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ON THE CAUCHY PROBLEM FOR QUASILINEAR HYPERBOLIC-PARABOLIC COUPLED SYSTEMS IN HIGHER DIMENSIONAL SPACES

By

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0. Introduction

In this paper, we consider the Cauchy problem for quasilinear hyperbolic-parabolic coupled systems of second order.

(0.1) $$\partial_t^2\vec{u}-\partial_i A^i(t, x, \theta+T_0, \nabla\vec{u})=\vec{f}(t, x, \theta+T_0, \vec{u}, \nabla\vec{u}, \partial_t\vec{u}),$$

(0.2) $$(\theta+T_0)\partial_t N(t, x, \theta+T_0, \nabla\vec{u})-\partial_i Q^i(t, x, \nabla\theta, \nabla\vec{u})$$
$$=g(t, x, \theta+T_0, \nabla\theta, \vec{u}, \nabla\vec{u}, \partial_t\vec{u}),$$

(0.3) $$\vec{u}(0, x)=\vec{u}_0(x),\ \partial_t\vec{u}(0, x)=\vec{u}_1(x),\ \theta(0, x)=\theta_0(x),$$

where $x\in \boldsymbol{R}^m$ and $t\in[0, T]$. Here and hereafter $\vec{v}={}^t(v_1, \cdots, v_n)$ (${}^tM$ means the transpose M); $\vec{u}$ and θ are unknown functions; T_0 is a positive constant. $\partial_i=\partial/\partial x_i$ and $\partial_t=\partial/\partial t$; ∇ denotes the gradient in x, the summation convention is understood such as sub and superscripts i and j take all values 1 to m; $A^i={}^t(A_1^i, \cdots, A_n^i)$, $\vec{f}={}^t(f_1, \cdots, f_n)$, N, Q^i and g are given nonlinear functions.

(0.1) and (0.2) can be easily extended to the certain kind of hyperbolic and parabolic equations respectively and arise from the thermoelastodynamics theory (cf [**9**]). On this kind of coupled systems with hyperbolic systems of second order and parabolic systems of second order, Slemrod [**14**] studied in the 1-dimensional case. We will show the local and unique solvability of (0.1), (0.2) and (0.3) with certain assumptions.

Our approach is the following: apply the existence theorems of solutions to linear coupled systems of 2nd order hyperbolic system and parabolic equations and estimations obtained by the energy method. Another approach is the following; reduce (0.1), (0.2) and (0.3) to the coupled systems of 1st order hyperbolic systems and 2nd order parabolic equations, and apply the theory due to Kawashima [8] or Zheng [**17**]. But it seems that our approach is the best regarding the minimal order of Sobolev spaces in which solutions exist. Another advantage of our approach is that we can handle the complicated nonlinear

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boundary conditions (e.g. Neumann type) in a similar fashion.

This paper is divided into six sections. Section 1 explains notations, main theorem and a theorem of linear hyperbolic-parabolic coupled systems, Section 2 presents some lemmas, Section 3 presents an energy inequality, Section 4 presents a result and a proof of linear hyperbolic-parabolic coupled systems with smooth coefficients and data, Section 5 presents a proof of linear theorem, and Section 6 presents a proof of main theorem.

1. Notations, assumptions and main results

1.1. Notations

First, we explain basic notations used throughout this paper. We always assume that functions are real-valued. Let $[\cdot]$ be Gauss' symbol. For each integer $s\geqq 0$, $H^s(\boldsymbol{R}^m)$ denotes the space of all $u\in L^2(\boldsymbol{R}^m)$ such that all distribution derivatives $\partial^\alpha u$ with $|\alpha|\leqq s$ belong to $L^2(\boldsymbol{R}^m)$, and we denote its inner product and norms by

$$(u, v)_s=\sum_{|\alpha|\leq s}(\partial^\alpha u, \partial^\alpha v)_0,$$

$$\|u\|_s=(u, u)_s^{1/2},$$

respectively. Here $(\cdot, \cdot)_0$ means the $L^2(\boldsymbol{R}^m)$ inner product.

For any spaces S equipped with norm $|\cdot|$, we denote the product space $S\times\cdots\times S$ and its norm by S and $|\cdot|$ simply. For $\vec{v}$ and $\vec{u}\in H^s(\boldsymbol{R}^m)$, put

$$(\vec{u}, \vec{v})_s=\sum_{a=1}^{m}(u_a, v_a)_s\,;\ |\vec{u}|_s=(\vec{u}, \vec{u})_s^{1/2}.$$

For simplicity, we write L^2 and H^s instead of $L^2(\boldsymbol{R}^m)$ and $H^s(\boldsymbol{R}^m)$.

$\vec{u}\cdot\vec{v}$ denotes the usual inner product in $\boldsymbol{R}^n$. Put $A^{ij}=(A^{ij}_{ab})$ where subscripts a and b denote the row and column, respectively.

Let B^s be the set of all bounded continuous functions whose derivatives up to s are also bounded continuous. We denote its norm by $\|\cdot\|_{B^s}$.

