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THE CENTER OF CROSSED PRODUCTS OVER SIMPLE RINGS

To Professor Tachikawa on the occasion of his sixtieth birthday

By

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Abstract. Let $R*G$ be the crossed product of an arbitrary group G over a simple ring R. Since G acts on $Z(R)$ and R is simple, $Z(R)$ is a G-field and the fixed field $Z(R)^{G}$ of G is contained in $Z(R*G)$. The main result of this paper exhibits a distinguished basis for $Z(R*G)$ over the field $Z(R)^{G}$. A number of applications is also provided. Our method is based on the theory of similinear monomial representations. In this way we obtain conceptual proofs of results which otherwise require lengthy computations and ad hoc arguments.

1. Introduction.

In the past ten years there have been a tremendous surge of activity in the theory of graded rings and their important special case, namely crossed products. For a detailed account of the theory, we refer the reader to [4]. The principal object of this paper is to provide a further development, which is to describe the center of crossed products over simple rings. We then apply our result to count nonisomorphic irreducible modules over such crossed products. Among other applications, we provide information on the number of linearly nonequivalent irreducible projective crossed representations of a finite group over fields.

To describe the main idea and method, let us first recall the following piece of information. Let A be a ring and let G be a multiplicative group. Given additive subgroups X and Y of A, we write XY for the additive subgroup of A consisting of all finite sums

$$\Sigma x_i y_i \qquad x_i \in X,\ y_i \in Y\,.$$

We say that A is a *G-graded ring*, provided there exists a family $\{A_g \mid g \in G\}$ of additive subgroups of A indexed by G such that the following two conditions

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hold:

$$A=\bigoplus_{g\in G} A_g,$$

$$A_x A_y \subseteq A_{xy} \qquad \text{for any} \quad x, y\in G.$$

It is immediate that A_1 is a subring of A with $1\in A_1$. Let $U(A)$ denote the unit group of A. We say that a unit $u\in U(A)$ is *graded* if it lies in A_g for some $g\in G$. We shall refer to such g as the degree of u and write

$$g=\deg(u).$$

It is clear that the set $GrU(A)$ of all graded units of A is a subgroup of $U(A)$ and that the sequence of group homomorphisms

$$1 \longrightarrow U(A_1) \longrightarrow GrU(A) \xrightarrow{\deg} G \longrightarrow 1 \tag{1}$$

is always exact except possibly at G. We say that A is a *crossed product* of G over A_1, written $A=A_1*G$, provided the sequence (1) is exact. In case (1) is an exact splitting sequence, we shall refer to A as a *skew group ring* of G over A_1. The ring A is said to be a *twisted group ring* of G over A_1, if for all $g\in G$, there exists $\bar{g}\in A_g\cap U(A)$ such that $\bar{g}$ centralizes A_1. In the particular case where $A_1\subseteq Z(A)$ we shall refer to A_1*G as a *twisted group algebra* of G over A_1. For any subset X of A, let $C_A(X)$ be the centralizer of X in A. If G acts on a ring R, we say that R is a *G-ring* and write R^G for the fixed ring of G defined by

$$R^G=\{r\in R \mid {}^g r=r \text{ for all } g\in G\}.$$

Now let us look at the question of the justification for restricting our attention to crossed products over simple rings.

Assume that A is a G-graded ring. What can be said about the center of A? The following general observation is due to Dade [1].

Fix $g\in G$, write $1=\sum_{i=1}^{n} a_i b_i$ for a suitable positive integer n and suitable $a_i\in A_g$, $b_i\in A_{g^{-1}}$, $1\leqq i\leqq n$, and for any $y\in C_A(A_1)$, put

$${}^g y=\sum_{i=1}^{n} a_i y b_i.$$

Manifestly, if A is a crossed product of G over A_1, we may put ${}^g y=\bar{g}y\bar{g}^{-1}$ where $\bar{g}\in U(A)\cap A_g$. Then ${}^g y$ is a unique element of A satisfying $a_g y={}^g y a_g$ for all $a_g\in A_g$. Furthermore, ${}^g y\in C_A(A_1)$ and, provided $y\in Z(A_1)$, ${}^g y\in Z(A_1)$. The group G acts as automorphisms of the rings $C_A(A_1)$ and $Z(A_1)$, with any $g\in G$ sending any $y\in C_A(A_1)$ and $y\in Z(A_1)$, respectively, into ${}^g y$. It is then immediate that

$$Z(A)=C_A(A_1)^G.$$

Unfortunately, nothing more can be said about $Z(A)$ under these general circumstances. Since the problem of description of $Z(A)$ seems so intractible one needs to impose more hypotheses to make any progress. The situation where A is a crossed product of G over a simple ring A_1 is favourable for the following two reasons. First of all, $Z(A_1)$ is a G-field and the fixed field $Z(A_1)^G$ of G is contained in $Z(A)$. Thus we may attempt to describe $Z(A)$ by exhibiting a distinguished basis over $Z(A_1)^G$.

