TSUKUBA J. MATH.
Vol. 13 No. 2 (1989), 505—512

COMPACTNESS CRITERIA FOR RIEMANNIAN MANIFOLDS WITH COMPACT UNSTABLE MINIMAL HYPERSURFACES

By

Kazuo AKUTAGAWA

1. Introduction

In this paper, we shall prove the following Theorem.

THEOREM A. *Let N be a complete Riemannian manifold with a compact embedded unstable minimal hypersurface M. Suppose that there exists a positive constant s_0 such that along each unit speed geodesic $\gamma: [0, \infty)\to N$ emanating from each point in the tubular neighborhood $U_{s_0}(M):=\{q\in N; \mathrm{dist}_N(q, M)<s_0\}$ the Ricci curvature satisfies*

$$\liminf_{r\to\infty}\int_0^r \mathrm{Ric}_N(d\gamma/dt,\ d\gamma/dt)dt\geqq 0.$$

Then N is compact.

The Myers' theorem [11] is one of the most well-known results relating the curvature and the topology of a complete Riemannian manifold N, which states that if the Ricci curvature has a positive lower bound then N is compact. In [1], Ambrose proved a generalization of Myers' theorem, that is, if there is a point $q\in N$ such that along each unit speed geodesic $\gamma: [0, \infty)\to N$ emanating from q the Ricci curvature satisfies

$$\int_0^\infty \mathrm{Ric}_N(d\gamma/dt,\ d\gamma/dt)dt=+\infty$$

then N is compact. It should be pointed out that in this result the Ricci curvature is not required to be everywhere nonnegative. Further developments can be found in Galloway [9] and different sorts of extensions of Myers' theorem can be found in Avez [3], Calabi [5] and Shiohama [12].

Theorem A is an Ambrose-type theorem for Riemannian manifolds with compact embedded unstable hypersurfaces (see also Remark in section 3). It should be also pointed out that in Theorem A the existence of the global unit normal vector field on M is not required.

Received February 12, 1988. Revised November 16, 1988.

The author would like to express his sincere gratitude to the referee for kind advices.

§ 2. Definitions and formulas

Let $N=(N, g)$ be a complete Riemannian manifold of dimension $n\geqq 2$ with a compact embedded hypersurface M. We choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ in N such that, restricted to M, the vectors $\{e_1, \cdots, e_{n-1}\}$ are tangent to M. Let denote the Levi-Civita connection of N by ∇, the component normal to M by $(\cdot)^{\perp}$ and the restriction of e_n to M by ν. The second fundamental form A_M of M is defined by

$$A_M(X, Y)\nu=(\nabla_X Y)^{\perp},$$

where X and Y are local vector fields on M. M is called *minimal* if $H_M=\mathrm{Trace}\, A_M$ is identically zero.

We shall derive the equation $H_M=0$ by another elegant way. For a smooth function $f\in C_0^{\infty}(\mathcal{D}(\nu))$ with compact support in $\mathcal{D}(\nu)$ and a small positive constant δ, let $\{M(\varepsilon f\,;\nu)\}_{\varepsilon\in(-\delta,\delta)}$ denote the one-parameter family of hypersurfaces $\{S(\varepsilon f\,;\nu)\cup\{M-\mathcal{D}(\nu)\}\}_{\varepsilon\in(-\delta,\delta)}$, where $\mathcal{D}(\nu)$ is the domain of ν and $S(\varepsilon f\,;\nu)=\{\exp_x \varepsilon f(x)\nu\in N;\, x\in\mathcal{D}(\nu)\}$. We then get a local deformation $\{M(\varepsilon f\,;\nu)\}_{\varepsilon\in(-\delta,\delta)}$ of M. Let $\mathcal{A}(\cdot)$ denote the $(n-1)$-dimensional area functional of hypersurfaces. Then $\mathcal{A}(M(\varepsilon f\,;\nu))$ is class of C^{∞} with respect to ε and

$$\frac{d}{d\varepsilon}\mathcal{A}(M(\varepsilon f\,;\nu))\Big|_{\varepsilon=0}=-\int_M f\cdot H_M dv_g,$$

where dv_g is the induced volume element of M. If M is a critical point of $\mathcal{A}$, then $H_M=0$.