What $f\in B^s(\boldsymbol{R}^m)+H^s$ means that f is expressed as $f=g+h$ with some $g\in B^s(\boldsymbol{R}^m)$ and $h\in H^s$, and we define its norms by

$$|f|_s=\inf\{\|g\|_{B^s}+\|h\|_s\,;\ f=g+h,\ g\in B^s(\boldsymbol{R}^m),\ h\in H^s\}.$$

For $I=[0, T]$, what $f\in B^s(I\times\boldsymbol{R}^m)+L^\infty(I\,;H^s)$ is defined similarly, and we define their norms by

$$|||f|||_s=\inf\{\sup_{t\in I}\|g(t, \cdot)\|_{B^s}+\operatorname{ess-sup}_{t\in I}\|h(t)\|_s\,;$$
$$f=g+h,\ g\in B^s(I\times\boldsymbol{R}^m,\ h\in L^\infty(I\,;H^s)\}.$$

1.2. Assumptions and main results.

Let u_{ja}, θ_j, u_a and u_{0a} be independent variables corresponding to $\partial_j u_a$, $\partial_j\theta$, u_a and $\partial_t u_a$, respectively. Put

$$\zeta=(u_{11}, \cdots, u_{ja}, \cdots, u_{mn});\ \eta=(u_1, \cdots, u_n);$$
$$\eta'=(u_{01}, \cdots, u_{0n});\ \xi=(\theta_1, \cdots, \theta_m).$$

Let L be a positive constant and put

$$D_1=[0, \infty)\times\boldsymbol{R}^m\times\{(\theta, \zeta);\ |\theta|<L,\ |\zeta|<L\},$$

$$D_2=[0, \infty)\times\boldsymbol{R}^m\times\{(\xi, \zeta);\ |\xi|<L,\ |\zeta|<L\},$$

$$D_3=[0, \infty)\times\boldsymbol{R}^m\times\{(\theta, \eta, \zeta, \eta');\ |\theta|<L,\ |\eta|<L,\ |\zeta|<L,\ |\eta'|<L\},$$

$$D_4=[0, \infty)\times\boldsymbol{R}^m\times\{(\theta, \xi, \eta, \zeta, \eta');\ |\theta|<L,\ |\xi|<L,\ |\eta|<L,\ |\zeta|<L,\ |\eta'|<L\}.$$

(A.1) Let T_0 be a positive constant. Assume that

$A_a^i(t, x, \theta+T_0, \zeta)$ and $N(t, x, \theta+T_0, \zeta)\in B^\infty(D_1)$; $Q_i(t, x, \xi, \zeta)\in B^\infty(D_2)$;

$f_a(t, x, \theta+T^0, \eta, \zeta, \eta')\in B^\infty(D_3)$; $g(t, x, \theta+T_0, \xi, \eta, \zeta, \eta')\in B^\infty(D_4)$.

Here the A_a^i, N, Q^i, f_a and g are nonliear functions appearing in (0.1) and (0.2).

Let us define the notations to represent some derivatives of functions above.

DEFINITION 1.1.

$$B_a^i(t, x, \theta+T_0, \zeta)=\frac{\partial}{\partial\theta}A_a^i(t, x, \theta+T_0, \zeta);$$

$$A_{ab}^{ij}(t, x, \theta+T_0, \zeta)=\frac{\partial}{\partial\zeta_{jb}}A_a^i(t, x, \theta+T_0, \zeta);$$

$$P(t, x, \theta+T_0, \zeta)=(\theta+T_0)\frac{\partial}{\partial\theta}N(t, x, \theta+T_0, \zeta);$$

$$N_b^j(t, x, \theta+T_0, \zeta)=(\theta+T_0)\frac{\partial}{\partial\zeta_{jb}}N(t, x, \theta+T_0, \zeta);$$

$$Q^{ij}(t, x, \xi, \zeta)=\frac{\partial}{\partial\xi_j}Q^i(t, x, \xi, \zeta);$$

$$R_b^{ij}(t, x, \xi, \zeta)=\frac{\partial}{\partial\zeta_{jb}}Q^i(t, x, \xi, \zeta);$$

$$F_a(t, x, \theta+T_0, \eta, \zeta, \eta')=f_a(t, x, \theta+T_0, \eta, \zeta, \eta')+\frac{\partial}{\partial x_i}A_a^i(t, x, \theta+T_0, \eta);$$

$$G(t, x, \theta+T_0, \xi, \eta, \zeta, \eta')=g(t, x, \theta+T_0, \xi, \eta, \zeta, \eta')$$
$$-(\theta+T_0)\frac{\partial}{\partial t}N(t, x, \theta+T_0, \eta)+\frac{\partial}{\partial x_i}Q^i(t, x, \xi, \eta).$$

Put

$$A^{ij}=(A^{ij}_{ab});\ \vec{B}^i={}^t(B^i_1, \cdots, B^i_n);\ \vec{N}^i=(N^i_1, \cdots, N^i_n);$$

$$\vec{R}^{ij}=(R^{ij}_1, \cdots, R^{ij}_n);\ \vec{F}={}^t(F_1, \cdots, F_n).$$

Here A^{ij} are $n\times n$ matrices whose subscripts a and b denote the row and column, respectively.

