TSUKUBA J. MATH.
Vol. 14 No. 1 (1990), 39—52

REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE IN TERMS OF HOLOMORPHIC DISTRIBUTION

By

Sadahiro MAEDA and Seiichi UDAGAWA

0. Introduction.

Real hypersurfaces in a complex projective space have been studied by many differential geometers (for example, see [1], [2], [3], [7], [14] and [15]). In this paper, we study real hypersurfaces in $P_n(\boldsymbol{C})$ from the point of view of holomorphic distribution, where $P_n(\boldsymbol{C})$ denotes an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4.

R. Takagi ([13]) showed that all homogeneous real hypersurfaces in $P_n(\boldsymbol{C})$ are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or 2. Namely, he proved the following

THEOREM A ([13]). *Let M be a homogeneous real hypersurface of $P_n(\boldsymbol{C})$. Then M is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere (, that is, a tube over a hyperplane $P_{n-1}(\boldsymbol{C})$),*

(A_2) *a tube over a totally geodesic $P_k(\boldsymbol{C})$ $(1\leqq k\leqq n-2)$,*

(B) *a tube over a complex quadric $\boldsymbol{Q}_{n-1}$,*

(C) *a tube over $P_1(\boldsymbol{C})\times P_{(n-1)/2}(\boldsymbol{C})$ and $n(\geqq 5)$ is odd,*

(D) *a tube over a complex Grassmann $G_{2,5}(\boldsymbol{C})$ and $n=9$,*

(E) *a tube over a Hermitian symmetric space $SO(10)/U(5)$ and $n=15$.*

On the other hand, Kimura ([4], [5]) constructed a certain class of non-homogeneous real hypersurfaces in $P_n(\boldsymbol{C})$, which are called *ruled* real hypersurfaces in $P_n(\boldsymbol{C})$.

Let M be a real hypersurface of $P_n(\boldsymbol{C})$ and denote by TM the tangent bundle of M. Set $\xi=-JN$, where J is the complex structure tensor of $P_n(\boldsymbol{C})$ and N is a local unit normal vector field of M in $P_n(\boldsymbol{C})$. Then we may write as $T_xM=T_x^0M+\boldsymbol{R}\{\xi_x\}$ at any fixed point x of M, where T_x^0M is a J-invariant subspace of T_xM. Let A_2 be the second fundamental form for the subbundle

Received April 18, 1989.

T^0M in $TP_n(\boldsymbol{C})$ over M (see §3), where $TP_n(\boldsymbol{C})$ is the tangent bundle of $P_n(\boldsymbol{C})$. Set $A^0=A_2|_{T^0M}$. Then A^0 may be interpreted as a smooth section of Hom(T^0M, Hom(T^0M, N^0M)), where N^0M is the orthogonal complement of T^0M in $TP_n(\boldsymbol{C})$ with respect to the metric on $TP_n(\boldsymbol{C})$, which is also a subbundle of $TP_n(\boldsymbol{C})$. Each of T^0M and N^0M has a connection induced from $TP_n(\boldsymbol{C})$ and hence Hom(T^0M, Hom(T^0M, N^0M)) has a connection, which is denoted by ∇^0 (cf. [6]).

In Section 3, we show the condition that $\nabla^0_X A^0=0$ for any $X\in T^0M$ implies that either ξ is a principal curvature vector and the shape operator A of M in $P_n(\boldsymbol{C})$ is η-parallel or T^0M is integrable, hence either M is locally a homogeneous real hypersurface of type A_1, A_2 or B, or M is foliated by complex hypersurface of $P_n(\boldsymbol{C})$ with parallel second fundamental form, which is $P_{n-1}(\boldsymbol{C})$ or a complex hyperquadric $\boldsymbol{Q}_{n-1}(\boldsymbol{C})$ by the well-known result of Nakagawa-Takagi ([10]). Moreover, we determine real hypersurfaces M's (in $P_n(\boldsymbol{C})$) which satisfy the condition "T^0M is a curvature invariant subspace of TM and ξ is not a principal curvature vector" by using Kimura's work [4].

In Section 2, we give some characterizations of homogeneous real hypersurfaces of type A_1 and A_2.

