equal to one of vertices 1 and 5. We shall consider the three cases below, depending on it. Now any rectangle region in F^2 contains no vertex and is divided into two triangles by a diagonal, by Lemma 7.

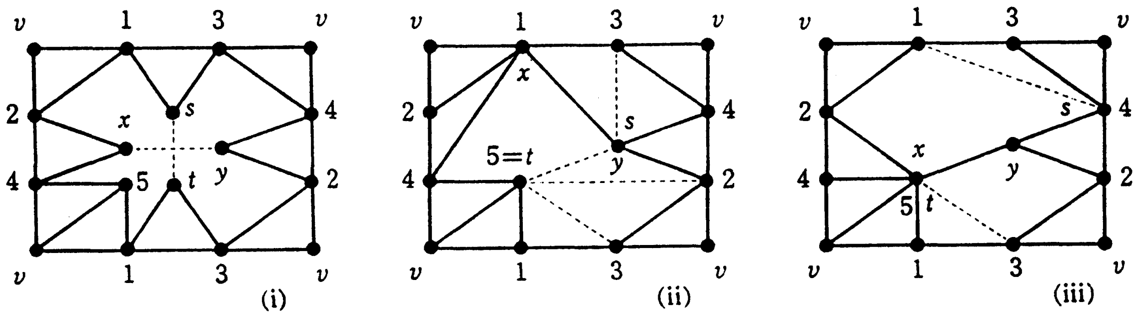

Figure 12.

If $x=1$ (Figure 12(ii)), then we have to draw the edges indicated by dashed lines but no more edge can be added to the rectangle $145s$ which also contains no vertex. Thus, it is impossible to construct the whole of G in this case.

When $x=5$ (Figure 12(iii)), we add first edges 14, 53 and next $3y$ in the rectangle $532y$ and finally a vertex z of degree 5 in the pentagon $125y4$ to complete the triangulation. The resulting triangulation (Figure 13(i)) is another pseudo-minimal triangulation of the Klein bottle with 8 vertices. However, it can be transformed into one that which has a vertex of degree 4 by switching $5y(=ty)$. This implies that G is equivalent to the previous normal form with a vertex of degree 4.

When $x \neq 1, 5$, we have $5=t$ and $4=s$ after adding 52, 53 and 14 to Figure

12(i). The rectangle containing xy as its diagonal must be $1x5y$ to forbid the switching of xy. Then the triangulation in Figure 13(ii) will be obtained. It is also pseudo-minimal, but we can decrease the degree of v to 4, replacing 25 with $3y$ and next $1v$ with 25. Thus, G is equivalent to the normal form in Figure 5.

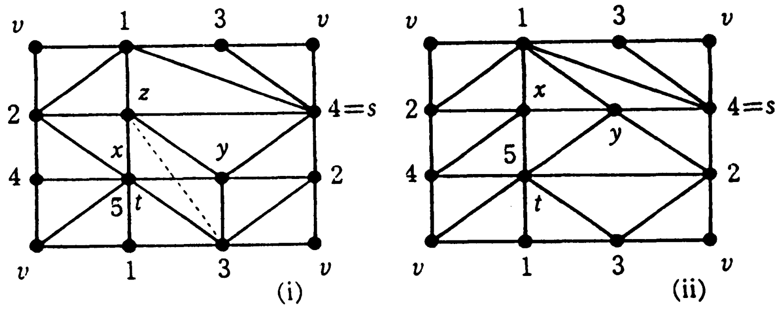

Figure 13.

Finally, suppose that v has degree 6. Then F_6 with v at the center is embedded in F^2 as shown in Figure 14. Since any rectangle contains no vertex now, the whole of G has to be obtained by adding diagonals to 1245, 3564 and 6132. It is however impossible; first add 36 to 3564, then there is no diagonal which can be added to 6132.

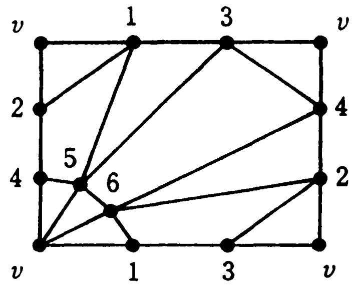

Figure 14.

