

KILLING VECTOR FIELDS AND THE HOLONOMY ALGEBRA IN SEMIRIEMANNIAN MANIFOLDS

By

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Abstract In this paper we generalize some results of Kostant [2] to semiriemannian manifolds of signature s . We also prove that any Killing vector field on a semiriemannian homogeneous compact flat manifold is parallel.

0. Introduction.

Let (M_s^n, g) be a semiriemannian manifold of dimension n and signature s . Let X be a Killing vector field on M . The A_X -operator provides a skew symmetric endomorphism of TM . It is well known that

$$\nabla_Y A_X = R_{XY}.$$

This fact and the Ambrose-Singer theorem (Wo) show that the A_X -operator lies infinitesimally in the holonomy algebra \mathfrak{h} of M .

We ask ourselves whether or not A_X lies in \mathfrak{h} .

In the riemannian case the question has an affirmative answer on compact manifolds [2]. We obtain here a similar result in the semiriemannian case.

Finally we study the holonomicity of a Killing vector field on semiriemannian manifolds of constant curvature. If the curvature is non zero, the holonomy algebra can be represented as $\mathfrak{po}(n, s)$, that is the skew symmetric endomorphisms of TM . In this case each Killing vector field is holonomic.

There are flat manifolds and Killing vector fields on them such that the A_X -operator does not lie in the holonomy algebra, that is $A_X \notin \mathfrak{h}$. Take, for instance, R_s^n . In the usual coordinates on R_s^n , X is a Killing vector field if

$$X = \sum_{i,j} \varepsilon_i K_i^j x_i \frac{\partial}{\partial x_j}$$

where $K_i^j = -K_j^i$ are constants, $\varepsilon_i = g(\partial/\partial x_i, \partial/\partial x_i) = \pm 1$ and $x_0 = 1$. There are nonholonomic Killing vector fields on R_s^n : nonparallel vector fields are nonholo-

monic because flatness implies $h=0$.

However, the assumption of compactness and homogeneity of M allows us to state that any Killing vector field on a compact homogeneous semiriemannian flat manifold is parallel.

1. Main theorem.

Let (M_s^n, g) be a semiriemannian manifold of dimension n and signature s . If $p \in M$ and $A, B \in \text{End}(T_p M)$ we denote by ϕ the trace form

$$\phi(A, B) = -\text{trace}(A \cdot B).$$

Note that,

- i) ϕ is nondegenerate on $\mathfrak{po}(n, s)$,
- ii) ϕ is parallel.

From now on for any $\Omega \subset \mathfrak{po}(n, s)$, Ω^\perp will denote its orthogonal complementary with respect to ϕ .

THEOREM 1. *Let (M_s^n, g) be a semiriemannian manifold compact orientable manifold and X a Killing vector field on M . If ϕ is nondegenerate on the holonomy algebra \mathfrak{h} , then the A_X -operator decomposes as*

$$A_X = h + B_X$$

where $h \in \mathfrak{h}$, $B_X \in \mathfrak{h}^\perp$ and $\phi(B_X, B_X) = 0$.

PROOF. The nondegenerate character of ϕ on \mathfrak{h} allows us to decompose

$$\mathfrak{po}(n, s) = \mathfrak{h} + \mathfrak{h}^\perp \quad \text{and} \quad A_X = h + B_X$$

in a unique way.

For any field Y on M .

$$R_{XY} = \nabla_Y A_X = \nabla_Y h + \nabla_Y B_X.$$

R_{XY} and $\nabla_Y h$ lie in \mathfrak{h} and $\nabla_Y B_X$ lies in \mathfrak{h}^\perp . Thus $\nabla_Y B_X = 0$ and B_X is parallel.