1. Preliminaries.

Let M be a real hypersurface of $P_n(\boldsymbol{C})$. In a neighborhood of each point, we choose a unit normal vector field N in $P_n(\boldsymbol{C})$. The Riemannian connections $\tilde{\nabla}$ in $P_n(\boldsymbol{C})$ and ∇ in M are related by the following formulas for arbitrary vector fields X and Y on M:

$$\tilde{\nabla}_X Y=\nabla_X Y+g(AX, Y)N, \tag{1.1}$$

$$\tilde{\nabla}_X N=-AX, \tag{1.2}$$

where g denotes the Riemannian metric of M induced from the Fubini-Study metric G of $P_n(\boldsymbol{C})$ and A is the shape operator of M in $P_n(\boldsymbol{C})$. An eigenvector X of the shape operator A is called a *principal curvature vector*. Also an eigenvalue λ of A is called a *principal curvature*. In what follows, we denote by V_λ the eigenspace of A associated with eigenvalue λ. It is known that M has an almost contact metric structure induced from the complex structure J of $P_n(\boldsymbol{C})$, that is, we define a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η on M by $g(\phi X, Y)=G(JX, Y)$ and $g(\xi, X)=\eta(X)=G(JX, N)$. Then we have

$$\phi^2X=-X+\eta(X)\xi\,, \qquad g(\xi, \xi)=1\,, \qquad \phi\xi=0\,. \tag{1.3}$$

From (1.1), we easily have

(1.4) $$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

(1.5) $$\nabla_X \xi = \phi AX.$$

Let $\tilde{R}$ and R be the curvature tensors of $P_n(\boldsymbol{C})$ and M, respectively. Since the curvature tensor $\tilde{R}$ has a nice form, we have the following Gauss and Codazzi equations:

(1.6) $$\begin{aligned} g(R(X, Y)Z, W) = {} & g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ & + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) \\ & - 2g(\phi X, Y)g(\phi Z, W) + g(AY, Z)g(AX, W) \\ & - g(AX, Z)g(AY, W), \end{aligned}$$

(1.7) $$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi.$$

It is well-known that there does not exist a real hypersurface M of $P_n(\boldsymbol{C})$ satisfying $\nabla A=0$ (, that is, the second fundamental form of M is parallel). Here we recall the following notion: The second fundamental form is called *η-parallel* if $g((\nabla_X A)Y, Z)=0$ for any X, Y and Z which are orthogonal to ξ. We note that the second fundamental form of homogeneous real hypersurfaces of type A_1, A_2, B and ruled real hypersurfaces is η-parallel (cf. Theorem 5). We say that M is a *ruled* real hypersurface if there is a foliation of M by complex hyperplanes $P_{n-1}(\boldsymbol{C})$. More precisely, let T^0M be the distribution defined by $T_x^0M=\{X \in T_xM : X \perp \xi\}$ for $x \in M$. Then T^0M is integrable and its integral manifold is a totally geodesic submanifold $P_{n-1}(\boldsymbol{C})$. In the following, we use the same terminology and notations as above unless otherwise stated. Now we prepare without proof the following in order to prove our Theorems:

THEOREM B ([11], [12]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the following are equivalent*:

(i) *M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.*

(ii) *$L_\xi g=0$, where L is the Lie derivative. Namely, ξ is an infinitesimal isometry.*

(iii) *$\phi A = A\phi$.*

THEOREM C ([5]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the second fundamental form of M is η-parallel and ξ is a principal curvature vector if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 and B.*

THEOREM D ([5]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the second fundamental form of M is η-parallel and the holomorphic distribution $T^0M(=\{X\in TM: X\perp\xi\})$ is integrable if and only if M is locally congruent to a ruled real hypersurface of $P_n(\boldsymbol{C})$.*

PROPOSITION A ([9]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

PROPOSITION B ([9]). *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX=rX$ for $X\perp\xi$, then we have $A\phi X=((\alpha r+2)/(2r-\alpha))\phi X$.*

PROPOSITION C ([9]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the following are equivalent:*

(i) *M is locally congruent to one of homogeneous ones of type A_1 and A_2.*

(ii) *$g((\nabla_X A)Y, Z)=-\eta(Y)g(\phi X, Z)-\eta(Z)g(\phi X, Y)$ for any vector fields X, Y and Z on M.*

PROPOSITION D ([5]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the following are equivalent:*

(i) *The holomorphic distribution $T^0M=\{X\in TM: X\perp\xi\}$ is integrable.*

(ii) *$g((\phi A+A\phi)X, Y)=0$ for any $X, Y\in T^0M$.*

2. Homogeneous real hypersurfaces of type A_1 and A_2.

In this section we provide some characterizations of homogeneous real hypersurfaces of type A_1 and A_2 in $P_n(\boldsymbol{C})$. Motivated by Theorem B, first of all we prove the following

