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A CLASS OF FINITE GROUPS ADMITTING CERTAIN SHARP CHARACTERS I

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By

Takashi MATSUHISA

1. Introduction.

Let G be a finite group and χ a character of G of degree n, and let L be the image of χ on G, and $L^{\#}=L\setminus\{n\}$. H.F. Blichfeldt ([1]) and P.J. Cameron—M. Kiyota ([2]) showed that the number $B(n)$ defined by $(1/|G|)\prod_{l\in L^{\#}}(n-l)$ is an integer when χ is faithful. Such (G, χ) is called a *sharp pair of type* $L^{\#}$ and χ *a sharp character* provided $B(n)=1$.

This concept generalizes that of sharply multiply—transitive permutation groups, where a rich theory of sharp pairs has been developed and many examples are known ([2], [3], [6]). Moreover, F. Blichfeldt showed that the number $B(l)$ defined by $(a(l)/|G|)\prod_{k\in L\setminus\{l\}}(l-k)$ is an algebraic integer for any $l\in L$ whenever χ may not be faithful (where $a(l)$ denotes the number of elements x of G with $\chi(x)=l$).

Recently M. Kiyota ([5]) introduced a class of pairs of finite groups and its characters which included that of sharp pairs of finite groups; a triple (G, χ, l) is called a *sharp triple of rank* r provided $B(l)$ is a unit in the ring of algebraic integers ($r=|L^{\#}|$, $l\in L$). He posed the problem of determining sharp triples when a finite subset $L^{\#}$ of $\boldsymbol{C}$ is given. This problem in the case of sharp pairs has been studied by several authors (cf. [2], [6]).

In this note we will investigate sharp triples of rank 2. There are two cases: (i) the two elements of $L^{\#}$ are algebraic conjugates (cf. Theorem 1). (ii) all elements of L are integers. However, it seems difficult to classify completely even sharp pairs in the case (ii), so we will treat it under the condition that χ is irreducible or sharp (cf. Proposition 3 and Theorem 4). In particular, we will note that all sharp triples of type $\{-1, 1\}$ are sharp pairs except for two cases in the last section (cf. Theorem 5).

Our notations are largely standard (cf. [4], [2]). Throughout this note, $\mathcal{O}$

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will denote the ring of algebraic integers, $\mathcal{O}^{\times}$ its units group, and $\mathfrak{G}$ a Galois group of $\boldsymbol{Q}(\omega)$ over $\boldsymbol{Q}$ $\left(\omega\equiv\frac{2\pi\sqrt{-1}}{|G|}\right)$. $f_K(X)=\prod_{k\in K}(X-k)\in\boldsymbol{C}[X]$ for each finite subset K of $\boldsymbol{C}$. A character χ of G is called *normalized* if $\langle\chi, 1_G\rangle=0$, where 1_G is the principal character of G.

2. Preliminaries.

We will recall a theorem of H. F. Blichfeldt and terminology in this section after M. Kiyota ([5]). $\tilde{\chi}$ will denote an $\mathcal{O}$-generalized character $f_{L\setminus\{l\}}(\chi)=\prod_{k\in L\setminus\{l\}}(\chi-k1_G)$, $(l\in L)$. It follows immediately from definitions that $B(l)=\langle\tilde{\chi}, 1_G\rangle$, which is an algebraic integer. In particular, $B(n)$ is an integer since a Galois group $\mathfrak{G}$ acts on the set L. Thus we get the following fact known as H. F. Blichfeldt's and P. J. Cameron—M. Kiyota's;

PROPOSITION 1 ([1], [2]). *Let G be a finite group, and χ a character of degree n. Then* (i) *the number $B(l)$ is an algebraic integer for each $l\in L$,* (ii) *in particular, $B(n)$ is an integer.*

REMARK 1. *Moreover, $B(n)$ is a positive integer.* In fact, since any element l of $L^{\#}$ is the sum of a n roots of unity, $|l|\leqq n$. If l is real, $n-l>0$. If not, its complex conjugate $\bar{l}$ is in $L^{\#}$, and $(n-l)(n-\bar{l})>0$, thus $B(n)>0$.