Case 2. Now suppose that all of $vu_iu_{i+2}(i=1, 2, \cdots, n)$ are 1-sided loops in the Klein bottle F^2, and cut open F^2 along vu_1u_3 (Figure 15(i)). Then one of

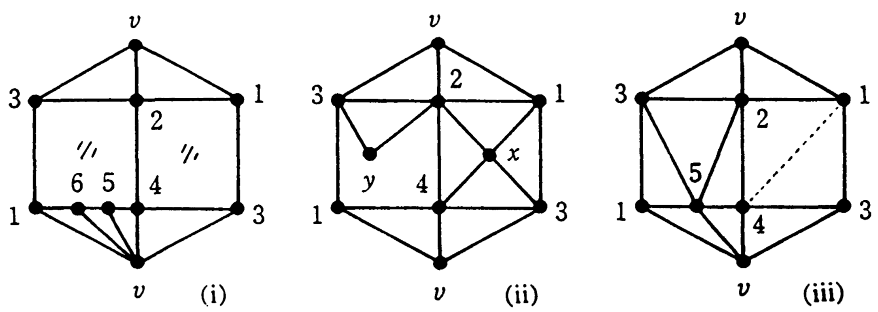

Figure 15.

cycles 1324(56) and 1243 bounds a cross cap (=Möbius band) and the other bounds a 2-cell.

First suppose that v has degree 4 (Figure 15(ii)). Then we may assume that 1243 bounds a 2-cell and 1324 bounds a cross cap, up to symmetry. Since 1243 cannot contains a diagonal, there is a unique vertex x of degree 4 in 1243. Let $2v3y$ be the rectangle containing 23 as its diagonal. Now y coincides with neither 1 nor 4 since multiple edges would arise if not. So we can replace the diaonal 23 with vy and next $4x$ with 23 in $243x$ so that $deg(x)=3$ afterward, contrary to G being pseudo-minimal.

Now suppose that 1243 bounds a cross cap and that $deg(v)=5$ or 6. If $deg(v)=6$, then 64 and 51 could not be placed simultaneously in the 2-cell bounded by 132456. If $deg(v)=5$, then the 2-cell bounded by 13245 is triangulated by edges 53, 52 and the cross cap contains edge 14 (Figure 15(iii)). In this case, if the cycle 124 did not bound a face, we could switch the diagonal 24 in 254* and next $3v$ in $34v2$ to decrease the degree of v, contrary to the assumption of v. Thus, triangles 124 and similarly 134 have to bound faces, but this implies that F^2 would be a projective plane, a contradiction.

Therefore, $deg(v)=5$ or 6 and 1243 has to bound a 2-cell which contains the diagonal 14. By the symmetry, the cycles $\{i, i+1, i+3, i+2\}$ bounds 2-cells with diagonals $(i, i+3)\,(i\equiv 1, 2, 3, 4, 5 \bmod 5)$. This is however possible only when both 1243 and 13245 bound 2-cells like Figure 15(iii), which implies that F^2 would be the projective plane.

Since any situation under Case 2 implies a contradiction, any pseudo-minimal triangulation of the Klein bottle is equivalent to the normal form recognized in Case 1 and Theorem 4 follows. ■

4. Remarks

We conjecture that any two triangulations in a given closed surface are equivalent under diagonal triangulations. To prove this, it suffices to observe that any two pseudo-minimal triangulations are equivalent, as our strategy in this paper. Unfortunately, if one carries out arguments similar to ours in Section 3, a tedious and long proof will be obtained in general.

A triangulation G in a closed surface F^2 is said to be *minimal* if the number of faces (or of vertices equivalently) of G is the smallest among all the triangulations of F^2. If the complete graph K_n has a triangular embedding in F^2, then the embedding is a minimal triangulation of F^2 and K_n is the unique graph which induces a minimual triangulation. (See [4] and [5] for minimal triangulations.)

Since no diagonal transformation can be applied to K_n, our conjecture is false for F^2 if such K_n has two or more inequivalent embeddings in F^2. For example, the embeddings of K_6 and K_7 in Figures 3 and 4 are minimal triangulations of the projective plane and the torus, respectively, and they are uniquely embeddable, up to homeomorphism, in each surface. On the other hand, the minimal trinagulations of the Klein bottle are not complete and not unique, but they are equivalent.

Every minimal triangulation is pseudo-minimal, but the converse is not so clear. If there is a pseudo-minimal triangulation G of F^2 which is not minimal, then our conjecture is not true again. For G is not equivalent to any triangulation obtained from a minimal triangulation by adding vertices of degree 3 in order. It is however not so difficult to show that for any two triangulations G_1 and G_2 of F^2 with possibly different number of vertices, there is a common triangulation which can be transformed into G_1 and G_2 by sequences of diagonal transformations and deletions of vertices of degree 3.

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