The second reason can be explained as follows. For each $g\in G$, fix a unit $\bar{g}$ of A in A_g with $\bar{1}=1$, and denote by G_0 the normal subgroup of G consisting of those $g\in G$ for which conjugation by $\bar{g}$ induces an inner automorphism of A_1. Of course, the definition of G_0 does not depend upon a choice of units $\bar{g}$, $g\in G$. Without loss of generality we may assume that $\bar{g}\in C_A(A_1)$ for all $g\in G_0$. It turns out that if A_1 is simple, then $C_A(A_1)$ is a twisted group algebra of G_0 over the field $Z(A_1)$. Thus $C_A(A_1)$ is a vector space over the field $Z(A_1)$ with distinguished basis $\{\bar{g}\mid g\in G_0\}$. The latter fact allows us to use the theory of semilinear monomial representations on graded vector spaces developed in Section 2, to provide the desired description of $Z(A)$. This has the advantage of preparing the way for dealing with more complicated situations, where ad hoc arguments are less easy to find.

2. Similinear monomial representations on graded vector spaces.

Let X be an arbitrary set. By an *X-graded space* over a field F we understand a pair $(V, (V_x))$, where V is a vector space over F and (V_x) is a family of one-dimensional subspaces of V indexed by X such that

$$V=\bigoplus_{x\in X} V_x.$$

Let V be a vector space over a field F. A *semilinear transformation* of V is any additive homomorphism $f: V\to V$ for which there exists an automorphism ψ of F such that

$$f(\lambda v)=\psi(\lambda)f(v) \qquad \text{for all} \quad \lambda\in F,\ v\in V.$$

Note that the automorphism ψ is uniquely determined by f. A semilinear transformation f of V is said to be *nonsingular* if f is a bijection. It is clear that under the composition of mappings the set of all nonsingular semilinear transformations of V constitutes a group; we denote this group by $GS(V)$ and refer to it as the *general semilinear group* of V. For each $f\in GS(V)$, let ψ_f

be the associated automorphism of F. Then the map

$$\begin{cases} GS(V) \longrightarrow \operatorname{Aut} F \\ \quad f \longmapsto \phi_f \end{cases}$$

is a homomorphism whose kernel is the general linear group $GL(V)$ on V. In particular, $GL(V) \lhd GS(V)$.

Let X be an arbitrary set and let $(V, (V_x))$ be an X-graded space over a field F. *By a semilinear monomial representation* on a group G on $(V, (V_x))$ we mean a homomorphism

$$\Gamma : G \longrightarrow GS(V)$$

such that for all $g \in G$, $\Gamma(g)$ permutes the V_x, $x \in X$. Given such a Γ, F becomes a G-field and we write $\lambda \mapsto {}^g\lambda$ for the automorphism of F corresponding to $\Gamma(g)$. Note also that Γ determines a homomorphism γ from G to the permutation group of the set X, where for all $g \in G$ and $x, y \in X$

$$\gamma(g)x = y \quad \text{if and only if} \quad \Gamma(g)V_x = V_y .$$

Thus G acts on the set X and we denote by $G(x)$ the stabilizer of $x \in X$, that is

$$G(x) = \{g \in G \mid \gamma(g)x = x\} .$$

We say that an element x of X is *Γ-regular* if there exists a nonzero v_x in V_x such that

$$\Gamma(g)v_x = v_x \qquad \text{for all} \quad g \in G(x) .$$

We shall refer to a G-orbit of X as being *Γ-regular* if each element of this orbit is *Γ-regular*. By the *fixed-point space* of Γ we understand the set of those $v \in V$ for which

$$\Gamma(g)v = v \qquad \text{for all} \quad g \in G .$$

It is clear that the fixed-point space of Γ is a vector space over F^G, the fixed field of G.

We have now accumulated all the information necessary to prove the following result. Its future application will dispel any notion that semilinear monomial representations form an exotic class of representations.

THEOREM 1. *Let X be an arbitrary set, let $(V, (V_x))$ be an X-graded space over a field F and let*

$$\Gamma : G \longrightarrow GS(V)$$

be a semilinear monomial representation of G on $(V, (V_x))$. Let Z be a full set *of representatives for the finite Γ-regular orbits of X and, for each $z\in Z$, let L_z be the sum of one-dimensional subspaces of V indexed by the elements of the orbit containing z. For each $z\in Z$, fix $0\neq v_z\in V_z$ with $\Gamma(g)v_z=v_z$ for all $g\in G(z)$.*

