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A NOTE ON MULTIVALENT FUNCTIONS

By

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The author would like to dedicate this paper to the memory of the late Professor Shigeo Ozaki.

It is well known that if $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is analytic in $E=\{z:|z|<1\}$ and $Re f'(z)>0$ in E, then $f(z)$ is univalent in E.

Ozaki [3, Theorem 2] extended the above result to the following: if $f(z)$ is analytic in a convex domain D and

$$Re(e^{i\alpha}f^{(p)}(z))>0 \quad \text{in} \quad D$$

where α is a real constant, then $f(z)$ is at most p-valent in D.

This shows that if $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and

$$Re f^{(p)}(z)>0 \quad \text{in} \quad E,$$

then $f(z)$ is p-valent in E.

DEFINITION. Let $F(z)$ be analytic and univalent in E and suppose that $F(E)=D$. If $f(z)$ is analytic in E, $f(0)=F(0)$, and $f(E)\subset D$, then we say that $f(z)$ is subordinate to $F(z)$ in E, and we write

$$f(z)\prec F(z).$$

In this paper, we need the following lemmata.

LEMMA 1. *If $p(z)$ is analytic in E, with $p(0)=1$, and*

$$p(z)+zp'(z)\prec\left(\frac{1+z}{1-z}\right)^{3/2} \quad \textit{in} \quad E,$$

then we have

$$p(z)\prec\frac{1+z}{1-z} \quad \textit{in} \quad E.$$

We owe this lemma to [1, Theorem 5 and its remark].

REMARK. From Lemma 1, it is trivial that if $p(0)=1$ and

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$$|arg(p(z)+zp'(z))|<\frac{3}{4}\pi \quad \text{in} \quad E,$$

then we have

$$|arg p(z)|<\frac{\pi}{2} \quad \text{in} \quad E.$$

LEMMA 2. *Let $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n$ be analytic in E. If there exists a $(p-k+1)$-valent starlike function $g(z)=z^{p-k+1}+\sum_{n=p-k+2}^{\infty}b_n z^n$ that satisfies*

$$Re\frac{zf^{(k)}(z)}{g(z)}>0 \quad in \quad E,$$

then $f(z)$ is p-valent in E.

We owe this lemma to [2, Theorem 8].

MAIN THEOREM. Let $p\geqq 2$. If $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n$ is analytic in E and

$$(1) \qquad |arg f^{(p)}(z)|<\frac{3}{4}\pi \quad \text{in} \quad E,$$

then $f(z)$ is p-valent in E.

PROOF. Let us put

$$p(z)=f^{(p-1)}(z)/(p!z).$$

Then from the assumption (1) and by an easy calculation, we have

$$p(z)+zp'(z)=\frac{f^{(p)}(z)}{p!}\prec\left(\frac{1+z}{1-z}\right)^{3/2} \quad \text{in} \quad E$$

and $p(0)=1$. Then, from Lemma 1, we have

$$\frac{f^{(p-1)}(z)}{p!z}\prec\frac{1+z}{1-z} \quad \text{in} \quad E.$$

This shows that

$$(2) \qquad Re\frac{f^{(p-1)}(z)}{z}=Re\frac{zf^{(p-1)}(z)}{z^2}>0 \quad \text{in} \quad E.$$

Moreover, it is trivial that $g(z)=z^2$ is 2-valently starlike in E. Therefore, from Lemma 2 and (2), we have that $f(z)$ is p-valent in E.

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References

[1] Miller, S.S. and Mocanu, P.T., Marx-Strohhäcker differential subordination systems, Proc. Amer. Math. Soc. **99** (1987), 527-534.

[2] Nunokawa, M., On the theory of mutivalent functions, Tsukuba J. Math. **11** (1987), 273-286.

[3] Ozaki, S., On the theory of multivalent functions, Sci. Rep. Tokyo Bunrika Daigaku A, **2** (1935), 167-188.

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