Suppose that M is minimal. Then

$$(1)\qquad \frac{d^2}{d\varepsilon^2}\mathcal{A}(M(\varepsilon f\,;\nu))\Big|_{\varepsilon=0}=\int_M[\,|\nabla^M f|^2-(\mathrm{Ric}_N(\nu, \nu)+|A_M|^2)f^2]dv_g,$$

where $\nabla^M f=\sum_{i=1}^{n-1}e_i(f)\cdot e_i$ and $|A_M|^2=\sum_{i=1}^{n-1}[A_M(e_i, e_i)]^2$. M is called *unstable* if there exist a local unit normal vector field ν on M and a smooth function $f\in C_0^{\infty}(\mathcal{D}(\nu))$ such that

$$\frac{d^2}{d\varepsilon^2}\mathcal{A}(M(\varepsilon f\,;\nu))\Big|_{\varepsilon=0}<0.$$

For later references, we also give the second variational formula of arc length functional of rays with respect to special variations. Let $\gamma:[0, \infty)\to N$ be a ray satisfying $\gamma(0)\in M$ and $\mathrm{dist}_N(M, \gamma(t))=\mathrm{dist}_N(\gamma(0), \gamma(t))\ (=t)$ for all $t\geqq 0$. Let $\mathcal{L}(\cdot)$ denote the arc length functional. We note that for each $r>0$ $\gamma|_{[0,r]}$

is a critical point of $\mathcal{L}$. Choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ in N around $\gamma(0)$ such that, restricted to M, the vectors $\{e_1, \cdots, e_{n-1}\}$ are tangent to M and the vector $\nu=e_n|_M$ satisfies $\nu(\gamma(0))=(d\gamma/dt)(0)$. Let $\gamma_{i,r}$: $[0, r]\times(-\delta, \delta)\to N$ be a variation of $\gamma|_{[0,r]}$ satisfying $\gamma_{i,r}(\{0\}\times(-\delta, \delta))\subset M$, $\gamma_{i,r}(\{r\}\times(-\delta, \delta))=\gamma(r)$ and $\frac{\partial}{\partial\varepsilon}\gamma_{i,r}(t, \varepsilon)\Big|_{\varepsilon=0}=\cos\frac{\pi t}{2r}\cdot e_i(t)$, where each $e_i(t)$ is the parallel translate vector of $e_i(\gamma(0))$ along γ. We then obtain (cf. [4, Chapter 11])

$$(2)\qquad \frac{d^2}{d\varepsilon^2}\sum_{i=1}^{n-1}\mathcal{L}(\gamma_{i,r}([0, r]\times\{\varepsilon\}))\Big|_{\varepsilon=0}$$

$$=(n-1)\pi^2/8r-\int_0^r \mathrm{Ric}_N(d\gamma/dt, d\gamma/dt)\left(\cos\frac{\pi t}{2r}\right)^2 dt-H_M(\gamma(0)),$$

where H_M is the mean curvature of M with respect to ν.

§3. Proof of Theorem A

Theorem A is an immediate consequence of the following.

THEOREM B. *Let $N=(N, g)$ be a complete Riemannianman ifold with a compact embedded unstable minimal hypersurface M. Suppose that there exist positive constants s_0 and θ such that along each unit speed geodesic $\gamma: [0, \infty)\to N$ satisfying $\gamma(0)\in M$ and $|g((d\gamma/dt)(0), V)|\geqq 1-\theta$, the Ricci curvature satisfies*

$$(3)\qquad \liminf_{r\to\infty}\int_s^r \mathrm{Ric}_N(d\gamma/dt, d\gamma/dt)dt\geqq 0$$

for all $0\leqq s<s_0$, where V is a unit vector normal to M at $\gamma(0)$. Then N is compact.