THEOREM 1. *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the following are equivalent:*

(i) *M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.*

(ii) *$L_\xi\phi=0$, that is, ξ is an infinitesimal automorphism of ϕ.*

PROOF. For any $X\in TM$, we have

$$\begin{aligned}(L_\xi\phi)(X)&=[\xi, \phi X]-\phi([\xi, X])\\&=\nabla_\xi(\phi X)-\nabla_{\phi X}\xi-\phi(\nabla_\xi X-\nabla_X\xi)\\&=(\nabla_\xi\phi)X-\nabla_{\phi X}\xi+\phi(\nabla_X\xi)\end{aligned}$$

$$=\eta(X)A\xi-g(A\xi,X)\xi-\phi A\phi X+\phi^2AX \text{ (from (1.4) and (1.5))}$$
$$=\eta(X)A\xi-g(A\xi,X)\xi-\phi A\phi X-AX+\eta(AX)\xi \text{ (from (1.3))}$$
$$=\eta(X)A\xi-\phi A\phi X-AX.$$

Since $(L_\xi\phi)(\xi)=0$, the above calculation asserts that $L_\xi\phi=0$ is equivalent to

$$AX=-\phi A\phi X \quad \text{for any } X(\perp\xi). \tag{2.1}$$

From (1.3) and (2.1) we find

$$\phi AX=A\phi X-\eta(A\phi X)\xi \quad \text{for any } X(\perp\xi). \tag{2.2}$$

Then we see

$$\phi^2AX=-AX+\eta(AX)\xi \text{ (from (1.3))}$$
$$=\phi A\phi X \quad \text{(from (1.3) and (2.2))}$$
$$=-AX \quad \text{(from (2.1))},$$

that is, $\eta(AX)=0$ for any $X(\perp\xi)$ so that ξ is a principal curvature vector. And hence, we get $\eta(A\phi X)=g(A\phi X,\xi)=g(\phi X,A\xi)=0$. Here we suppose that $L_\xi\phi=0$. Then from (2.2) we obtain $\phi AX=A\phi X$ for any $X(\perp\xi)$. Moreover, from the fact that ξ is a principal curvature vector, it follows that $\phi A\xi=A\phi\xi(=0)$. Then "$L_\xi\phi=0$" implies "$\phi A=A\phi$". On the other hand "$\phi A=A\phi$" yields the equation (2.1), that is, "$L_\xi\phi=0$". Therefore by virtue of Theorem B, we get our conclusion. Q. E. D.

Nom let T^0M^C be a complexification of T^0M. Then we have $T^0M^C=T^0M^{(1,0)}\oplus T^0M^{(0,1)}$ with respect to ϕ, where

$$T^0M^{(1,0)}=\{Z\in T^0M^C:\phi Z=\sqrt{-1}Z\}=\{X-\sqrt{-1}\phi X:X\in T^0M\}$$

and

$$T^0M^{(0,1)}=\{Z\in T^0M^C:\phi Z=-\sqrt{-1}Z\}=\{X+\sqrt{-1}\phi X:X\in T^0M\}.$$

We are now in a position to prove the following

THEOREM 2. *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the following are equivalent:*

(i) *M is locally equivalent to one of homogeneous real hypersurfaces of type A_1 and A_2.*

(ii) *ξ is a principal curvature vector and $\nabla_Z\xi$ is a $(0,1)$-vector for any $Z\in T^0M^{(0,1)}$.*

PROOF. For any $Z(=X+\sqrt{-1}\phi X)\in T^0M^{(0,1)}$, from (1.5) we have

$$\nabla_Z\xi=\phi AX+\sqrt{-1}\phi A\phi X\in T^0M^C\,,\qquad \text{where } X\in T^0M. \tag{2.3}$$

(i)⇒(ii): Since $\phi A=A\phi$, ξ is a principal curvature vector. Then from (2.3) we get

$$\begin{aligned}\nabla_Z\xi&=\phi AX+\sqrt{-1}\phi^2AX\\&=\phi AX+\sqrt{-1}(-AX+\eta(AX)\xi)\quad\text{(from (1.3))}\\&=\phi AX-\sqrt{-1}\,AX.\end{aligned}$$

Then we find

$$\begin{aligned}\phi(\nabla_Z\xi)&=\phi(\phi AX-\sqrt{-1}\,AX)\\&=-AX+\eta(AX)\xi-\sqrt{-1}\,\phi AX\\&=-\sqrt{-1}(\phi AX-\sqrt{-1}\,AX),\end{aligned}$$

which shows that $\nabla_Z\xi$ is a (0, 1)-vector with respect to ϕ.