Furthermore we assume that

(A.2)

(1) For $i, j=1, \cdots, m$, ${}^tA^{ij}=A^{ji}$, $Q^{ij}=Q^{ji}$.

(2) There exists a $\delta_1>0$ such that for all $\nu\in\boldsymbol{R}^m$

$$A^{ij}(t, x, \theta+T_0, \zeta)\nu_i\nu_j\geqq\delta_1|\nu|^2 I_n,$$

$$Q^{ij}(t, x, \xi, \zeta)\nu_i\nu_j\geqq\delta_1|\nu|^2,$$

uniformly in $(t, x, \theta, \zeta)\in D_1$ and $(t, x, \xi, \zeta)\in D_2$, where I_n is the unit $n\times n$ matrix.

(3) Thre exists a $\delta_2>0$ such that

$$\frac{\partial}{\partial\theta}N(t, x, \theta+T_0, \zeta)\geqq\delta_2,$$

for $(t, x, \theta, \zeta)\in D_1$.

(4) $$\vec{F}(t, x, T_0, 0, 0, 0)=0, \quad G(t, x, T_0, 0, 0, 0, 0)=0,$$

for all $t\in[0, \infty)$ and $x\in\boldsymbol{R}^m$.

Our purpose is to prove

THEOREM 1.2. *Let $s\geqq[m/2]+1$ be an integer. Assume* (A.1) *and* (A.2). *If the initial data $\vec{u}_0$, $\vec{u}_1$, θ_0 satisfy the conditions*

$$\vec{u}_0\in H^{s+3},\quad \vec{u}_1\in H^{s+2},\quad \theta_0\in H^{s+2},$$

$$\|\vec{u}_0\|_{L^\infty},\ \|\nabla\vec{u}_0\|_{L^\infty},\ \|\vec{u}_1\|_{L^\infty},\ \|\nabla\theta_0\|_{L^\infty}<L,$$

$$\|\theta_0\|_{L^\infty}<\min\{T_0, L\},$$

then there is a $T>0$ such that the Cauchy problem (0.1), (0.2) *and* (0.3) *admits a unique pair of solutions*

$$\vec{u}\in C(I; H^{s+3})\cap C^1(I; H^{s+2})\cap C^2(I; H^{s+1}),$$

$$\theta\in L^2(I; H^{s+3})\cap C(I; H^{s+2})\cap C^1(I; H^s),$$

where $I=[0, T]$.

The method of our proof of this theorem is standard: apply the successive approximation to solutions of a related linear problem. For this reason we shall need to examine the linear system

(1.1) $$\partial_t^2\vec{u}-A^{ij}(t, x)\partial_i\partial_j\vec{u}-\vec{B}^i(t, x)\partial_i\theta=\vec{F}(t, x),$$

(1.2) $$P(t, x)\partial_t\theta-Q^{ij}(t, x)\partial_i\partial_j\theta+\vec{R}^{ij}(t, x)\cdot\partial_i\partial_j\vec{u} +\vec{N}^j(t, x)\cdot\partial_t\partial_j\vec{u}=G(t, x),$$

with initial conditions

(1.3) $$\vec{u}(0, x)=\vec{u}_0(x), \quad \partial_t\vec{u}(0, x)=\vec{u}_1(x), \quad \theta(0, x)=\theta_0(x).$$

Here $A^{ij}=(A_{ab}^{ij})$ are $n\times n$ matrices; $B^i={}^t(B_1^i, \cdots, B_n^i)$; $R^{ij}=(R_1^{ij}, \cdots, R_n^{ij})$; $N^j=(N_1^j, \cdots, N_n^j)$; $\vec{F}={}^t(F_1, \cdots, F_n)$; A^{ij}, B^i, P, Q^{ij}, R^{ij} and N^j are functions such that

(1.4) $$A^{ij} \text{ and } \vec{B}^i\in B^\infty(I\times \boldsymbol{R}^m)+L^\infty(I; H^{s+2})\cap C(I; H^{s+1});$$
$$\partial_t A^{ij}\in B^\infty(I\times \boldsymbol{R}^m)+L^\infty(I; H^s);$$
$$P, Q^{ij}, \vec{N}^j, \vec{R}^{ij}\in B^\infty(I\times \boldsymbol{R}^m)+L^\infty(I; H^{s+1})\cap C(I; H^s),$$

where $I=[0, T]$ and $s\geqq[m/2]+1$.

Assume that the following conditions hold;

(1.5) $${}^tA^{ij}=A^{ji}, \quad Q^{ij}=Q^{ji} \ (i, j=1, \cdots, m).$$

(1.6) There exists a $\delta_3>0$ such that for $\nu\in\boldsymbol{R}^m$

$$A^{ij}(t, x)\nu_i\nu_j\geqq\delta_3|\nu|^2\boldsymbol{I}_n,$$

$$Q^{ij}(t, x)\nu_i\nu_j\geqq\delta_3|\nu|^2,$$

for $t\in[0, T]$ and $x\in\boldsymbol{R}^m$.