To prove Theorem B, we will suppose that N is noncompact and, finally, lead a contradiction.

Since N is noncompact, there exists a ray $\gamma: [0, \infty)\to N$ satisfying $\gamma(0)\in M$ and

$$(4)\qquad \mathrm{dist}_N(M, \gamma(t))=\mathrm{dist}_N(\gamma(0), \gamma(t))=t$$

for all $t\geqq 0$.

From the unstability of M, we will first construct C^0-hypersurfaces $\{M(\varepsilon u; \bar{\nu})\}_{\varepsilon\in(0,\sigma)}$ near M, which are smooth and have positive mean curvature around $\gamma\cap M(\varepsilon u; \bar{\nu})$.

LEMMA 1. *There exist a continuous nonnegative function $u\in C(M)$, a local unit normal vector field $\bar{\nu}$ on M and a positive constant σ such that*

(i) $\gamma(0)\in\mathscr{D}(\bar{\nu})=\{x\in M;\ u(x)>0\}$,

(ii) *u is smooth in* $\mathscr{D}(\bar{\nu})$,

(iii) $M(\varepsilon u;\bar{\nu})\subset U_{s_0}(M)$,

(iv) $H_{M(\varepsilon u;\bar{\nu})}>0$ *in* $\{\exp_x t\bar{\nu}\in N;\ x\in W,\ 0\leqq t<s_0\}$

for all $\varepsilon(0<\varepsilon<\sigma)$, *where* $W=\{x\in M; u(x)>\frac{1}{2}u(\gamma(0))\}\subset\mathscr{D}(\bar{\nu})$.

PROOF. From the unstability of M, there exist a local unit normal vector field $\tilde{\nu}$ on M and a function $f\in C_0^\infty(\mathscr{D}(\tilde{\nu}))$ such that

(5)
$$\frac{d^2}{d\varepsilon^2}\mathcal{A}(M(\varepsilon f;\tilde{\nu}))\Big|_{\varepsilon=0}<0.$$

We may assume that the closure $\bar{\mathscr{D}}(\tilde{\nu})$ is contained in a coordinate neighborhood of M. Let ν be a local unit normal vector field on M around $\gamma(0)$ satisfying $(d\gamma/dt)(0)=\nu(\gamma(0))$. Replacing $\tilde{\nu}$ by $-\tilde{\nu}$ if necessary, we can choose a local unit normal vector field $\bar{\nu}$ on M, which is an extension of $\tilde{\nu}$, ν and satisfies that $\mathscr{D}(\bar{\nu})$ is connected with C^∞-boundary $\partial\mathscr{D}(\bar{\nu})$.

Consider the functional

$$I_{\bar{\nu}}(\phi)=\int_M[|\nabla^M\phi|^2-(\mathrm{Ric}_N(\bar{\nu},\bar{\nu})+|A_M|^2)\phi^2]dv_g$$

and define $\lambda=\inf I_{\bar{\nu}}(\phi)$ for all $\phi\in C_0^\infty(\mathscr{D}(\bar{\nu}))$ satisfying $\phi=0$ on $M-\mathscr{D}(\bar{\nu})$ and $\int_M\phi^2dv_g=1$. From (1) and (5) we then obtain a continuous function $u\in C(M)$ satisfying $\lambda=I_{\bar{\nu}}(u)<0$, which u has the following properties (cf. [2], [7] and [8])

(6) $u>0$ in $\mathscr{D}(\bar{\nu})$ and $u|_{\partial\mathscr{D}(\bar{\nu})}=0$,