(i) *If $x\in X$ is Γ-regular, then so are all the elements in the G-orbit of x*

(ii) *If W is the fixed-point space of Γ, then*

(a) $W=\bigoplus_{z\in Z}(W\cap L_z)$

(b) $W\cap L_z=\{\sum_{g\in T_z}{}^{g}\lambda\Gamma(g)v_z\mid\lambda\in F^{G(z)}\}$

where T_z is a left transversal for $G(z)$ in G containing 1

(c) *If $\{\lambda_i\mid i\in I\}$ is an F^G-basis of $F^{G(z)}$, then*

$$\Big\{\sum_{g\in T_z}{}^{g}\lambda_i\Gamma(g)v_z\mid i\in I\Big\}$$

is an F^G-basis of $W\cap L_z$

(d) *If $\dim_F V<\infty$ and G is a finite group, then $\dim_{F^G} W<\infty$ and*

$$\dim_{F^G} W=\sum_{z\in Z}\Big(\dim_{F^G} F^{G(z)}\Big)$$

Proof. (i) Let $x\in X$ be Γ-regular let $y\in X$ be any element in the G-orbit of x. Then there exists $g\in G$ such that

$$\Gamma(g)V_x=V_y \text{ and } G(y)=gG(x)g^{-1}.$$

Since x is Γ-regular, there is a non-zero v_x in V_x fixed by all $\Gamma(h)$ with $h\in G(x)$. Because $\Gamma(g^{-1})V_g=V_x$, we may write $v_x=\Gamma(g^{-1})v_y$ for some non-zero v_y in V_y.

Now assume that $t\in G(y)$, say $t=ghg^{-1}$ with $h\in G(x)$. Then we have

$$\Gamma(t)v_y=\Gamma(g)\Gamma(h)\Gamma(g^{-1})v_y=\Gamma(g)\Gamma(h)v_x=\Gamma(g)v_x=v_y,$$

proving (i).

(ii) Denote by Y a full set of representatives for the orbits of X and, for each $y\in Y$, let L_y be the sum of one-dimensional subspaces of V indexed by the elements of the orbit containing y. Then

$$V=\bigoplus_{y\in Y}L_y$$

is a decomposition of V into direct sum of G-invariant subspaces. Hence

$$W=\bigoplus_{y\in Y}(W\cap L_y).$$

Let $v=\sum_{x\in X}v_x$, $v_x\in V_x$, belong to W and assume that $v_t\neq 0$ for some $t\in X$. Then for all $g\in G$, $\Gamma(g)v_t\in V_{\gamma(g)t}$ which ensures, in view of the equality

$\Gamma(g)v=v$, that

$$0\neq\Gamma(g)v_t=v_{\gamma(g)t}.$$

In particular, if $g\in G(t)$, then $\Gamma(g)v_t=v_t$, proving that t is Γ-regular and hence, by (i), that t belongs to a Γ-regular orbit. Moreover, since the number of $0\neq v_x\in V_x$ is finite, t belongs to a finite Γ-regular orbit. Thus

$$W=\bigoplus_{z\in Z}(W\cap L_z),$$

proving (a).

Given $v\in L_z$, we may write uniquely

$$v=\sum_{g\in T_z}\lambda_g\Gamma(g)v_z \qquad (\lambda_g\in F)$$

since $\{\Gamma(g)v_z \mid g\in T_z\}$ is an F-basis for U_z. Hence $v\in W\cap L_z$ if and only if

$$\Gamma(h)v=\sum_{g\in T_z}{}^h\lambda_g\Gamma(hg)v_z=\sum_{g\in T_z}\lambda_g\Gamma(g)v_z \qquad \text{for all} \quad h\in G \tag{1}$$

Given $h\in G$ and $g\in T_z$, let $g_h\in T_z$ be defined by $hg\in g_hG(z)$. Then

$$hg=g_ht_h \qquad \text{for some} \quad t_h\in G(z)$$

and therefore

$$\Gamma(hg)v_z=\Gamma(g_h)\Gamma(t_h)v_z=\Gamma(g_h)v_z.$$

Thus (1) is equivalent to

$$\sum_{g\in T_z}{}^h\lambda_g\Gamma(g_h)v_z=\sum_{g\in T_z}\lambda_g\Gamma(g)v_z \qquad \text{for all} \quad h\in G \tag{2}$$

Taking into account that $\{g_h \mid g\in T_z\}=T_z$, we deduce that (2) is equivalent to

$${}^h\lambda_g=\lambda_{g_h} \qquad \text{for all} \quad h\in G,\ g\in T_z. \tag{3}$$

Now put $\lambda=\lambda_1$ and assume that (3) holds. Then, taking $g=1$, $h\in T_z$ and $h\in G(z)$, we obtain

$${}^h\lambda=\lambda_h \quad \text{and} \quad \lambda\in F^{G(z)} \qquad \text{for all} \quad h\in T_z. \tag{4}$$

Conversely, suppose that (4) holds. Fix $h\in G$, $g\in T_z$ and write $hg=g_ht_h$ for some $t_h\in G(z)$. Then we have

$${}^h\lambda_g={}^h({}^g\lambda)={}^{hg}\lambda={}^{g_ht_h}\lambda={}^{g_h}\lambda=\lambda_{g_h}$$

proving (3), and thus (b) is established.