(ii)⇒(i): From (2.3) we have

$$\phi(\nabla_Z\xi)=\phi(\phi AX+\sqrt{-1}\phi A\phi X)=-\sqrt{-1}(\phi AX+\sqrt{-1}\phi A\phi X).$$

This, together with (1.3), shows that

$$\begin{aligned}&-AX+\eta(AX)\xi+\sqrt{-1}(-A\phi X+\eta(A\phi X)\xi)\\&=-\sqrt{-1}\,\phi AX+\phi A\phi X\qquad\text{for any } X(\perp\xi).\end{aligned}\tag{2.4}$$

Since ξ is a principal curvature vector, the equation (2.4) is reduced to $-AX-\sqrt{-1}\,A\phi X=\phi A\phi X-\sqrt{-1}\,\phi AX$ for any $X(\perp\xi)$. Therefore we conclude that $\phi A=A\phi$. Q. E. D.

REMARK 1. Let M be a Kaehler manifold (with complex structure J). Then the following are equivalent:

(i) $L_XJ=0$.

(ii) ∇_ZX is a (0, 1)-vector for any (0, 1)-vector Z.

Motivated by this fact, we established Theorem 2.

Finally we prove the following

PROPOSITION 1. *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Suppose that ξ is a principal curvature vector and the corresponding principal curvature is non-zero. If $\nabla_\xi A=0$, then M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.*

PROOF. By hypothesis we may put $A\xi=\alpha\xi$. Then from Proposition A, (1.3) and (1.5) we have

$$(\nabla_\xi A)\xi=\nabla_\xi(A\xi)-A\nabla_\xi\xi=(\xi\alpha)\xi+\alpha\nabla_\xi\xi=0\,.$$

And hence "$\nabla_\xi A=0$" implies

$$g((\nabla_\xi A)X,\ Y)=0 \quad (\text{for any } X,\ Y\perp\xi)\,. \tag{2.5}$$

On the other hand, for any $X(\in V_r=\{X: AX=rX,\ X\perp\xi\})$ we get

$$\begin{aligned} g((\nabla_\xi A)X,\ Y) &= g((\nabla_X A)\xi+\phi X,\ Y) \quad (\text{from (1.7)}) \\ &= g(\nabla_X(A\xi)-A\nabla_X\xi+\phi X,\ Y) \\ &= g(\alpha\phi AX-A\phi AX+\phi X,\ Y) \quad (\text{from Proposition A and (1.5)}) \\ &= g(\alpha r\phi X-rA\phi X+\phi X,\ Y) \\ &= \left\{r\left(\alpha-\frac{\alpha r+2}{2r-\alpha}\right)+1\right\}g(\phi X,\ Y) \quad (\text{from Proposition B}) \end{aligned}$$

Therefore the equation (2.5) asserts that

$$r\left(\alpha-\frac{\alpha r+2}{2r-\alpha}\right)+1=0\,.$$

Namely we find $\alpha(r^2-\alpha r-1)=0$. Since $\alpha\neq 0$, we have $r^2-\alpha r-1=0$ so that $r(2r-\alpha)=\alpha r+2$, that is, $r=(\alpha r+2)/(2r-\alpha)$. Therefore $\phi V_r=V_r$ so that our real hypersurface M must be locally congruent to one of homogeneous ones of type A_1 and A_2 (cf. [8]). Of course a homogeneous real hypersurface of type A_1 and A_2 satisfies the condition "$\nabla_\xi A=0$" (cf. Proposition C). Q. E. D.

REMARK 2. "$A\xi=0$" implies "$\nabla_\xi A=0$" (see the proof of Proposition 1).

REMARK 3. By an easy calculation we find the following:
$\nabla_\xi\xi=0$ (, that is, ξ is principal)$\Leftrightarrow(\nabla_\xi\phi)X=0$ for any $X\in TM\Leftrightarrow(\nabla_\xi\phi)(\xi)=0$.