(1.7) There exists a $\delta_4>0$ such that $P(t, x)\geqq\delta_4$. Furthermor there exists $P_\infty\in B^\infty(I\times\boldsymbol{R}^m)$ and $P_s\in L^\infty(I; H^{s+1})\cap C(I; H^s)$ such that $P=P_\infty+P_s$ and $P_\infty(t, x)\geqq\delta_5$ for some $\delta_5>0$ uniformly in t and x.

THEOREM 1.3 (Linear theorem). *Let* $s\geqq[m/2]+1$ *and* $T>0$. *Put* $I=[0, T]$. *If*

$$\vec{F}\in L^2(I; H^{s+2})\cap C(I; H^{s+1}), \quad G\in L^2(I; H^{s+1})\cap C(I; H^s),$$

and the initial data satisfy

$$\vec{u}_0\in H^{s+3}, \quad \vec{u}_1\in H^{s+2}, \quad \theta_0\in H^{s+2},$$

then for any T' *of* $0<T'<T$, *the Cauchy problem* (1.1), (1.2) *and* (1.3) *admits a unique pair of solutions*

$$\vec{u}\in C(I'; H^{s+3})\cap C^1(I'; H^{s+2})\cap C^2(I'; H^{s+1}),$$

$$\theta\in L^2(I'; H^{s+3})\cap C(I'; H^{s+2})\cap C^1(I'; H^s),$$

where $I'=[0, T']$.

The strategy of our proof of Theorem 1.2 is the following: First we regularize the coefficients and data. Secondly we prove the existence of solutions to such regularized problems by using the abstract theory of evolution equations. Finally we prove the convergence of such a family of solutions by using the energy inequality.

2. Some lemmas

We list below the Sobolev embedding theorems and the Gårding's inequality. The proofs may be omitted.

LEMMA 2.1 (Sobolev embedding theorem). *Let* s *be an integer* $\geqq[m/2]+1$.

(1) *If* k *be a non negative integer, then* $H^{s+k}(\boldsymbol{R}^m)\subset B^k(\boldsymbol{R}^m)$, *where* $\subset$ *implies the continuity of the embedding.*

(2) *Suppose that* $p_j\geqq 0$, $j=1, \cdots, k$, *and that*

$$\min_{1\leq h\leq k}\min_{j_1<\cdots<j_h}\{p_{j_1}+\cdots+p_{j_h}-(h-1)([m/2]+1)\}=r\geqq 0.$$

Then $\sum_{j=1}^{k} H^{p_j}\subset H^r$.

LEMMA 2.2 (Gårding's inequality). *Let* s *be an integer* $\geqq[m/2]+1$. *For* $i, j=1, \cdots, m$ *let* $A^{ij}\in H^{s+1}(\boldsymbol{R}^m)$. *Assume that there exists a* $\delta>0$ *such that*

$$A^{ij}(x)\nu_i\nu_j\geqq\delta|\nu|^2\boldsymbol{I}_n$$

for $\nu\in\boldsymbol{R}^m$ *and* $x\in\boldsymbol{R}^m$, *and that*

$$^tA^{ij}=A^{ji}.$$

Then for any $0\leqq r\leqq s+1$ (*resp.* $r=0$) *and* $0<\delta'<\delta$, *there exists a constant* $\lambda>0$ *depending on* $s, m, n, r, \delta, \delta'$ *and* $\|A^{ij}\|_{s+1}$ (*resp.* $\|A^{ij}\|_{B^1}$) *such that*

$$\lambda(\vec{u}, \vec{u})_r+(A^{ij}\partial_i\vec{u}, \partial_j\vec{u})_r\geqq\delta'(\vec{u}, \vec{u})_{r+1} \tag{2.1}$$

for any $\vec{u}={}^t(u_1, \cdots, u_n)\in H^{r+1}$.

3. Energy inequality

In this section we present an inequality useful in Section 5 and 6. For preparation we list two lemmas below. A proof of first one may be omitted.

LEMMA 3.1. *Let* $s\geqq[m/2]+1$, $A\in B^{\infty}(\boldsymbol{R}^m)+H^{s+1}$ *and* $u\in H^s$. *Let* $\phi\in C_0^{\infty}(\boldsymbol{R}^m)$ *such that* $\operatorname{supp}\phi\subset\{x\in\boldsymbol{R}^m;\ |x|\leqq 1\}$, $\phi(x)\geqq 0$ *for* $x\in\boldsymbol{R}^m$ *and* $\int_{\boldsymbol{R}^m}\phi(x)dx=1$. *For* $\delta>0$ *put* $\phi_\delta(x)=\delta^{-m}\phi(\delta^{-1}x)$. *Then*

(1) $$\|\phi_\delta*(Au)-A(\phi_\delta*u)\|_{s+1}\leqq C\|u\|_s\ (\delta>0),$$

where C *is some constant depending on* s, m, ϕ *and* $|A|_{s+1}$,

(2) $$\|\phi_\delta*(Au)-A(\phi_\delta*u)\|_{s+1}\longrightarrow 0\ (\delta\to 0).$$

LEMMA 3.2. *Let* s *be an integer* $\geqq[m/2]+1$.