And accordingly

$$\text{div } B_X X = \text{trace}(B_X \cdot B_X) = \phi(B_X, B_X).$$

But $\phi(B_X, B_X)$ is constant because

$$Y \phi(B_X, B_X) = 2\phi(\nabla_Y B_X, B_X) = 0.$$

Finally, the integral of $\text{div } B_X X$ on M gives

$$0 = \int_M \operatorname{div} (B_X X) = k \operatorname{vol} (M).$$

That is

$$0 = k = \phi(B_X, B_X). \tag{Q. E. D.}$$

REMARK. This theorem still holds without the assumption of orientability because the covering of the orientations is (2:1) and is also a local isometry.

We gave in [1] some examples of compact semiriemannian manifolds with nonholonomic Killing vector fields.

2. The flat case.

Let (M_s^n, g) be a compact flat manifold. If X is a Killing vector field on M , by Theorem 1

$$\phi(A_X, A_X) = 0.$$

On the assumption of homogeneity, we will see in this section that $A_X = 0$.

We recall

LEMMA 2 [2]. *Let (M_s^n, g) be a semiriemannian manifold; if X is a Killing vector field on M , assume that*

$$2f = g(X, X).$$

Then,

- i) $\operatorname{grad} f = A_X X.$
- ii) $H^f(V, W) = g(\nabla_V X, \nabla_W X) + g(R_{XV} X, W) = g(\nabla_V(A_X X), W)$
- iii) $\Delta f = -\phi(A_X, A_X) - \operatorname{Ric}(X, X).$

PROPOSITION 3 (Marsden) [2]. *A homogeneous compact semiriemannian manifold is complete.*

COROLLARY 4. *A homogeneous compact flat semiriemannian manifold is geodesically convex. (i.e. given any two points there is a geodesic joining them).*

PROOF. By Proposition 3 the universal covering of M is a flat complete simply connected manifold; thus it is R_s^n . In order to obtain a geodesic σ joining $p \in M$ and $q \in M$, take \tilde{p} , in the fiber of p and \tilde{q} in the fiber of q and project on M the straight line $\tilde{p}\tilde{q}$. (Q. E. D.)

PROPOSITION 5. *Let X be a Killing vector field on a homogeneous compact flat semiriemannian manifold (M_s^n, g) . The product $g(X, X)$ is constant on M .*

Using parallel transport and because of the flatness we can assume that we have a coordinate system $x_1, y_1, \dots, x_r, y_r, z_1, \dots, z_t$ on R_s^n such that the matrix of the $A_{\tilde{X}}$ -operator in the associated frame is (*). In this coordinate system the nonparallel part of \tilde{X} is

$$x_1 \frac{\partial}{\partial y_1} + \dots + x_r \frac{\partial}{\partial y_r}.$$

There is no loss of generality in assuming that

$$\tilde{X} = x_1 \frac{\partial}{\partial y_1} + \dots + x_r \frac{\partial}{\partial y_r}.$$

4th Step. Let us now consider the new system

$$(x_1, \dots, x_r, y_1, \dots, y_r, z_1, \dots, z_t).$$

Let μ be an element of Γ ; because of the homogeneity of M_s^n , Γ is a group of pure translations (see [4] pg. 135). In our new coordinate system

$$\mu = \begin{pmatrix} x & y & z \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} + \begin{pmatrix} M \\ N \\ U \end{pmatrix}$$

where I is the identity matrix.

The Γ -invariance of \tilde{X} is reflected on the μ -matrix by the fact that $M=0$, so that the dimension of the subspace spanned by the translation components of the elements of Γ is not greater than $n-r$. If $r \neq 0$ the translation components of the elements of Γ do not generate R_s^n . But this is impossible because M is compact.

Consequently, $r=0$ and \tilde{X} and X are parallel. (Q. E. D.)

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Bibliography

- [1] Fossas, E., Killing vector fields on semiriemannian manifolds, To appear in Tsukuba J. Math.
- [2] Kostant, B., Holonomy and the Lie algebra of infinitesimal motions of a riemannian manifold, Trans. Amer. Math., 80 (1955), 528-542.

- [3] O'Neill, B., Semi-Riemannian geometry, Academic Press, 1983.
- [4] Wolf, J., Spaces of constant curvature, McGraw-Hill, 1967.

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