3. Main results.

To state our results, we prepare some fundamental equations of subbundles (cf. [6]). Let F be a vector bundle over a Riemannian manifold M. Assume that F has a metric connection. Then any subbundle E of F has an induced metric connection. Denote by ∇^F and ∇^E the connections of F and E, respectively. Then we have

$$\nabla^F_X v=\nabla^E_X v+A(X)(v) \qquad \text{for any } v\in C^\infty(E) \text{ and } X\in TM, \tag{3.1}$$

where A is a Hom(E, $E^{\perp}$)-valued 1-form on M and $E^{\perp}$ is the orthogonal complement of E in F with respect to the metric on F. A is called the *second fundamental form* of subbundle E in F. $E^{\perp}$ is also given a connection induced from F. Denote it by $\nabla^{E^{\perp}}$. Then we see that

$$\nabla_X^F w=\nabla_X^{E^{\perp}} w+B(X)(w) \quad \text{for any } w\in C^{\infty}(E^{\perp}) \text{ and } X\in TM, \tag{3.2}$$

where B is a Hom($E^{\perp}$, E)-valued 1-form on M. It is easily seen that $A=-{}^tB$, where ${}^tB$ is the transpose of B with respect to the metric on F.

Now let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then TM is a subbundle of $TP_n(\boldsymbol{C})$ over M and $T^0M=\{X\in TM: X\perp\xi\}$ is a subbundle of TM. Thus each of TM and T^0M has a metric connection induced from $TP_n(\boldsymbol{C})$. The orthogonal complement of T^0M in $TP_n(\boldsymbol{C})$ with respect to the metric on $TP_n(\boldsymbol{C})$ is denoted by N^0M, which is also a subbundle of $TP_n(\boldsymbol{C})$ with the induced metric connection.

Denote by ∇^0 and $\nabla^{\perp}$ the connections of T^0M and N^0M, respectively. By (3.1) we have

$$\nabla_X Y=\nabla_X^0 Y+A_1(X)(Y) \tag{3.3}$$

$$\tilde{\nabla}_X Y=\nabla_X^0 Y+A_2(X)(Y) \quad \text{for any } Y\in C^{\infty}(T^0M) \text{ and } X\in TM, \tag{3.4}$$

where A_1 and A_2 are the second fundamental forms of the subbundle T^0M in TM and $TP_n(\boldsymbol{C})$, respectively. Note that the second fundamental form of TM in $TP_n(\boldsymbol{C})$ coincides with the ordinary second fundamental form of the immersion $M\to P_n(\boldsymbol{C})$. A_2 is interpreted as a smooth section of Hom(TM, Hom(T^0M, N^0M)). Set $A^0=A_2|_{T^0M}$, which is a smooth section of Hom(T^0M, Hom(T^0M, N^0M)). Note that any ruled real hypersurfaces in $P_n(\boldsymbol{C})$ may be characterized by the condition $A^0\equiv 0$. We here consider the covariant derivative of A^0 with respect to the connection on Hom(T^0M, Hom(T^0M, N^0M)) induced from $TP_n(\boldsymbol{C})$. First of all we show the following fundamental relations.

PROPOSITION 2.

(i) $A_1(X)(Y)=-g(\phi AX, Y)\xi$,

(ii) $A_2(X)(Y)=g(AX, Y)N-g(\phi AX, Y)\xi$,

(iii) $\nabla^0\phi=0$,

(iv) $\nabla_X^{\perp}\xi=g(AX, \xi)N$,

(v) $\nabla_X^{\perp}N=-g(AX, \xi)\xi$,

where $X\in TM$ *and* $Y\in C^{\infty}(T^0M)$.

PROOF. For any $X\in TM$ and $Y\in C^{\infty}(T^0M)$, we have

(i) $g(A_1(X)(Y), \xi)=g(\nabla_X Y, \xi)=-g(Y, \phi AX)$,

(ii) $g(A_2(X)(Y), \xi)=G(\tilde{\nabla}_X Y, \xi)=g(\nabla_X Y, \xi)=-g(Y, \phi AX)$,

$G(A_2(X)(Y), N)=G(\tilde{\nabla}_X Y, N)=g(AX, Y)$,

(iii)
$$\begin{aligned}(\nabla^0_X\phi)(Y)&=\nabla^0_X\phi(Y)-\phi(\nabla^0_X Y)\\&=\nabla_X\phi(Y)-A_1(X)(\phi(Y))-\phi(\nabla_X Y-A_1(X)(Y))\\&=(\nabla_X\phi)(Y)+g(\phi AX, \phi Y)\xi\\&=0,\end{aligned}$$

where we have used (1.1)~(1.5).

(iv) $\tilde{\nabla}_X\xi=\nabla_X\xi+g(AX, \xi)N=\phi AX+g(AX, \xi)N$,

which, together with (3.2), implies $\nabla^{\perp}_X\xi=g(AX, \xi)N$.