Given $\lambda\in F^{G(z)}$, we may write $\lambda=\sum_{i=1}^n\mu_i\lambda_i$ for a unique $n\geq 1$ and unique $\mu_1,\cdots,\mu_n$ in F^G. Then

$$\sum_{g\in T_z}{}^g\lambda\Gamma(g)v_z=\sum_{g\in T_z}\left(\sum_{i=1}^n\mu_i{}^g\lambda_i\right)\Gamma(g)v_z$$

$$=\sum_{i=1}^{n}\mu_i\Big(\sum_{g\in T_z}\lambda_i\Gamma(g)v_z\Big) \tag{5}$$

and therefore, by (b), $W\cap L_z$ is the F^G-linear span of

$$\Big\{\sum_{g\in T_z}{}^{g}\lambda_i\Gamma(g)v_z \,|\, i\in I\Big\}.$$

Furthermore, if the equality

$$\sum_{i=1}^{n}\mu_i\Big(\sum_{g\in T_z}{}^{g}\lambda_i\Gamma(g)v_z\Big)=0$$

holds, then by (5) we have $\sum_{g\in T_z}{}^{g}\lambda\Gamma(g)v_z=0$. But then $\lambda=0$ and hence each $\mu_i=0$, proving (c).

Finally, assume that $\dim_F V<\infty$ and that G is a finite group. Then X is a finite set, hence so is Z and, since $\dim_{F^G} F^{G(z)}<\infty$ for all $z\in Z$, (d) follows by appealing to (a) and (c). ■

3. The center of crossed products over simple rings.

Throughout this section, $R*G$ denotes a crossed product of a (possibly infinite) group G over a simple ring R. For each $g\in G$, we fix a unit $\bar{g}$ of $R*G$ in $(R*G)_g$ with $\bar{1}=1$ and define

$$\alpha: G\times G \longrightarrow U(R)$$

by

$$\alpha(x, y)=\bar{x}\bar{y}\overline{xy}^{-1} \qquad (x, y\in G).$$

We write G_0 for the normal subgroup of G consisting of all those $g\in G$ for which conjugation by $\bar{g}$ induces an inner automorphism of R. It is clear that the definition of G_0 does not depend upon a choice of units $\bar{g}$, $g\in G$. For each $g\in G_0$, let $\lambda_g\in U(R)$ be such that

$$\bar{g}r\bar{g}^{-1}=\lambda_g^{-1}r\lambda_g \qquad \text{for all} \quad r\in R.$$

Then $\tilde{g}=\lambda_g\bar{g}$ is clearly in $C_{R*G}(R)$. Thus we may, and from now on we shall, assume that

$$\bar{g}\in C_{R*G}(R) \qquad \text{for all} \quad g\in G_0. \tag{1}$$

As has been observed earlier, the formula ${}^{g}r=\bar{g}r\bar{g}^{-1}$, $r\in Z(R)$ or $r\in C_{R*G}(R)$, $g\in G$, provides an action of G on $Z(R)$ and $C_{R*G}(R)$. Since G acts on $Z(R)$ and R is simple, $Z(R)$ is a G-field and the fixed field $Z(R)^G$ of G is contained in $Z(R*G)$. Our aim in this section is to provide a distinguished basis for $Z(R*G)$ over the field $Z(R)^G$. The following two preliminary results will clear our path.

LEMMA 2. *With the notation above, the following properties hold*:
(i) $C_{R*G}(R)=Z(R)*G_0$ *is a twisted group algebra of* G_0 *over the field* $Z(R)$.
(ii) $Z(R*G)=(Z(R)*G_0)^G$.

PROOF. (i) It follows from (1) that

$$Z(R)*G_0=\left\{\sum_{g\in G_0} x_g\bar{g} \mid x_g\in Z(R)\right\}\subseteqq C_{R*G}(R).$$

Conversely, let $x=\sum_{g\in G} x_g\bar{g}\in C_{R*G}(R)$ and let $x_g\neq 0$ for some $g\in G$. Then

$$rx_g=x_g{}^g r \quad \text{for all} \quad r\in R.$$

Hence $Rx_g=x_gR$ is a nonzero ideal of R and thus $R=Rx_g=x_gR$. It follows that x_g is a unit of R such that

$${}^g r=x_g^{-1}rx_g \quad \text{for all} \quad r\in R.$$

Therefore $g\in G_0$ and, by (1),

$$r={}^g r=x_g^{-1}rx_g \quad \text{for all} \quad r\in R$$

which shows that $x_g\in Z(R)$. This proves that $C_{R*G}(R)\subseteqq Z(R)*G_0$ as required.