(1) *Let* $P=a+b$ *where* $a\in B^{\infty}(\boldsymbol{R}^m)$ *and* $b\in H^s$ *such that* $a(x)\geqq\tau>0$ *and* $P(x)\geqq\delta>0$ *for some constants* τ *and* δ. *Then* $P^{-1}\in B^{\infty}(\boldsymbol{R}^m)+H^s$ *and*

$$|P^{-1}|_s\leqq C\tau^{-1}(1+\tau^{-s}\|a\|_{B^{\infty}}^s)(1+\delta^{-s-1}|P|_s^{s+1})$$

where C *is a constant depending on* s *and* m.

(2) *If* $P,Q\in B^{\infty}(\boldsymbol{R}^m)+H^s$, *then* $PQ\in B^{\infty}(\boldsymbol{R}^m)+H^s$ *and*

$$|PQ|_s\leqq C'|P|_s|Q|_s$$

where C' *is a constant depending on* s *and* m.

PROOF. (1) Note that $P^{-1}=a^{-1}-a^{-1}(a+b)^{-1}b$. Obviously, we see that $a^{-1}\in B^{\infty}(\boldsymbol{R}^m)$ and

(3.1) $$\|a^{-1}\|_{B^s}\leqq C\tau^{-1}(1+\tau^{-s}\|a\|_{B^s}^s).$$

Put $g=(a+b)^{-1}b$. Our task is to show that $g\in H^s$ and

(3.2) $$\|g\|_s\leqq C\delta^{-1}\|b\|^s(1+\delta^{-s}|P|_s^s).$$

By direct calculation, we have for $|\alpha|\leqq s$

(3.3) $$\partial^\alpha g=\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\{\sum C_{\alpha_1\cdots\alpha_k}(a+b)^{-k-1}\partial^{\alpha_1}P\cdots\partial^{\alpha_k}P\}\partial^{\alpha-\beta}b$$

where summation in braces above is taken over all combinations of multi-indices $\alpha_1>0,\cdots,\alpha_k>0$ such that $\alpha_1+\cdots+\alpha_k=\beta$. By Sobolev embedding theorem, each term in left side of (3.3) belongs to L^2, so we get $g\in H^s$ and (3.2). From (3.1) and (3.2) we obtain

(3.4) $$|P^{-1}|_s\leqq C\tau^{-1}(1+\tau^{-s}\|a\|_{B^s}^s)(1+\delta^{-s-1}|P|_s^{s+1}).$$

(2) Since the proof is easy, we may omit it.

For the simplicity of notation, in Sections 3, 4 and 5, we use the same letter c to denote various constants depending on some of the following: δ_3, δ_4, δ_5, s, m, n, r and the norms describing the smoothness of the coefficients appearing in assumptions of Theorem 1.3.

Put for $0\leqq r\leqq s+2$

$$\|U(t)\|_r^2=\|\partial_t\vec{u}(t)\|_r^2+\|\vec{u}(t)\|_{r+1}^2+\|\theta(t)\|_r^2, \tag{3.5}$$

$$E_r(t)=\|\partial_t\vec{u}(t)\|_r^2+\sum_{|\alpha|\leq r}(A^{ij}(t)\partial^\alpha\partial_i\vec{u}(t),\ \partial^\alpha\partial_j\vec{u}(t))_0+\lambda_0\|\vec{u}(t)\|_r^2+\|\theta(t)\|_r^2, \tag{3.6}$$

where λ_0 is some constant such that (2.1) with $\delta'=\delta_3/2$ and $r=0$ is satisfied uniformly in $t\in I$, and it depends on s, m, n, δ_3 and $\operatorname{ess-sup}_{t\in I}\|A^{ij}(t)\|_{B^1}$.

The result of this section is

THEOREM 3.3 (Energy inequality). *Under the assumptions of Theorem* 1.3., *if linear Cauchy problem* (1.1), (1.2) *and* (1.3) *has a solution such that*

$$\vec{u}\in L^\infty(I';H^{s+3})\cap C(I';H^{s+2})\cap C^1(I';H^{s+1}), \tag{3.7}$$

$$\partial_t\vec{u}\in L^\infty(I';H^{s+2}), \tag{3.8}$$

$$\theta\in L^2(I';H^{s+3})\cap L^\infty(I';H^{s+2})\cap C(I';H^{s+1}), \tag{3.9}$$

where $I'=[0,T']$ *of* $0<T'\leqq T$, *then for* $1\leqq r\leqq s+2$ *and* $t\in I'$,

$$\|U(t)\|_r^2+2e^{ct}\int_0^t\|\theta\|_{s+3}^2dt\leqq 2/\delta_3\left\{E_r(0)+\int_0^t(\|\vec{F}\|_r^2+c\|G\|_{r-1}^2)dt\right\}e^{ct}. \tag{3.10}$$

PROOF. Let (1.2)′ be an equality dividing both side of (1.2) by P and put $\tilde{Q}^{ij}=P^{-1}Q^{ij}$, $\tilde{\vec{R}}^{ij}=P^{-1}\vec{R}^{ij}$, $\tilde{\vec{N}}^j=P^{-1}\vec{N}^j$ and $\tilde{G}=P^{-1}G$.