(v) $\tilde{\nabla}_X N=-AX$,

which, combined with (3.2), implies $\nabla^{\perp}_X N=-g(AX, \xi)\xi$. Q.E.D.

The connection on $\mathrm{Hom}(T^0M, \mathrm{Hom}(T^0M, N^0M))$ is also denoted by ∇^0. The covariant derivative of A^0 is defined by

$$(\nabla^0_X A^0)(Y)(Z)=\nabla^{\perp}_X A^0(Y)(Z)-A^0(\nabla^0_X Y)(Z)-A^0(Y)(\nabla^0_X Z) \tag{3.5}$$

for any $X\in TM$ and $Y, Z\in C^{\infty}(T^0M)$.

Now we prove

PROPOSITION 3. *For any $X\in TM$ and $Y, Z\in C^{\infty}(T^0M)$,*

$$(\nabla^0_X A^0)(Y)(Z)=\Psi(X, Y, Z)N+\Psi(X, Y, \phi Z)\xi, \tag{3.6}$$

where Ψ is the trilinear tensor defined by

$$\begin{aligned}\Psi(X, Y, Z)=&\,g((\nabla_X A)(Y), Z)-\eta(AX)g(\phi AY, Z)\\&-\eta(AY)g(\phi AX, Z)-\eta(AZ)g(\phi AX, Y).\end{aligned} \tag{3.7}$$

PROOF. We have from Proposition 2

$$\begin{aligned}(\nabla^0_X A^0)(Y)(Z)=&\,\nabla^{\perp}_X A^0(Y)(Z)-A^0(\nabla^0_X Y)(Z)-A^0(Y)(\nabla^0_X Z)\\=&\,\{g(\nabla_X(AY), Z)+g(AY, \nabla_X Z)\}N-\eta(AX)g(AY, Z)\xi\\&-\{g(\nabla_X(\phi AY), Z)+g(\phi AY, \nabla_X Z)\}\xi-\eta(AX)g(\phi AY, Z)N\end{aligned}$$

$$-g(A(\nabla^0_X Y), Z)N+g(\phi A(\nabla^0_X Y), Z)\xi-g(AY, \nabla^0_X Z)N$$
$$+g(\phi AY, \nabla^0_X Z)\xi$$
$$=\{g((\nabla_X A)(Y), Z)-\eta(AY)g(\phi AX, Z)-\eta(AX)g(\phi AY, Z)$$
$$-\eta(AZ)g(\phi AX, Y)\}N+\{-\eta(AX)g(AY, Z)-\eta(AY)g(AX, Z)$$
$$-g(\phi(\nabla_X(AY)), Z)+g(\phi A(\nabla_X Y), Z)-\eta(A\phi Z)g(\phi AX, Y)\}\xi,$$

which implies (3.6). Q.E.D.

Recall the definition of η-parallelity of A. We say that A^0 is *η-parallel* if $\nabla^0_X A^0\equiv 0$ for any $X\in C^\infty(T^0M)$.

The main purpose of this paper is to prove the following

THEOREM 3. *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Assume that A^0 is η-parallel. Then M is locally congruent to one of the following:*

(1) *a homogeneous real hypersurface of type A_1,*

(2) *a homogeneous real hypersurface of type A_2,*

(3) *a homogeneous real hypersurface of type B,*

(4) *a real hypersurface in which T^0M is integrable and its integral manifold is a totally geodesic $P_{n-1}(\boldsymbol{C})$ (, that is, M is a ruled real hypersurface),*

(5) *a real hypersurface in which T^0M is integrable and its integral manifold is a complex quadric $\boldsymbol{Q}_{n-1}$.*

PROOF. By Proposition 3, A^0 is η-parallel if and only if $\Psi(X, Y, Z)=0$ for any $X, Y, Z\in C^\infty(T^0M)$, that is,

$$(3.8)\qquad g((\nabla_X A)(Y), Z)=\eta(AX)g(\phi AY, Z)+\eta(AY)g(\phi AX, Z)$$
$$+\eta(AZ)g(\phi AX, Y)\quad \text{for any } X, Y, Z\in C^\infty(T^0M).$$