(iii) Direct consequence of (i) and the fact that $Z(R*G)$ consists of all elements of $C_{R*G}(R)$ which commute with all $\bar{g}$, $g\in G$. ■

The discussion has now reached a point where, in order to make further progress, we need to bring in the notion of α-regularity.

We say that $g\in G$ is α-regular, provided g satisfies the following two conditions:

(a) $g\in G_0$
(b) There exists a nonzero v in $(Z(R)*G_0)_g$ such that $\bar{x}v=v\bar{x}$ for all $x\in C_G(g)$

Since each nonzero v in $(Z(R)*G_0)_g$ is of the form $v=\lambda\bar{g}$ for som $0\neq\lambda\in Z(R)$ and some $g\in G_0$, we see that $g\in G_0$ is α-regular if and only if there exists $0\neq\lambda\in Z(R)$ such that

$${}^x\lambda\alpha(x, g)=\lambda\alpha(g, x) \quad \text{for all} \quad x\in C_G(g). \tag{2}$$

Thus, if G acts trivially on $Z(R)$, then $g\in G_0$ is α-regular if and only if

$$\alpha(x, g)=\alpha(g, x) \quad \text{for all} \quad x\in C_G(g)$$

while if $R*G$ is a skew group ring of G over R (i.e. if $\alpha(x, y)=1$ for all $x, y\in G$) then each $g\in G_0$ is α-regular.

The following observation will enable us to take full advantage of Theorem 1.

LEMMA 3. *Let $F=Z(R)$, $V=F*G_0$ and, for each $g\in G_0$, put $V_g=\{\lambda\bar{g}\mid\lambda\in F\}$*

(i) *$(V, (V_g))$ is a G_0-graded space over the field F,*

(ii) *For each $g\in G$, the map $\Gamma(g)\colon V\to V$ defined by*

$$\Gamma(g)(v)=\bar{g}v\bar{g}^{-1}$$

is a nonsingular similinear transformation of V,

(iii) *The map $\Gamma\colon G\to GS(V)$ is a semilinear monomial representation of G on $(V, (V_g))$ such that,*

(a) *For each $x\in G_0$, $G(x)=C_G(x)$,*

(b) *An element $g\in G_0$ is Γ-regular if and only if g is α-regular. In particular by Theorem 1, if $g\in G_0$ is α-regular, then so is any G-conjugate of g,*

(c) *$Z(R*G)$ is equal to the fixed-point space of V.*

PROOF. (i) Direct consequence of the fact that $\{\bar{g}\mid g\in G_0)$ is an F-basis of $F*G_0$

(ii) The map Γ(g) obviously additive and is a bijection. Since for all $\lambda\in F$, $v\in V$,

$$\Gamma(g)(\lambda v)=(\bar{g}\lambda\bar{g}^{-1})(\bar{g}v\bar{g}^{-1})={}^{g}\lambda\Gamma(g)v$$

the assertion follows.

(iii) Owing to (ii), each $\Gamma(g)\in GS(V)$ and since $\Gamma(g)$ permutes the V_x, $x\in G_0$, Γ is in fact a semilinear monomial representation of G on $(V, (V_g))$. Let γ denote the corresponding homomorphism from G to the permutation group of the set G_0. Then, for each $g\in G$, $x\in G_0$, $\gamma(g)=gxg^{-1}$ and thus $G(x)=C_G(x)$. This proves (a) and (b), by applying (a) and the definitions of α-regularity and Γ-regularity. Property (c) being a consequence of Lemma 2(ii), the result follows. ■

We say that a conjugacy class C of G contained in G_0 is *α-regular* if g is α-regular for some (hence for all) g in C.

We are at last in a position to attain our main objective, which is to prove the following result, a particular case of which is due to Yamazaki (5).

THEOREM 4. *Let $R*G$ be a crossed product of a group G over a simple ring R and let Z be a full set of representatives for finite α-regular classes of G. For each $z\in Z$, choose $0\neq r_z\in Z(R)$ such that*

$${}^{g}r_z\alpha(g, z)=r_z\alpha(z, g) \qquad \textit{for all} \quad g\in C_G(z)$$

let $\{\lambda_{i,z}\mid i\in I_z\}$ be a $Z(R)^G$-basis of $Z(R)^{C_G(z)}$, let T_z be a left transversal for $C_G(z)$ in G containing 1, and put

$$v_{i,z}=\sum_{g\in T_z}{}^{g}(\lambda_{i,z}r_z)(\bar{g}\bar{z}\bar{g}^{-1}) \qquad (i\in T_z)$$

(i) $\bigcup_{z\in Z}\{v_{i,z}|i\in I_z\}$ *is a* $Z(R)^G$*-basis of* $Z(R*G)$

(ii) *If* G *is finite, then* $\dim_{Z(R)^G} Z(R*G)$ *is also finite and is given by the following formula*

$$\dim_{Z(R)^G} Z(R*G)=\sum_{z\in Z}\left(\dim_{Z(R)^G} Z(R)^{C_G(z)}\right).$$

(iii) $Z(R*G)=Z(R)^G$ *is and only if* $\{1\}$ *is the only finite* α*-regular class of* G.