(7) u is smooth in $\mathscr{D}(\bar{\nu})$,

(8) $Lu:=-\Delta_M u-(\mathrm{Ric}_N(\bar{\nu},\bar{\nu})+|A_M|^2)u=\lambda u\ (<0)$ in $\mathscr{D}(\bar{\nu})$,

where $\Delta_M u=\sum_{i=1}^{n-1}g(e_i,\nabla_{e_i}\nabla^M u)$. In particular, the property (6) is an immediate consequence of Courant's nodal domain theorem for the linear elliptic operator of second order L (cf. [6, Chapter 1], [7, VI-§6]). From (6)-(8) and an easy calculation we obtain

(9)
$$\frac{\partial}{\partial\varepsilon}H_{M(\varepsilon u;\bar{\nu})}\Big|_{\varepsilon=0}=\Delta_M u+(\mathrm{Ric}_N(\bar{\nu},\bar{\nu})+|A_M|^2)u=-\lambda u>0\quad\text{in}\quad\mathscr{D}(\bar{\nu}).$$

It follows from (6), (7) and (9) that there exists a positive constant σ such that for any $\varepsilon(0<\varepsilon<\sigma)$ $M(\varepsilon u;\bar{\nu})\subset U_{s_0}(M)$ and $H_{M(\varepsilon u;\bar{\nu})}=\int_0^\varepsilon\Big(\frac{\partial}{\partial\rho}H_{M(\rho u;\bar{\nu})}\Big|_{\rho=s}\Big)ds>0$ in $\{\exp_x t\bar{\nu}\in N;\ x\in W,\ 0\leqq t<s_0\}$. This completes the proof of Lemma 1.

LEMMA 2. *There exist positive constants* $\varepsilon_0(0<\varepsilon_0<\sigma)$, $t_0(0<t_0<s_0)$ *and a unit*

speed geodesic $\bar{\gamma}:[0,\infty)\to N$ *such that*

(i) $\bar{\gamma}(t_0)\in M(\varepsilon_0 u;\bar{\nu})\cap\{\exp_x t\bar{\nu}\in N;\ x\in W,\ 0\leqq t<s_0\}$,

(ii) $\bar{\gamma}(0)\in W\subset\mathcal{D}(\bar{\nu})$,

(iii) $g((d\bar{\gamma}/dt)(0),\ \bar{\nu}(\bar{\gamma}(0)))\geqq 1-\theta$,

(iv) $\mathrm{dist}_N(M(\varepsilon_0 u;\bar{\nu}),\ \bar{\gamma}(t))=\mathrm{dist}_N(\bar{\gamma}(t_0),\ \bar{\gamma}(t))=t-t_0$ *for all* $t\geqq t_0$.

PROOF. Take $\varepsilon(0<\varepsilon<\sigma)$ arbitrarily and fix it. For each $i\in\boldsymbol{N}$, there exists a minimizing geodesic $\gamma_{\varepsilon,i}$, emanating from $M(\varepsilon u;\bar{\nu})$, between $M(\varepsilon u;\bar{\nu})$ and $\gamma(i)$. Put $\widetilde{W}=\{x\in M;\ u(x)\geqq u(\gamma(0))\}\subset W\subset\mathcal{D}(\bar{\nu})$. Suppose that there exists $j_1\in\boldsymbol{N}$ such that

$$\gamma_{\varepsilon,j_1}(0)\notin M(\varepsilon u;\bar{\nu})\cap\{\exp_x t\bar{\nu}\in N;\ x\in\widetilde{W},\ 0\leqq t<s_0\}. \tag{10}$$

From (4), (10) and Lemma 1-(iii) we have

$$\begin{aligned}\mathrm{dist}_N(M,\ \gamma(j_1))&\leqq\mathrm{dist}_N(M,\ \gamma_{\varepsilon,j_1}(0))+\mathcal{L}(\gamma_{\varepsilon,j_1})\\&<\mathcal{L}(\gamma|_{[0,j_1]})=\mathrm{dist}_N(M,\ \gamma(j_1)).\end{aligned}$$