First we assume $\vec{u}$ and θ are smooth in x. Then we see easily that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_r=&(\partial_t^2\vec{u},\ \partial_t\vec{u})_r+\frac{1}{2}\sum_{|\alpha|\leq r}\{(\partial_tA^{ij}\partial^\alpha\partial_i\vec{u},\ \partial^\alpha\partial_j\vec{u})_0\\&+(A^{ij}\partial_t\partial^\alpha\partial_i\vec{u},\ \partial^\alpha\partial_j\vec{u})_0+(A^{ij}\partial^\alpha\partial_i\vec{u},\ \partial_t\partial^\alpha\partial_j\vec{u})_0\}\\&+\lambda_0(\partial_t\vec{u},\ \vec{u})_r+(\partial_t\theta,\ \theta)_r.\end{aligned} \tag{3.11}$$

By (1.1) and (1.2)′, it follows that

$$\begin{aligned}(\partial_t^2\vec{u},\ \partial_t\vec{u})_r=&\sum_{|\alpha|\leq r}\{(A^{ij}\partial^\alpha\partial_i\partial_j\vec{u},\ \partial_t\partial^\alpha\vec{u})_0\\&+\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta A^{ij}\partial^{\alpha-\beta}\partial_i\partial_j\vec{u},\ \partial_t\partial^\alpha\vec{u})_0\}+(B^i\cdot\partial_i\theta,\ \partial_t\vec{u})_r\\&+(\vec{F},\ \partial_t\vec{u})_r,\end{aligned} \tag{3.12}$$

and

(3.13) $$(\partial_t\theta,\ \theta)_r=(\tilde{Q}^{ij}\partial_i\partial_j\theta+\tilde{R}^{ij}\cdot\partial_i\partial_j\vec{u}-\tilde{N}^j\cdot\partial_t\partial_j\vec{u}+\tilde{G},\ \theta)_r\,.$$

By assumptions on A^{ij} it follows that

(3.14) $$\frac{1}{2}(A^{ij}\partial_t\partial^\alpha\partial_i\vec{u},\ \partial^\alpha\partial_j\vec{u})_0+\frac{1}{2}(A^{ij}\partial^\alpha\partial_i\vec{u},\ \partial_t\partial^\alpha\partial_j\vec{u})_0$$
$$+(A^{ij}\partial^\alpha\partial_i\partial_j\vec{u},\ \partial_t\partial^\alpha\vec{u})_0=-(\partial_jA^{ij}\partial^\alpha\partial_j\vec{u},\ \partial_t\partial^\alpha\vec{u})_0$$

for $|\alpha|\leqq r$. On the other hand, it follows that

(3.15) $$(\tilde{Q}^{ij}\partial_i\partial_j\theta,\ \theta)_r=(\tilde{Q}^{ij}\partial_i\partial_j\theta,\ \theta)_{r-1}+\sum_{|\alpha|=r}(\partial^\alpha(\tilde{Q}^{ij}\partial_i\partial_j\theta),\ \partial^\alpha\theta)_0\,.$$

Dividing $\alpha=\alpha_1+\alpha_2$ with $\alpha_1>0$, $\alpha_2>0$ and $|\alpha_1|=1$, then we get

(3.16) $$(\partial^\alpha(\tilde{Q}^{ij}\partial_i\partial_j\theta),\ \partial^\alpha\theta)_0=-(\partial^{\alpha_2}(\tilde{Q}^{ij}\partial_i\partial_j\theta),\ \partial^{\alpha+\alpha_1}\theta)_0$$
$$=-\sum_{0<\beta\leqq\alpha_2}\binom{\alpha_2}{\beta}(\partial^\beta\tilde{Q}^{ij}\partial^{\alpha_2-\beta}\partial_i\partial_j\theta,\ \partial^{\alpha+\alpha_1}\theta)_0$$
$$+(\partial^{\alpha_1}\tilde{Q}^{ij}\partial^{\alpha_2}\partial_i\partial_j\theta,\ \partial^\alpha\theta)_0-(\tilde{Q}^{ij}\partial^\alpha\partial_i\theta,\ \partial^\alpha\partial_j\theta)_0$$
$$-(\partial_j\tilde{Q}^{ij}\partial^\alpha\partial_i\theta,\ \partial^\alpha\theta)_0\,.$$

Substituting (3.12)-(3.16) into (3.11), integrating by parts and applying Lemma 3.2 we obtain