PROOP. (i) Keeping the notation of Lemma 3, put $v_z=r_z\bar{z}$. Then our choice of r_z ensures that $v_z\neq 0$ is in V_z and that $\Gamma(g)v_z=v_z$ for all $g\in C_G(z)$. Moreover, for each $\lambda\in F^{C_G(z)}$ and $g\in T_z$,

$$\begin{aligned}{}^{g}\lambda\Gamma(g)v_z={}^{g}\lambda\Gamma(g)(r_z\bar{z})&={}^{g}\lambda{}^{g}r_z(\bar{g}\bar{z}\bar{g}^{-1}),\\ &={}^{g}(\lambda r_z)(\bar{g}\bar{z}\bar{g}^{-1}).\end{aligned}$$

The desired conclusion is therefore a consequence of Theorem 1(a), (c) and Lemma 3(a), (c).

(ii) If G is finite, then $\dim_F V=|G_0|$ is also finite. Hence the required assertion follows from Theorem 1(ii) and Lemma 3(a), (c).

(iii) Direct consequence of (i) ■

4. Applications.

The aim of this section is to provide a number of applications of Theorem 4. All conventions and notations introduced in Section 3 remain in force. In particular, $R*G$ denotes the crossed product of an arbitrary group G over a simple ring R, Z a full set of representatives for finite α-regular classes of G and, for each $z\in Z$, T_z is a left transversal for $C_G(z)$ in G containing 1.

THEOREM 5. *Assume that* G *acts trivially on* $Z(R)$ *(e.g.* $R*G$ *is a twisted group ring of* G *over* R*). Then*

(i) $\left(\sum_{g\in T_z}\bar{g}\bar{z}\bar{g}^{-1}|z\in Z\right)$ *is a* $Z(R)$*-basis of* $Z(R*G)$. *In particular, if* G_0 *is finite, then* $\dim_{Z(R)} Z(R*G)$ *is also finite and is equal to the number of* α*-regular classes of* G.

(ii) *If* G *is abelian, then* Z *is a subgroup of* G *and* $\{\bar{z}|z\in Z\}$ *a* $Z(R)$*-basis of* $Z(R*G)$.

PROOF. (i) Keep the notation of Theorem 4. Since G acts trivially on $Z(R)$, we can choose $r_z=1$ for all $z\in Z$. Futhermore, we also have

$$Z(R)^G=Z(R)=Z(R)^{C_G(z)}.$$

Hence $|I_z|=1$ and we can choose $\lambda_{i,z}=1$. Now apply Theorem 4(i).

(ii) Assume that G is abelian. Then, Z consists of all α-regular elements of G and, for each $z\in Z$, $C_G(z)=G$, so as T_z we can choose $\{1\}$. This proves that $\{\bar{z}|z\in Z\}$ is a $Z(R)$-basis of $Z(R*G)$, by applying (i).

Assume that $z_1, z_2\in Z$. Then $\bar{z}_1\bar{z}_2=\alpha(z_1, z_2)\overline{z_1z_2}$ and $\alpha(z_1, z_2)\in Z(R)$ since $z_1, z_2\in G_0$. Taking into account that $\bar{z}_i\in Z(R*G)$ and $Z(R)\subseteq Z(R*G)$, $i=1, 2$, we conclude that $\overline{z_1z_2}\in Z(R*G)$. Thus, by definition z_1z_2 is α-regular. Finally, assume that $z\in Z$. Since

$$\bar{z}\bar{z}^{-1}=\alpha(z, z^{-1})\cdot\bar{1}\in Z(R)\subseteq Z(R*G),$$

we see that $\bar{z}^{-1}\in Z(R*G)$. Hence z^{-1} is α-regular and therefore $z^{-1}\in Z$ as required. ■

THEOREM 6. *Let $R*G$ be a skew group ring of a group G over a simple ring R, let Z be a full set of representatives of finite conjugacy classes of G contained in G_0 and, for each $z\in Z$, let $\{\lambda_{i,z}|i\in I_z\}$ be a $Z(R)^G$-basis of $Z(R)^{C_G(z)}$. Put*

$$v_{i,z}=\sum_{g\in T_z}{}^g\lambda_{i,z}(\bar{g}\bar{z}\bar{g}^{-1}) \qquad (i\in I_z)$$

Then

$$\bigcup_{z\in Z}\{v_{i,z}|i\in I_z\} \text{ is a } Z(R)^G\text{-basis of } Z(R*G).$$

PROOF. Since $R*G$ is a skew group ring, each $g\in G_0$ is α-regular. Furthermore, in the notation of Theorem 4, we can put $r_z=1$. Now apply Theorem 4(i). ■

Our next application of Theorem 4 is concerned with counting nonisomorphic irreducible $R*G$-modules.