Therefore we must study real hypersurfaces (in $P_n(\boldsymbol{C})$) which satisfy the equation (3.8). Since the Codazzi equation (1.7) tells us that $g((\nabla_X A)Y, Z)$ is symmetric for any X, Y and $Z(\in T^0M)$, exchanging X and Y in (3.8), we obtain $g(Y, \phi AX)\eta(AZ)=g(X, \phi AY)\eta(AZ)$ so that

$$(3.9)\qquad \eta(AZ)g((A\phi+\phi A)X, Y)=0\qquad \text{for any } X, Y, Z(\in T^0M).$$

Now we assume that $\eta(AZ)=0$ for any $Z(\in T^0M)$, that is, ξ is a principal curvature vector. Then the equation (3.8) shows that $g((\nabla_X A)Y, Z)=0$ for any $X, Y, Z(\in T^0M)$, that is, the second fundamental form A of M is η-parallel. And hence our real hypersurface M is locally congruent to one of homogeneous ones of type A_1, A_2 and B (cf. Theorem C). Next we assume that ξ is not a principal curvature vector. Then the equation (3.9) tells us that the holomorphic distribution T^0M is integrable (cf. Proposition D). Of course the integral manifold M^0 of T^0M is a complex hypersurface (with complex structure ϕ) in $P_n(\boldsymbol{C})$. Moreover, the second fundamental form A^0 of M^0 is parallel (, which is equivalent to (3.8)). Therefore we conclude that M^0 is locally congruent to $P_{n-1}(\boldsymbol{C})$ or $\boldsymbol{Q}_{n-1}$ (cf. [10]). Q.E.D.

As an immediate consequence of Theorem C and (3.8), we get

THEOREM 4. *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then A^0 is η-parallel and ξ is a principal curvature vector if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 and B.*

In addition, from Theorem C, Theorem D and Theorem 3, we find

THEOREM 5. *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then A^0 is η-parallel and the second fundamental form of M is η-parallel if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 and B or a ruled real hypersurface.*

REMARK 4. We now denote by H the sectional curvature of a holomorphic 2-plane (with respect to ϕ) on a real hypersurface M. Kimura ([4]) determined real hypersurfaces (in $P_n(\boldsymbol{C})$) on which H is constant. He showed the following

THEOREM E ([4]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$ ($n\geqq 3$) on which H is constant. Then M is one of the following*:

(a) *a homogeneous real hypersurface of type A_1 ($H>4$)*,

(b) *a real hypersurface in which T^0M is integrable and its integral manifold is a totally geodesic $P_{n-1}(\boldsymbol{C})$ (, that is, M is a ruled real hypersurface) ($H=4$)*,

(c) *a real hypersurface in which there is a foliation contained in some complex hyperplane $P_{n-1}(\boldsymbol{C})$ as a ruled real hypersurface ($H=4$)*.

Our aim here is to give a characterization of the cases (b), (c) in Theorem E. We prove

PROPOSITION 4. *Let M be a real hypersurface of $P_n(\boldsymbol{C})$ ($n\geqq 3$). If T^0M is a curvature invariant subspace of TM and ξ is not a principal curvature vector, then M is locally congruent to one of the cases* (b), (c) *in Theorem E.*

PROOF. Since $R(T^0M, T^0M)T^0M\subset T^0M$, the equation (1.6) yields

$$0=g(R(X, Y)Z, \xi)$$
$$=g(AY, Z)g(AX, \xi)-g(AX, Z)g(AY, \xi)$$

for any $X, Y, Z\in T^{0}M$ and $\xi=-JN$.

Then we have

$$\eta(AX)\phi AY=\eta(AY)\phi AX \quad \text{for any } X, Y\in T^{0}M. \tag{3.10}$$

We here consider a linear transformation $\phi A: T^{0}M\to T^{0}M$. Note that

$$\operatorname{rank}(\phi A)\leqq 1 \quad \text{at each point of } M. \tag{3.11}$$

(Suppose that $\operatorname{rank}(\phi A)\geqq 2$ at a certain point x of M. Then there exist $X, Y\in T_{x}^{0}M$ such that

$$\phi AX\neq 0,\quad \phi AY\neq 0 \quad \text{and} \quad g(\phi AX, \phi AY)=0. \tag{3.12}$$

So from (3.10) and (3.12) we see

$$\eta(AX)=0. \tag{3.13}$$

It follows from (3.10) and (3.13) that

$$\eta(AY)=0 \quad \text{for any } Y(\perp X). \tag{3.14}$$

Therefore, from (3.13) and (3.14) we find that ξ is a principal curvature vector at x, which is a contradiction.)

