TSUKUBA J. MATH.
Vol. 13 No. 2 (1989), 419—422

REFLEXIVE MODULES AND RINGS WITH SELF-INJECTIVE DIMENSION TWO

By

Mitsuo HOSHINO

Let R be a left and right noetherian ring and M a finitely generated left R-module with $\mathrm{Ext}_R^i(M, R)=0$ for $i\geqq 1$. Is then M reflexive ? This is a stronger version of the generalized Nakayama conjecture posed by Auslander and Reiten [2]. In this note, we ask when every finitely generated left R-module M with $\mathrm{Ext}_R^i(M, R)=0$ for $i=1, 2$ is reflexive. Our main aim is to show that if R is a left and right noetherian ring then $\mathrm{inj\,dim}_R R=\mathrm{inj\,dim}\, R_R\leqq 2$ if and only if for a finitely generated left R-module M the following conditions are equivalent: (1) M is reflexive; (2) there is an exact sequence $0\to M\to P_1\to P_0$ of left R-modules with the P_i projective; and (3) $\mathrm{Ext}_R^i(M, R)=0$ for $i=1, 2$. We will show also that if R is a commutative noetherian ring then it is a Gorenstein ring of dimension at most two if and only if the ring of total quotients of R is a Gorenstein ring and every finitely generated R-module M with $\mathrm{Ext}_R^i(M, R)=0$ for $i=1, 2$ is reflexive.

In what follows, R stands for a ring with identity, and all modules are unital R-modules. We denote by $(\ )^*$ both the R-dual functors, and for a module M we denote by $\varepsilon_M: M\to M^{**}$ the usual evaluation map. Recall that a module M is said to be torsionless if ε_M is a monomorphism and to be reflexive if ε_M is an isomorphism. Also, a module M is said to be finitely presented if it admits an exact sequence $P_1\to P_0\to M\to 0$ with the P_i finitely generated and projective. Note that if R is left noetherian then every finitely generated left module is finitely presented.

1. Preliminaries

In this section, we prepare several lemmas which we need in the next section.

LEMMA 1.1. *The following are equivalent*:

(1) *Every finitely presented left module M with* $\mathrm{Ext}_R^i(M, R)=0$ *for* $i=1, 2$

Received August 8, 1988.

is reflexive.

(2) *For any finitely presented reflexive right module N we have* $\mathrm{Ext}_R^i(N, R)=0$ *for* $i=1, 2$.

PROOF. Let M be a left module with a finite presentation $P_1 \xrightarrow{f} P_0 \to M \to 0$ and put $N=\mathrm{Cok}\, f^*$. Then we have a finite presentation $P_0^* \xrightarrow{f^*} P_1^* \to N \to 0$ with $\mathrm{Cok}\, f^{**} \cong \mathrm{Cok}\, f = M$. Fix these notations. By Auslander [1, Proposition 6.3], $\mathrm{Ker}\, \varepsilon_M \cong \mathrm{Ext}_R^1(N, R)$ and $\mathrm{Cok}\, \varepsilon_M \cong \mathrm{Ext}_R^2(N, R)$. Similarly, $\mathrm{Ker}\, \varepsilon_N \cong \mathrm{Ext}_R^1(M, R)$ and $\mathrm{Cok}\, \varepsilon_N \cong \mathrm{Ext}_R^2(M, R)$.

(1) $\Rightarrow$ (2). Suppose that N is reflexive. Then $\mathrm{Ext}_R^i(M, R)=0$ for $i=1, 2$, and M is reflexive. Thus $\mathrm{Ext}_R^i(N, R)=0$ for $i=1, 2$.

(2) $\Rightarrow$ (1). Suppose $\mathrm{Ext}_R^i(M, R)=0$ for $i=1, 2$. Then N is reflexive, and $\mathrm{Ext}_R^i(N, R)=0$ for $i=1, 2$. Thus M is reflexive.