THEOREM 7. *Let $R*G$ be a crossed product of a finite group G over a simple ring R and assume that*

$$\dim_{Z(R)^G} R<\infty \text{ and that char } R\nmid |G|.$$

Denote by $n(R*G)$ the number of nonisomorphic irreducible $R*G$-modules. Then

$$n(R*G) \leqq \sum_{z \in Z} \left(\dim_{Z(R)^G} Z(R)^{C_G(z)} \right)$$

with equality if $Z(R)^G$ *is a splitting field for the* $Z(R)^G$*-algebra* $R*G$.

PROOF. By hypothesis, $R*G$ is a finite-dimensional algebra over the field $Z(R)^G$. Furthermore, since $\operatorname{char} R \nmid |G|$, $R*G$ is semisimple by Maschke's theorem [4]. Hence

$$n(R*G) \leqq \dim_{Z(R)^G} Z(R*G)$$

with equality if $Z(R)^G$ is a splitting field for $R*G$. Now apply Theorem 4(ii). ■

As an easy consequence of Theorem 7, we derive

COROLLARY 8. *Let* $R*G$ *be a crossed product of a finite group* G *over a simple ring* R. *Assume that the following three conditions hold*:

(i) G *acts trivially on* $Z(R)$ (*e.g.* $R*G$ *is a twisted group ring of* G *over* R).

(ii) R *is finite-dimensional over* $Z(R)$.

(iii) $\operatorname{char} R \nmid |G|$

Then the number of nonisomorphic irreducible $R*G$*-modules does not exceed the number of* α*-regular classes of* G. *The equality holds if* $Z(R)$ *is a splitting field for the* $Z(R)$*-algebra* $R*G$.

PROOF. By hypothesis, $Z(R)^G = Z(R)$ and so the result follows by virtue of Theorem 7. ■

5. Projective crossed representations.

Throughout this section, G denotes a finite group, V a finite-dimensional vector space over a field F and $Z^2(G, F^*)$ the group of all 2-cocycles of G over F^* defined with respect to a specified action of G on F. Given $\alpha \in Z^2(G, F^*)$, we write $F^\alpha G$ for the corresponding crossed product of G over F. Thus $F^\alpha G$ is a free left F-module with basis $\{\bar{g} \mid g \in G\}$ and with multiplication defined distributively by using the identities

$$\bar{x}\bar{y} = \alpha(x, y)\overline{xy} \qquad \text{for all} \quad x, y \in G$$

$$\bar{x}\lambda = {}^x\lambda\bar{x} \qquad \text{for all} \quad x \in G, \lambda \in F$$

where ${}^x\lambda$ denotes the image of λ under the automorphism of F corresponding to x. In what follows we always choose $\bar{1} = 1$ so that $\alpha(g, 1) = \alpha(1, g) = 1$ for all $g\in G$.

The concept of a projective crossed representation of G over F was introduced by Jacobson [3] (under the name projective representation). An important application of projective crossed representations was provided by Isaacs [2], whose used the matrix form of such representations.

Our aim in this section is to apply Theorem 4 in order to provide information on the number of linearly non-equivalent irreducible projective crossed representations. Since no adequate formal treatment of projective crossed representations is available in the literature, we will provide all relevant details which are required for our purposes.

In what follows we write $GS(V)$ for the general semilinear group of V, that is the group of all nonsingular semilinear transformations of V. A mapping

$$\rho: G \longrightarrow GS(V)$$

is called a *projective crossed representation* of G over F if there exists a mapping

$$\alpha: G\times G \longrightarrow F^*$$

such that

(i) $\rho(x)\rho(y)=\alpha(x, y)\rho(xy)$ for all $x, y\in G$

(ii) $\rho(1)=1_V$

To stress the dependence of ρ on V and α, we shall often refer to ρ as an *α-representation* of G on V. For each $g\in G$, let l_g be the automorphism of F determined by $\rho(g)$. Then one immediately verifies that

(a) The formula ${}^g\lambda=l_g(\lambda)$, $g\in G$, $\lambda\in F$, provides an action of G on F.

(b) $\alpha\in Z^2G, F^*)$, where $Z^2(G, F^*)$ is defined with respect to the action of G on F given in (a).

Assume that

$$\rho: G \longrightarrow GS(V)$$

is an α-representation of G on V. If $\alpha(x, y)=1$ for all $x, y\in G$, then we say that ρ is a *crossed representation* of G over F. Thus a crossed representation of G over F is just a homomorphism $\rho: G\rightarrow GS(V)$. In case each $\rho(g)\in GL(V)$, we refer to ρ as a projective *representation* of G over F. Hence ρ is a projective representation if and only if it determines the trivial action of G on F. Finally, if ρ is both crossed and projective representation, then ρ is nothing else but a linear representation of G over F.