(3.17) $$\frac{1}{2}\frac{d}{dt}E_r\leqq c\varepsilon^{-1}(\|\partial_t\vec{u}\|_r^2+\|\vec{u}\|_{r+1}^2+\|\theta\|_r^2)$$
$$+(c\varepsilon-\delta_3)\|\theta\|_{r+1}^2+\frac{1}{2}(\|\vec{F}\|_r^2+c\varepsilon^{-1}\|G\|_{r-1}^2)\,,$$

for any $\varepsilon>0$. By Gårding's inequality it follows that

(3.18) $$\delta_3/2(\|\partial_t\vec{u}\|_r^2+\|\vec{u}\|_{r+1}^2+\|\theta\|_r^2)\leqq E_r\,.$$

Put $\varepsilon=\delta_3/(2c)$ then by Gronwall's inequality we get

(3.19) $$E_r(t)+\delta^s e^{ct}\int_0^t\|\theta\|_{r+1}^2dt\leqq\left\{E_r(0)+\int_0^t(\|\vec{F}\|_r^2+c\|G\|_{r-1}^2)dt\right\}e^{ct}\,,$$

for $t\in I'$. Applying (3.18) again, then we obtain (3.10).

In the case $\vec{u}$ and θ satisfy (3.7)-(3.9), operating $\phi_\delta *$ to (1.1) and (1.2)′, and deforming them, then we get

$$(3.20)\qquad \partial_t^2\vec{u}-A^{ij}\partial_i\partial_j\vec{u}^\delta-\vec{B}^i\cdot\partial_i\theta^\delta=\vec{F}_\delta,$$

$$(3.21)\qquad \partial_t\theta^\delta-\tilde{Q}^{ij}\partial_i\partial_j\theta^\delta-\tilde{R}^{ij}\cdot\partial_i\partial_j\vec{u}^\delta-\tilde{N}^j\cdot\partial_t\partial_j\vec{u}^\delta=\tilde{G}_\delta,$$

where $\vec{u}^\delta=\vec{u}*\phi_\delta$, $\theta^\delta=\theta*\phi_\delta$ and

$$\vec{F}_\delta=\vec{F}*\phi_\delta+(A^{ij}\partial_i\partial_j\vec{u})*\phi_\delta-A^{ij}\cdot\partial_i\partial_j\vec{u}^\delta+(\vec{B}^i\cdot\partial_i\theta)*\phi_\delta-\vec{B}^i*\partial_i\theta^\delta,$$

$$\tilde{G}_\delta=\tilde{G}*\phi_\delta+(\tilde{Q}^{ij}\partial_i\partial_j\theta)*\phi_\delta-\tilde{Q}^{ij}\partial_i\partial_j\theta^\delta+(\tilde{R}^{ij}\cdot\partial_i\partial_j\vec{u}^\delta)*\phi_\delta$$
$$-\tilde{R}^{ij}\cdot\partial_i\partial_j\vec{u}^\delta-(\tilde{N}^j\cdot\partial_t\partial_j\vec{u})*\phi_\delta+\tilde{N}^j\cdot\partial_t\partial_j\vec{u}^\delta.$$

Since $\vec{u}^\delta$ and θ^δ are sufficiently smooth, we have

$$\|U^\delta(t)\|_r^2+2e^{ct}\int_0^t\|\theta^\delta\|_{r+1}^2dt\leqq 2/\delta_3\left\{E_r^\delta(0)+\int_0^t(\|\vec{F}_\delta\|_r^2+c\|\tilde{G}_\delta\|_{r-1}^2)d\tau\right\}e^{ct},$$

for $t\in I'$ where $U^\delta(t)$ and $E_r^\delta(0)$ are defined by replacing $\vec{u}$ and θ by $\vec{u}^\delta$ and θ^δ in (3.5) and (3.6). Letting $\delta\downarrow 0$, by Lemma 3.1 and the assumptions of $\vec{u}$ and θ we have (3.10).

Applying the theorem above, we obtain the following:

COROLLARY 3.4. *Under the assumptions of Theorem* 1.3 *if* (1.1), (1.2) *and* (1.3) *has a solution such that*

$$\vec{u}\in L^\infty(I';H^{s+3})\cap C(I';H^{s+2})\cap C^1(I';H^{s+1}),$$
$$\partial_t\vec{u}\in L^\infty(I';H^{s+2}),$$
$$\theta\in L^\infty(I';H^{s+2})\cap C(I';H^{s+1}),$$

where $I'=[0,T']$ *of* $0<T'\leqq T$, *then it follows that*

$$\vec{u}\in C(I';H^{s+3})\cap C^1(I';H^{s+2})\cap C^2(I';H^{s+1}),$$
$$\theta\in C(I';H^{s+2})\cap C^1(I';H^s).$$

PROOF. Let $\vec{u}^\delta$ and θ^δ be the same as in the proof of Theorem 3.3. For $\delta,\delta'>0$, applying Theorem 3.3 to $\vec{u}^\delta-\vec{u}^{\delta'}$ and $\theta^\delta-\theta^{\delta'}$ and using Lemma 3.1, we have

$$\sup_{t\in I'}(\|\partial_t(\vec{u}^\delta-\vec{u}^{\delta'})\|_{s+2}^2+\|\vec{u}^\delta-\vec{u}^{\delta'}\|_{s+3}^2+\|\theta^\delta-\theta^{\delta'}\|_{s+2}^2)\longrightarrow 0$$

as $\delta,\delta'\downarrow 0$. Thus $\partial_t\vec{u}^\delta(t)$, $\vec{u}^\delta(t)$ and $\theta^\delta(t)$ are Cauchy sequences on δ in H^{s+2}, H^{s+3} and H^{s+2} respectively, and convergences are uniform in $t\in I'$. The Corollary follows easily from the following lemma.