LEMMA 1.2. *Let R be left noetherian. Suppose* $\mathrm{inj\,dim}\, R_R \leqq 2$. *Then every finitely generated left module M with* $\mathrm{Ext}_R^i(M, R)=0$ *for* $i=1, 2$ *is reflexive.*

PROOF. Let N be a finitely presented reflexive right module. Note that N^* is finitely presented. Take a finite presentation $P_1 \to P_0 \to N^* \to 0$ of N^*. Applying $(\ )^*$, we get an exact sequence $0 \to N \to P_0^* \to P_1^*$ with the P_i^* projective. Thus $\mathrm{Ext}_R^i(N, R)=0$ for $i \geqq 1$, since $\mathrm{inj\,dim}\, R_R \leqq 2$. By Lemma 1.1, we are done.

LEMMA 1.3. *For a module M, M^* is reflexive if and only if M^{**} is.*

PROOF. Note first that $\varepsilon_L^* \circ \varepsilon_{L^*} = id_{L^*}$ for any module L (see e.g. Jans [4]).

"Only if" part. Since $(\varepsilon_{M^*})^* \circ \varepsilon_{M^{**}} = id_{M^{**}}$, if ε_{M^*} is an isomorphism, so is $\varepsilon_{M^{**}}$.

"If" part. Note that $\mathrm{Ker}\, \varepsilon_M^* \cong (\mathrm{Cok}\, \varepsilon_M)^*$. Since $\varepsilon_M^* \circ \varepsilon_{M^*} = id_{M^*}$, we get $\mathrm{Cok}\, \varepsilon_{M^*} \cong \mathrm{Ker}\, \varepsilon_M^* \cong (\mathrm{Cok}\, \varepsilon_M)^*$. Applying this to M^*, we get $\mathrm{Cok}\, \varepsilon_{M^{**}} \cong (\mathrm{Cok}\, \varepsilon_{M^*})^* \cong (\mathrm{Cok}\, \varepsilon_M)^{**}$. Thus $(\mathrm{Cok}\, \varepsilon_M)^{**}=0$, which implies $(\mathrm{Cok}\, \varepsilon_M)^*=0$. Hence $\mathrm{Cok}\, \varepsilon_{M^*}=0$, and M^* is reflexive, since it is torsionless.

LEMMA 1.4. *Let R be left and right noetherian. The following are equivalent:*

(1) *The dual of a finitely generated left module is reflexive.*

(2) *The dual of a finitely generated right module is reflexive.*

PROOF. (1) $\Rightarrow$ (2). Let N be a finitely generated right module. Since N^* is finitely generated, N^{**} is reflexive. Thus, by Lemma 1.3, N^* is reflexive.

(2) $\Rightarrow$ (1). Similarly.

2. Main results

To begin with, we deal with the case of R being commutative.

PROPOSITION 2.1. *Let R be commutative and noetherian. Then R is a Gorenstein ring of dimension at most two if and only if the ring of total quotients of R is a Gorenstein ring and every finitely generated module M with* $\mathrm{Ext}_R^i(M, R)=0$ *for $i=1, 2$ is reflexive.*

PROOF. "Only if" part. The former assertion is well known (see e.g. Bass [3]). The latter assertion follows from Lemma 1.2.

"If" part. Let M be a module with a finite presentation $P_1 \xrightarrow{f} P_0 \to M \to 0$ and put $N=\mathrm{Cok}\, f^*$. Then $\mathrm{Ker}\, f \cong N^*$. By Bass [3, Proposition 6.1], N^* is reflexive. Thus, by Lemma 1.1, we get $\mathrm{Ext}_R^3(M, R) \cong \mathrm{Ext}_R^1(N^*, R)=0$. Hence $\mathrm{inj\,dim}_R R \leqq 2$.

In order to prove the main theorem, we need one more auxiliary result.