We shall call a triple (G, χ, l) *a sharp triple of rank* r $(r=|L^{\#}|)$ or *of type* $L^{\#}$, if $B(l)$ is a unit of $\mathcal{O}$ $(l\in L)$. In particlar, when χ is faithful we shall call a pair (G, χ) *a sharp pair of type $L^{\#}$* (*in the sense of* P. J. Cameron—M. Kiyota) if $B(n)=1$, and then χ *sharp.*

Our purpose is to investigate sharp triples for various $L^{\#}$, in particular, of its cardinality 2. Without loss of generality. We may assume that χ is normalized (See [2]). We shall give an example of rank 1, which is due to M. Kiyota:

PROPOSITION 2 ([5]). *Let (G, χ, l) be a sharp triple of rank* 1. *Assume χ is faithful and normalized.*

(i) *If $l=n$, G is arbitrary, and $\chi+\rho_G-1_G$, where ρ_G is the regular character of G.*

(ii) *If $l\neq n$, G is cyclic of order* 2, *and χ is the non-trivial irreducible character.*

PROOF. Let $L=\{n, \alpha\}$ $(n=\deg\chi,\ \alpha\in\mathcal{O})$. Since L is $\mathfrak{G}$-invariant, $\alpha\in\boldsymbol{Z}$.

(i) Put $\tilde{\chi}=(\chi-\alpha 1_G)\in\boldsymbol{Z}[\mathrm{Irr}(G)]$. As χ is normalized, $B(n)=\langle\tilde{\chi}, 1_G\rangle=-\alpha\in$

$\mathcal{O}^{\times}\cap\boldsymbol{Q}=\{\pm 1\}$, and $\alpha=-B(n)=-(1/|G|)(n-\alpha)$. Thus, by Remark 1, we have $\alpha=-1$, $B(n)=1$ and $|G|=n+1$. This (G, χ) is a sharp pair of type $\{-1\}$, so it follows from Proposition 1.2 in [2] that G is arbitrary, and $\chi=\rho_G-1_G$.

(ii) Put $\tilde{\chi}=(\chi-n1_G)\in\boldsymbol{Z}[\mathrm{Irr}(G)]$. $B(\alpha)=\langle\tilde{\chi}, 1_G\rangle=-n\in\mathcal{O}^{\times}$. Thus $n=1$, so $|G|=a(\alpha)(1-\alpha)$. As $\langle\chi, 1_G\rangle=0$, $\alpha a(\alpha)=-1$, so $\alpha=-1$ and $a(\alpha)=1$. Hence $|G|=2$, and then $\langle\chi, \chi\rangle=1$ i.e. $\chi\in\mathrm{Irr}(G)$.

3. Sharp triples of rank 2.

We shall investigate sharp triples of rank 2 in this section. Since $\mathfrak{G}$ acts on the image L of χ, it sufficies for us to treat the two cases of L mentioned in the introduction above. We shall always assume χ is *faithful* and *normalized.*

The Case where two elements in $L^{\#}$ are algebraic conjugates.

We prove the following:

THEOREM 1. *Assume $\mathfrak{G}$ acts transitively on $L^{\#}=\{\alpha, \beta\}$. If (G, χ, l) is a sharp triple of type $\{\alpha, \beta\}$, then G is a cyclic group of order 3 or 5, and χ is a linear character or the sum of a linear character and its complex conjugate.*

PROOF. We may assume $\mathfrak{G}=\mathrm{Gal}(\boldsymbol{Q}(\alpha)/\boldsymbol{Q})=\langle\sigma\rangle\cong\boldsymbol{Z}_2$. $\alpha^{\sigma}=\beta$.

Case $l=n$. As $B(n)$ is in $\mathcal{O}^{\times}\cap\boldsymbol{Q}=\{\pm 1\}$, $B(n)=1$ by Remark 1. (G, χ) is a sharp pair, so we get our conclusion from Theorem 4.1 in [2].

Case $l\neq n$. Say $l=\alpha$. We shall show (G, χ) is sharp. In fact, χ also takes the values α and β equally often (say, a times). $B(\alpha)=(a/|G|)(\alpha-n)(\alpha-\beta)\in\mathcal{O}^{\times}$. Let N, Tr be the norm and the trace from $\boldsymbol{Q}(\alpha)$ to $\boldsymbol{Q}$ respectively.

$$N(B(\alpha))=B(\alpha)B(\alpha)^{\sigma}=B(n)(-a)\Big(\frac{a}{|G|}(\alpha-\beta)^2\Big),$$

and

$$Tr(B(\alpha))=B(\alpha)+B(\alpha)^{\sigma}=\frac{a}{|G|}(\alpha-\beta)^2.$$

Thus $B(n)(-a)Tr(B(\alpha))=N(B(\alpha))=\pm 1$, so $B(n)=1$ and $a=1$. By the theorem cited above, we get our conclusion. (Note: In this case, $G\cong\boldsymbol{Z}_3$ since $a=1$, and so χ is linear.)