This is a contradiction. Then we obtain for all $i\in\boldsymbol{N}$

$$\gamma_{\varepsilon,i}(0)\in M(\varepsilon u;\bar{\nu})\cap\{\exp_x t\bar{\nu}\in N;\ x\in\widetilde{W},\ 0\leqq t<s_0\}\subset U_{s_0}(M). \tag{11}$$

We also note that for each $i\in\boldsymbol{N}$ the vector $(d\gamma_{\varepsilon,i}/dt)(0)$ is perpendicular to $TM(\varepsilon u;\bar{\nu})$ and

$$\gamma_{\varepsilon,i}\cap M(\varepsilon u;\bar{\nu})=\{\gamma_{\varepsilon,i}(0)\}. \tag{12}$$

Suppose that there exists $j_2\in\boldsymbol{N}$ such that

$$g((d\gamma_{\varepsilon,j_2}/dt)(0),\ (d(\exp t\bar{\nu})/dt)(\gamma_{\varepsilon,j_2}(0)))<0.$$

From (11) and (12) that there exists $c(0<c<\mathcal{L}(\gamma_{\varepsilon,j_2}))$ such that

$$\gamma_{\varepsilon,j_2}(c)\in\widetilde{W}\cup\{\exp_x t\bar{\nu}\in N;\ x\in\partial\widetilde{W},\ 0\leqq t<\varepsilon u(\gamma(0))\}. \tag{13}$$

It then follows from (4), (11) and (13) that

$$\begin{aligned}\mathrm{dist}_N(M,\ \gamma(j_2))&\leqq\mathrm{dist}_N(M,\ \gamma_{\varepsilon,j_2}(c))+\mathcal{L}(\gamma_{\varepsilon,j_2}|_{[c,\mathcal{L}(\gamma_{\varepsilon,j_2})]})\\&<\mathrm{dist}_N(M,\ \gamma_{\varepsilon,j_2}(c))+\mathcal{L}(\gamma_{\varepsilon,j_2})\\&<\mathcal{L}(\gamma|_{[0,j_2]})=\mathrm{dist}_N(M,\ \gamma(j_2)).\end{aligned}$$

This is a contradiction, too. Then we obtain for all $i\in\boldsymbol{N}$

$$g((d\gamma_{\varepsilon,i}/dt)(0),\ (d(\exp t\bar{\nu})/dt)(\gamma_{\varepsilon,i}(0)))\geqq 0. \tag{14}$$

Let $v_\varepsilon\in\{v\in TM(\varepsilon u;\bar{\nu})^\perp;\ \|v\|=1\}$ be an accumulation point of the sequence

$\{(d\gamma_{\varepsilon,i}/dt)(0)\}_{i\in N}$. Let γ_ε: $[0, \infty)\to N$ be the geodesic such that $\gamma_\varepsilon(0)=\mathcal{P}(v_\varepsilon)$ and $(d\gamma_\varepsilon/dt)(0)=v_\varepsilon$, where $\mathcal{P}$: $TN\to N$ is the bundle projection. Then γ_ε is a ray satisfying

$$\text{dist}_N(M(\varepsilon u\,;\bar{\nu}), \gamma_\varepsilon(t))=\text{dist}_N(\gamma_\varepsilon(0), \gamma_\varepsilon(t)) \tag{15}$$

for all $t\geqq 0$. We say that γ_ε is a *limit ray* of the sequence of minimizing geodesics $\{\gamma_{\varepsilon,i}\}_{i\in N}$. It then follows from (11) and (14) that

$$\gamma_\varepsilon(0)\in M(\varepsilon u\,;\bar{\nu})\cap\{\exp_x t\bar{\nu}\in\widetilde{W},\ 0\leqq t<s_0\}, \tag{16}$$

$$g((d\gamma_\varepsilon/dt)(0), (d(\exp t\bar{\nu})/dt)(\gamma_\varepsilon(0)))\geqq 0. \tag{17}$$