PROPOSITION 2.2. *Let R be left and right noetherian. Suppose* $\mathrm{inj\,dim}_R R \leqq 2$. *Then* $\mathrm{inj\,dim}_R R = \mathrm{inj\,dim}\, R_R$ *if and only if every finitely generated left module M with* $\mathrm{Ext}_R^i(M, R)=0$ *for $i=1, 2$ is reflexive.*

PROOF. "Only if" part. By Lemma 1.2.

"If" part. We claim $\mathrm{inj\,dim}\, R_R \leqq 2$. Let N be a right module with a finite presentation $P_1 \xrightarrow{f} P_0 \to N \to 0$ and put $M=\mathrm{Cok}\, f^*$. Applying $(\ )^*$, we get an exact sequence $0 \to N^* \to P_0^* \to P_1^* \to M \to 0$ with the P_i^* projective. Thus $\mathrm{Ext}_R^i(N^*, R) \cong \mathrm{Ext}_R^{i+2}(M, R)=0$ for $i \geqq 1$, and N^* is reflexive. Hence the dual of a finitely generated right module is reflexive, and by Lemma 1.4 M^* is reflexive. By Lemma 1.1 we have $\mathrm{Ext}_R^1(M^*, R)=0$. Since $\mathrm{Ker}\, f \cong M^*$, we get $\mathrm{Ext}_R^3(N, R) \cong \mathrm{Ext}_R^1(M^*, R)=0$. Therefore $\mathrm{inj\,dim}\, R_R \leqq 2$, and by Zaks [5, Lemma A], we are done.

We are now in a position to prove the main theorem.

THEOREM 2.3. *Let R be left and right noetherian. Then* $\mathrm{inj\,dim}_R R = \mathrm{inj\,dim}\, R_R \leqq 2$ *if and only if for a finitely generated left module M the following are equivalent:*

(1) *M is reflexive.*

(2) *There is an exact sequence $0 \to M \to P_1 \to P_0$ with the P_i projective.*

(3) $\mathrm{Ext}_R^i(M, R)=0$ *for $i=1, 2$.*

PROOF. "Only if" part. By Proposition 2.2, (3)⇒(1). Also inj $\dim_R R \leqq 2$ implies (2)⇒(3). Finally, by applying ( )* to a finite presentation of M^*, we get (1)⇒(2).

"If" part. Since (2)⇒(3), we get inj $\dim_R R \leqq 2$. Thus, by Proposition 2.2, (3)⇒(1) implies inj $\dim_R R =$ inj $\dim R_R \leqq 2$.

We end with making the following

REMARK. In Proposition 2.1, the condition that the ring of total quotients of R is a Gorenstein ring is really needed. Let $R=k[x, y]/(x^2, xy, y^2)$, where k is a field. Then R is not a Gorenstein ring, whereas every finitely generated module M with $\mathrm{Ext}_R^i(M, R)=0$ for $i=1, 2$ is free and thus reflexive. On the other hand, by a slight modification of Lemma 1.1, one can easily verify that if R is right noetherian then inj dim $R_R \leqq 1$ if and only if every finitely presented left module M with $\mathrm{Ext}_R^1(M, R)=0$ is torsionless.

References

[1] Auslander, M., Coherent functors, in: Proceedings of the conference on categorical algebra, Springer, Berlin, 1966, pp. 189-231.

[2] Auslander, M. and Reiten, I., On a generalized version of the Nakayama conjecture, Proc. Amer. Math. Soc. **52** (1975), 69-74.

[3] Bass, H., On the ubiquity of Gorenstein rings, Math. Z. **82** (1963), 8-28.

[4] Jans, J. P., Duality in noetherian rings, Proc. Amer. Math. Soc. **12** (1961), 825-835.

[5] Zaks, A., Injective dimension of semiprimary rings, J. Algebra **13** (1969) 73-86.

Institute of Mathematics
University of Tsukuba
Ibaraki, 305
Japan