Let $\rho: G\rightarrow GS(V)$ be an α-representation of G on V. The *degree* of ρ, written deg ρ, is defined as the dimension of V. A subspace W of V is said to be invariant if W is sent into itself by all semilinear transformations $\rho(g)$,

$g\in G$. We say that ρ is *irreducible,* if O and V are the only invariant subspaces of V. The representation ρ is said to be *completely reducible* if for any invariant subspace W there exists another such subspace W' such that $V=W\oplus W'$. We refer to ρ as being *indecomposable* if V cannot be written as a nontrivial direct sum of invariant subspaces.

Two projective crossed representations

$$\rho_i : G \longrightarrow GS(V_i)$$

are said to be *linearly equivalent* if there exists a vector space isomorphism

$$f : V_1 \longrightarrow V_2$$

such that

$$\rho_2(g)=f\rho_1(g)f^{-1} \qquad \text{for all} \quad g\in G.$$

It is an immediate consequence of the definition that linearly equivalent projective representations determine the same action of G on F and their corresponding cocycles are equal.

The following result shows that the study of α-representations with a fixed action of G on F is equivalent to the study of $F^{\alpha}G$-modules.

LEMMA 9. *Let F be a G-field and let $\alpha\in Z^2(G, F^*)$, where $Z^2(G, F^*)$ is defined with respect to the given action of G on F. Then, there is a bijective correspondence between α-representations of G which determine the given action of G on F and $F^{\alpha}G$-modules. This correspondence maps bijectively linearly equivalent (irreducible, completely reducible, indecomposable) α-representations into isomorphic (irreducible, completely reducible, indecomposable) $F^{\alpha}G$-modules.*

PROOF. Let ρ be an α-representation of G on the space V which gives rise to the given action of G on F. Then $\rho(g)(\lambda v)={}^{g}\lambda\rho(g)v$ for all $\lambda\in F$, $v\in V$, $g\in G$. Hence, a straightforward verification shows that the map

$$f : F^{\alpha}G \longrightarrow \mathrm{End}(V)$$

defined by

$$f\Big(\sum_{g\in G} x_g\bar{g}\Big)=\sum_{g\in G} x_g\rho(g) \qquad (x_g\in F)$$

is a ring homomorphism. Hence V becomes an $F^{\alpha}G$-module by setting

$$\Big(\sum_{g\in G} x_g\bar{g}\Big)v=\sum_{g\in G} x_g\rho(g)v \qquad x_g\in G,\ v\in V.$$

Conversely, given an $F^{\alpha}G$-module V, V is a vector space over F and we define $\rho(g)\in\mathrm{End}(V)$ by $\rho(g)v=\bar{g}v$. Then $\rho(g)$ is invertible and

$$\rho(g)(\lambda v)=\bar{g}(\lambda v)={}^{g}\lambda\bar{g}v={}^{g}\lambda\rho(g) \qquad \text{for all} \quad \lambda\in F,\ g\in G,\ v\in V$$

Thus each $\rho(g)$ lies in $GS(V)$ and the automorphism of F determined by $\rho(g)$ coincides with that determined by g. Furthermore, by the definition of ρ, we have $\rho(1)=1_V$ and $\rho(x)\rho(y)=\alpha(x, y)\rho(xy)$ for all $x, y\in G$. Thus ρ is an α-representation of G on V which determines the given action of G on F. This sets up the desired bijective correspondence.

Let ρ be an α-representation of G on the space V. A subspace W of V is invariant under all $\rho(g)$, $g\in G$ if and only if W is an $F^{\alpha}G$-submodule. Hence the correspondence maps bijectively irreducible (completely reducible, indecomposable) α-representations into irreducible (completely reducible, indecomposable) $F^{\alpha}G$-modules.

Finally, a straightforward argument shows that two α-representations are linearly equivalent if and only if the corresponding $F^{\alpha}G$-modules are isomorphic. So the lemma is true. ■

We are now ready to prove

THEOREM 10. *Let F be a G-field, let $\alpha\in Z^2(G, F^*)$, let X be a full set of representatives for the α-regular classes of G and let $\operatorname{char} F \nmid |G|$. Denote by n the number of linearly nonequivalent irreducible α-representations of G which determine the given action of G on F. Then*

$$n\leqq \sum_{x\in X} \dim_{F^G} F^{C_G(x)}$$

with equality if F^G is a splitting field for the F^G-algebra $F^{\alpha}G$.

PROOF. Since G is finite, the field extension F/F^G is also finite. The desired conclusion is therefore a consequence of Theorem 7 and Lemma 9. ■

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