Let $\tilde{\gamma}$ be a limit ray of the sequence of rays $\{\gamma_{1/i}\}_{i\geq i_0}$, where $1/i_0<\sigma$. It then follows from (15)-(17) that

$$\tilde{\gamma}(0)\in\widetilde{W}\subset W\subset\mathcal{D}(\bar{\nu}) \tag{18}$$

$$g((d\tilde{\gamma}/dt)(0), \bar{\nu}(\tilde{\gamma}(0)))\geqq 0, \tag{19}$$

$$\text{dist}_N(M, \tilde{\gamma}(t))=\text{dist}_N(\tilde{\gamma}(0), \tilde{\gamma}(t)) \tag{20}$$

for all $t\geqq 0$. Also from (19) and (20) $(d\tilde{\gamma}/dt)(0)=\bar{\nu}(\tilde{\gamma}(0))$ and then

$$g((d\tilde{\gamma}/dt)(0), \bar{\nu}(\tilde{\gamma}(0)))=1. \tag{21}$$

By the construction of $\tilde{\gamma}$, (18) and (21) there exists a positive constant $\varepsilon_0(\varepsilon_0=1/i, i\geqq i_0)$ such that

$$s_0>t_0:=\inf\{t>0\,;\gamma_{\varepsilon_0}^{-1}(t)\in W\}, \tag{22}$$

$$|g((d\gamma_{\varepsilon_0}^{-1}/dt)(t_0), \bar{\nu}(\gamma_{\varepsilon_0}^{-1}(t_0)))|\geqq 1-\theta, \tag{23}$$

where $\gamma_{\varepsilon_0}^{-1}(t)=\exp_{\gamma_{\varepsilon_0}(0)}(-t(d\gamma_{\varepsilon_0}/dt)(0))$.

Let $\bar{\gamma}$: $[0, \infty)\to N$ be the geodesic such that

$$\bar{\gamma}(t)=\begin{cases}\gamma_{\varepsilon_0}^{-1}(t_0-t) & \text{if}\quad 0\leqq t\leqq t_0\\ \gamma_{\varepsilon_0}(t-t_0) & \text{if}\quad t\geqq t_0.\end{cases}$$

It then follows from (15), (16), (22) and (23) that $\bar{\gamma}$ satisfies the properties (i)-(iv). This completes the proof of Lemma 2.

Let $\{\bar{e}_1, \cdots, \bar{e}_{n-1}\}$ be a local orthonormal frame field on $M(\varepsilon_0 u\,;\bar{\nu})$ around $\bar{\gamma}(t_0)$ and each $\bar{e}_i(t)$ be the parallel translate vector of $\bar{e}_i(\bar{\gamma}(t_0))$ along $\bar{\gamma}$ with the initial condition $\bar{e}_i(t_0)=\bar{e}_i(\bar{\gamma}(t_0))$. Let $\bar{\gamma}_{i,r}$: $[0, r]\times(-\delta, \delta)\to N$ be a variation of $\bar{\gamma}|_{[t_0,t_0+r]}$ satisfying $\bar{\gamma}_{i,r}(\{0\}\times(-\delta, \delta))\subset M(\varepsilon_0 u\,;\bar{\nu})$, $\bar{\gamma}_{i,r}(\{r\}\times(-\delta, \delta))=\bar{\gamma}(t_0+r)$ and $(\partial\bar{\gamma}_{i,r}/\partial\varepsilon)(t, \varepsilon)|_{\varepsilon=0}=\cos\dfrac{\pi t}{2r}\cdot\bar{e}_i(t_0+t)$. From (2) we then obtain

$$(24)\quad \frac{d^2}{d\varepsilon^2}\sum_{i=1}^{n-1}\mathcal{L}(\bar{\gamma}_{i,r}([0,r]\times\{\varepsilon\}))\Big|_{\varepsilon=0}$$

$$=(n-1)\pi^2/8r-\int_{t_0}^{t_0+r}\mathrm{Ric}_N(d\bar{\gamma}/dt,\ d\bar{\gamma}/dt)\left(\cos\frac{\pi(t-t_0)}{2r}\right)^2 dt-H_{M(\varepsilon_0 u;\,\bar{\nu})}(\bar{\gamma}(t_0)).$$