REMARK 2. Recently Prof. M. Kiyota has obtained a more general result as follows; *Suppose that $\mathfrak{G}$ acts transitively on $L^{\#}$ with $|L^{\#}|\geqq 3$. If (G, χ, l) is a sharp triple ($l\in L^{\#}$), G is the cyclic group of some prime order and χ is linear.*

The Case where L consists of integers.

First we prove the following:

THEOREM 2. *Let* (G, χ, α) *be a sharp triple of type* $\{0, l\}$ *with* $l\in\boldsymbol{Z}$. (i) *If* $\alpha\neq 0$, (G, χ) *is a sharp pair of type* $\{0, l\}$ *and* χ *is irreducible.* (ii) *If* $\alpha=0$, G *is the symmetric group of degree* 3 *and* χ *is irreducible of degree* 2 *or the sum of two irreducible characters of degree* 1 *and* 2. *In the latter case* χ *is not sharp.*

PROOF. Since $\langle\chi, 1_G\rangle=0$, we have

(1) $$l=-\frac{n}{a(l)},$$

which is a negative integer. It follows that both $B(n)$ and $B(l)$ are positive integers and $B(0)$ is a negative integer by definition.

Case $\alpha=n$. $B(n)=1$, i.e. (G, χ) is a sharp pair of type $\{0, l\}$ and χ is irreducible from Theorem 2.2 in [2].

Case $\alpha=l$. Since $B(l)=1$, we have $|G|=a(l)l(l-n)$.

$$B(n)=\frac{1}{|G|}n(n-l)=-\frac{n}{a(l)l}.$$

(1) yields $B(n)=1$, i.e. (G, χ) is a sharp pair of type $\{0, l\}$.

Case $\alpha=0$. Since $B(0)=(a(0)/|G|)ln\in\mathcal{O}^{\times}\cap\boldsymbol{Q}=(\pm 1)$, we have $B(0)=-1$, and $|G|=a(0)a(l)l^2$ from (1). Thus we have

(2) $$|G|=1+a(0)+a(l)=a(0)a(l)l^2.$$

We get

$$l^2=\frac{1}{a(0)a(l)}+\frac{1}{a(l)}+\frac{1}{a(0)}\leqq 3,$$

which yields $l=-1$ and $n=a(l)$. Thus (2) leads to

$$a(l)=1+\frac{2}{a(0)-1},$$

$a(0)=3$ or 2, and $n=a(l)=2$ or 3, respectively. If $a(0)=2$ and $n=3$, then $|G|=6$, and $\langle\chi, \chi\rangle=2$. Since $\deg\chi=3$, G is non-abelian. Thus we get $G\cong S_3$, and and χ is the sum of two irreducible characters of degrees 2 and 1. However, (G, χ) is not a sharp pair since $B(n)=2$. If $a(0)=3$ and $n=2$, then we get by the same way as above $G\cong S_3$, and $\chi\in\mathrm{Irr}(G)$. Moreover, since $B(n)=1$, (G, χ) is a sharp pair. This completes our proof.

REMARK 3. Sharp pairs of type $\{0, l\}$ have been studied in §2 of [2]; in particular, if $l=-1$ then G is a 2-trasitive Frobenius group and $\chi=\theta_G-1_G$ where θ_G is permutation character of G (Proposition 2.3), and it $l\neq-1$ then $-l$ is a prime power and $|G|$ is bounded by a function of l (Proposition 2.4).

PROPOSITION 3. *Let (G, χ, l) be a sharp triple of type $\{k, l\}$ with integers k and l. If χ is irreducible, then G and χ is one of the groups and of the characters appeared in Theorem* 2 *above.* (Note: Since we assume χ is irreducible, the last case in Theorem 2 does not occur).

PROOF. It suffices to show $L^{\#}=\{0, l\}$. If $n=\chi(1)>1$, this is valid by Theorem of Burnside (cf. Theorem (3.15) in [4]). Assume $n=1$. Then, since χ is rational valued, there holds $L\subseteqq\{\pm 1\}$, which contradicts the hypothesis.

THEOREM 4. *Let (G, χ, l) be a sharp triple of type $\{k, l\}$ with non-zero integers k, l. If χ is a sharp character, then G is one of the dihedral group of order* 8, *or the quaternion group, and χ is the sum of two irreducible characters with their degrees* 1 *and* 2.