Then (3.11) asserts that the Gauss equation (1.6) is reduced to

$$g(R(X, Y)Z, W)=g(Y, Z)g(X, W)-g(X, Z)g(Y, W)+g(\phi Y, Z)g(\phi X, W)$$
$$-g(\phi X, Z)g(\phi Y, W)-2g(\phi X, Y)g(\phi Z, W),$$

that is,

$$R(X, Y)Z=g(Y, Z)X-g(X, Z)Y+g(\phi Y, Z)\phi X-g(\phi X, Z)\phi Y$$
$$-2g(\phi X, Y)\phi Z \quad \text{for any } X, Y, Z\in T^{0}M.$$

Then we conclude that our real hypersurface M satisfies that $H=4$. Therefore Theorem E tells us that M is locally congruent to one of the cases (b), (c). Of course the cases (b), (c) satisfy the hypothesis of Proposition 4. Q. E. D.

We here provide a geometric meaning of the condition "the second fundamental form of M is η-parallel". The following is due to Nakagawa.

PROPOSITION 5. *Let M be a real hypersurface of $P_n(C)$. Then the following are equivalent:*

(i) *The second fundamental form of M is η-parallel.*

(ii) *Every geodesic $\gamma=\gamma(t)$ $(t\in I)$ of M such that $\gamma'(t)$ is orthogonal to ξ (for any $t\in I$), considered as a curve in $P_n(\boldsymbol{C})$, has constant first curvature along γ.*

PROOF. We find that the condition (ii) is equivalent to $g((\nabla_X A)X, X)=0$ for any $X(\in T^0M)$. On the other hand, the Codazzi equation shows that $g((\nabla_X A)Y, Z)$ is symmetric for any X, Y and $Z(\in T^0M)$. And hence the condition (i) is equivalent to the condition (ii). Q.E.D.

REMARK 5. The first author ([8]) proved the following:

Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then every geodesic γ of M, considered as a curve in $P_n(\boldsymbol{C})$, has constant first curvature along γ if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.

REMARK 6. The authors do not know how to construct a real hypersurface M with $M^0=\boldsymbol{Q}_{n-1}$ (, that is, M is of case (5) in Theorem 3).

References

[1] Cecil, T.E. and Ryan, P.J., Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. **269** (1982), 481-499.

[2] Ki, U.H., Nakagawa, H. and Suh, Y.J., Real hypersurfaces with harmonic Weyl tensor of a complex space form, a preprint.

[3] Kimura, M., Real hypersurfaces and complex submanifolds in complex projective space, Trans. Amer. Math. Soc. **296** (1986), 137-149.

[4] Kimura, M., Sectional curvatures of holomorphic planes on a real hypersurface in $P^n(\boldsymbol{C})$, Math. Ann. **276** (1987), 487-497.

[5] Kimura, M. and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. **202** (1989), 299-311.

[6] Kobayashi, S., Differential geometry of complex vector bundles, Publ. Math. Soc. Japan 15, Iwanami Shoten, Publ. and Princeton Univ. Press, 1987.

[7] M. Kon, Pseudo-Einstein real hypersufaces in complex space form, J. Diff. Geom. **14** (1979), 339-354.

[8] Maeda, S., Real hypersurfaces of complex projective spaces, Math. Ann. **263** (1983), 473-478.

[9] Maeda, Y., On real hypersurfaces of a complex projective space, J. Math. Soc. Japan **28** (1976), 529-540.

[10] Nakagawa, H. and Takagi, R., On locally symmetric Kaehler submanifolds in a complex projective space, J. Math. Soc. Japan **28** (1976), 638-667.

[11] Okumura, M., On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. **212** (1975), 355-364.

[12] Okumura, Compact real hypersurfaces of a complex projective space, J. Diff. Geom. **12** (1977), 595-598.

[13] Takagi, R., On homogeneous real hypersurfaces in a complex projective space,

Osaka, J. Math. 10 (1973), 495-506.
[14] Takagi, R., Real hypersurfaces in a complex projective space with constant principal curvatures I, II, J. Math. Soc. Japan 27 (1975), 43-53, 507-516.
[15] Udagawa, S., Bi-order real hypersurfaces in a complex projective space, Kodai Math. J. 10 (1987), 182-196.

Sadahiro Maeda
Department of Mathematics
Kumamoto Institute of Technology
Ikeda 4-22-1
Kumamoto 860, Japan

Seiichi Udagawa
Department of Mathematics
School of Medicine
Nihon University
Itabashi, Tokyo 173, Japan