It follows from (3), (24), Lemma 1, Lemma 2 and Lemma 3 below that there exists a large constant r_0 such that

$$(n-1)\pi^2/8r_0-\int_{t_0}^{t_0+r_0}\mathrm{Ric}_N(d\bar{\gamma}/dt,\ d\bar{\gamma}/dt)\left(\cos\frac{\pi(t-t_0)}{2r_0}\right)^2 dt$$
$$-H_{M(\varepsilon_0 u;\,\bar{\nu})}(\bar{\gamma}(t_0))<0.$$

This contradicts that $\bar{\gamma}|_{[t_0,\infty)}$ is a ray. This completes the proof of Theorem B.

LEMMA 3. *For each constant* K

$$\liminf_{r\to\infty}\int_0^r \mathrm{Ric}_N(d\bar{\gamma}/dt,\ d\bar{\gamma}/dt)dt\geqq K$$

implies

$$\liminf_{r\to\infty}\int_0^r \mathrm{Ric}_N(d\bar{\gamma}/dt,\ d\bar{\gamma}/dt)\left(\cos\frac{\pi t}{2r}\right)^2 dt\geqq K.$$

COROLLARY. *Let N be a complete Riemannian manifold of nonnegative Ricci curvature with a compact embedded minimal hypersurface M. Suppose that either*

(i) *M is unstable in N or*

(ii) *$(N-M)$ is connected.*

Then N is compact. In the case (ii) *it is also established that $(N-M)$ is isometric to a product Riemannian manifold $M\times(0, l)$, where l is a suitable positive constant.*

PROOF. In the case (ii), Corollary was proved by Ichida [10].

REMARK. Without the unstability of M it follows immediately from (2) and Lemma 3 that

"Let N be a complete Riemannian manifold with a compact embedded minimal hypersurface M. Suppose that along each unit speed geodesic $\gamma:[0,\infty)\to N$ emanating perpendicularly from each point in M the Ricci curvature satisfies

$$\liminf_{r\to\infty}\int_0^r \mathrm{Ric}_N(d\gamma/dt,\ d\gamma/dt)dt>0.$$

Then N is compact."

References

[1] Ambrose, W., A theorem of Myers, Duke Math. J., **24** (1957), 345-348.

[2] Aubin, T., Nonlinear Analysis on Manifolds, Monge-Ampère Equations, Springer, 1982.

[3] Avez, A., Riemannian manifolds with non-negative Ricci curvature, Duke Math. J., **39** (1972), 55-64.

[4] Bishop R. and Crittenden R., Geometry of Manifolds, Academic Press, 1964.

[5] Calabi, E., On Ricci curvature and geodesics, Duke Math. J., **34** (1967), 667-676.

[6] Chavel, I., Eigenvalues in Riemannian Geometry, Academic Press, 1984.

[7] Courant, R. and Hilbert D., Methods of Mathematical Physics, Vol. I, J. Wiley & Sons, 1953.

[8] Friedman, A., Partial Differential Equations, Holt, Rinehart and Winston, New York, 1969.

[9] Galloway, G. J., Compactness criteria for Riemannian manifolds, Proc. Amer. Math. Soc., **84** (1982), 106-110.

[10] Ichida, R., Riemannian manifolds with compact boundary, Yokohama Math. J., **29** (1981), 169-177.

[11] Myers, S. B., Riemannian manifolds with positive mean curvature, Duke Math. J., **8** (1941), 401-404.

[12] Shiohama, K., An extension of a theorem of Myers, J. Math. Soc. Japan, **27** (1975), 561-569.

Department of Mathematics
Miyakonojo National College of Technology
Miyazaki 885, Japan