PROOF. From the normality of χ, we get

$$n-a(k)k+a(l)l=0\,. \tag{1}$$

Since χ is sharp, it follows from definitions that

$$\frac{k-n}{a(l)(l-k)}=\frac{B(n)}{B(l)}=\frac{1}{B(l)}\in\mathcal{O}^{\times}\cap\boldsymbol{Q}=\{\pm 1\}\,. \tag{2}$$

If $B(l)=1$, by (1) and (2), we get $k\{1+a(k)+a(l)\}=0$, a contradiction, so $B(l)=-1$. Thus,

$$|G|=a(l)(n-l)(l-k)\,. \tag{3}$$

Put $\tilde{\chi}=(\chi-n1_G)(\chi-k1_G)$. From $B(l)=\langle\tilde{\chi}, 1_G\rangle$, we have, since $\langle\chi, 1_G\rangle=0$,

$$\langle\chi^2, 1_G\rangle+nk+1=0\,. \tag{4}$$

Thus (1), (3) and (4) yield

$$a(l)(l-k)=\frac{(k-n)}{k(n-l)+1}\,, \tag{5}$$

so we get, from (2) and $B(l)=-1$, that

$$k(n-l)=-2\,.$$

From this and (4), we have the two possibilities: (a) $n-l=1$, $k=-2$, and (b) $n-l=2$, $k=-1$. By (3), (5), both $|G|$ and $a(l)$ are determined in each case as follows;

(a) $|G|=n+2$, $a(l)=(n+2)/(n+1)$, and
(b) $|G|=2(n+1)$, $a(l)=(n+1)/(n-1)$.

In case (a), as $a(l)\in N$, $n=0$, a contradiction. In cade (b), we have $n=2$, $|G|=6$, or $n=3$, $|G|=8$. If $n=2$, $l=0$, a contradiction. If $n=3$, we have $\langle \chi, \chi\rangle=2$, and G is non-abelian, so $G\cong D_8$ or Q_8, and χ is the sum of two irreducible characters. Our proof is completed.

In some cases the characters in theorems above are sums of two irreducible characters. By the same way as above we shall characterize sharp triples admitting such characters;

REMARK 4. *Let (G, χ, l) be a sharp triple of type $\{k, l\}$ with non-zero integers k, l. If χ is a sum of two distinct irreducible characters, then G is the dihedral group of order 8 or the quaternion group.*

In fact, as $\langle \chi, \chi\rangle=2$, $B(l)=\langle(\chi-n1_G)(\chi-k1_G), 1_G\rangle=2+nk\in\mathcal{O}^{\times}\cap\boldsymbol{Q}=\{\pm1\}$. Thus, since $n>1$, we have $B(l)=-1$, and $nk=-3$. Hence we get $n=3$ and $k=-1$. From $\dfrac{B(n)}{B(l)}=\dfrac{4}{a(l)(l+1)}\in\boldsymbol{Z}$, the case $a(l)=2$, $l=1$ remains, so in it $G\cong D_8$ or Q_8.

4. Sharp triples of type $\{-1, 1\}$.

Last we will note the following:

THEOREM 5. *Any sharp triples (G, χ, l) of type $\{-1, 1\}$ are sharp pairs of the same type, or otherwise G is the dihedral group of order 8 or the quaternion group, and χ is the sum of three linear characters and the irreducible character of degree 2.*

By the theorem of Cameron—Kataoka—Kiyota, sharp pairs of type $\{-1, 1\}$ are completely determined (See [2], [3]).

PROOF. Let $a=a(-1)$, and $b=a(1)$.

Case $l=-1$. Put $\tilde{\chi}=(\chi-n1_G)(\chi-1_G)$. Since $B(-1)=(2a/|G|)(n+1)>0$, we have $\langle\tilde{\chi}, 1_G\rangle=B(-1)=1$, so $\langle\chi, \chi\rangle+n=1$, a contradiction.

Case $l=1$, i.e. $(G, \chi, 1)$ is sharp. Since $B(1)=(2b/|G|)(1-n)=-1$, $|G|=2b(n-1)$. Put $\tilde{\chi}=(\chi-n1_G)(\chi+1_G)$. Since $B(l)=\langle\tilde{\chi}, 1_G\rangle=-1$, we have $\langle\chi, \chi\rangle=n-1$, $a=n+b$ by the normality of χ. Thus $|G|=1+a+b$ and the above yield $2b(n-1)=n+1+2b$. Hence $n=5$ or 3, $|G|=8$, $\langle\chi, \chi\rangle=4$, or 2 respectively. If $n=3$, χ is the sum of two irreducible characters. However, if $n=5$, χ is the sum of 3 linear characters and the irreducible character of degree 2, which is not sharp but satisfies the hypothesis by looking at the character table.

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Department of Mathematics,
Ibaraki College of Technology
866, Fukayatsu Katsuta
Ibaraki 312
Japan