LEMMA 3.5. *Let X and Y be Banach spaces and Y be continuously embedded in X. Let $T>0$ be arbitrary and put $I=[0, T]$.*

Assume that $u_n \in C(I; Y) \cap C^1(I; X)$ for all positive integers n and that they satisfy

$$\|\partial_t u_n(t)\|_X \leqq C$$

uniformly in t and n where C is some constant. If

$$u_n \longrightarrow u \text{ in } Y \text{ as } n \longrightarrow \infty \text{ uniformly in } t \in I, \tag{3.22}$$

$$\partial_t u_n \longrightarrow v \text{ in } X \text{ as } n \longrightarrow \infty \text{ at each } t \in I \tag{3.23}$$

and $v \in C(I; X)$, then $\partial_t u = v$ and $u \in C(I; Y) \cap C^1(I; X)$.

PROOF. From (3.22) it follows that $u \in C(I; Y)$. For any t, $s \in I$ and positive integer n, we have

$$u_n(t) - u_n(s) = \int_s^t \partial_t u_n(\tau) d\tau.$$

Letting $n \to \infty$, by Lebesgue's convergence theorem, we get $\partial_t u = v$ and $u \in C(I; Y) \cap C^1(I; X)$.

4. A result on the linear hyperbolic-parabolic coupled systems with smooth coefficients and data.

For a preparation of our proof of Theorem 1.3, we consider linear problem (1.1), (1.2) and (1.3) with smooth coefficients and data. The result of this section is

LEMMA 4.1. *Let $s \geqq [m/2]+1$, $0 \leqq r \leqq s$ and $T>0$. Put $I=[0, T]$. Let*

$$A^{ij}, \vec{B}^i, Q^{ij}, \vec{R}^{ij}, \vec{N}^j \in B^\infty(I \times \boldsymbol{R}^m) + C^1(I; H^{s+1}),$$
$$P \equiv 1,$$

for $i, j=1, \cdots, m$. Furthermore, we assume (1.5) *and* (1.6) *on A^{ij} and Q^{ij}. If*

$$\vec{F} \in C(I; H^{r+1}), \quad G \in C(I; H^{r+2}),$$

and the initial data satisfy

$$\vec{u}_0 \in H^{r+2}, \quad \vec{u}_1 \in H^{r+1}, \quad \theta_0 \in H^{r+2},$$

then the Cauchy problem (1.1), (1.2) *and* (1.3) *admits a unique solution*

$$\vec{u} \in C(I; H^{r+2}) \cap C^1(I; H^{r+1}) \cap C^2(I; H^r),$$
$$\theta \in C(I; H^{r+2}) \cap C^1(I; H^r).$$

The reason why $P\equiv 1$ is that under the assumptions of Theorem 1.3 (Linear theorem), there is no essentially change of properties of coefficients if both sides of (1.2) is divided by P. We can see this by Lemma 3.2.

To prove Lemma 4.1, we use the following theorem of linear evolution equation.

THEOREM 4.2. *Let X and Y be Banach spaces and Y is continuously and densely embedded in X. Let us consider the equations*:

$$(L) \qquad \partial_t U(t)-A(t)U(t)=H(t), \quad U(0)=U_0.$$

We assume that

(1) $A=\{A(t); t\in I\}$ is a stable family of generators of C_0-semigroups on X with stability constants M and β. (The definition of this notion is given by [**4**].)

(2) The domain $D(A(t))\equiv Y$.

(3) $A(\cdot)\in C_*^1(I; L(Y:X))$, i.e. $A(t)\in L(Y:X)$ for $t\in I$ and $A(\cdot)U\in C^1(I; X)$ for $U\in Y$. ($L(Y:X)$ means the set of all bounded linear operators of Y into X. The symbol C_*^1 appears in [**6**].)

(4) $H\in C(I; Y)$.

(5) $U_0\in Y$.

Then (L) has a unique solution

$$U\in C(I; Y)\cap C^1(I; X).$$

REMARK. Under the weaker assumptions on $A(t)$, Kato [**4**], [**5**] proved Theorem 4.2.

Put

$$\partial_t\vec{u}=\vec{v}\,;\ U(t)=(\vec{u}(t),\ \vec{v}(t),\ \theta(t))\,;\ U_0=(\vec{u}_0,\ \vec{u}_1,\ \theta_0)\,;$$

$$A(t)U(t)=(\vec{v}(t),\ A^{ij}(t)\partial_i\partial_j\vec{u}+B_1(t)\theta,\ A_2(t)\theta+B_2(t)(\vec{u}(t),\ \vec{v}(t)))\,; \tag{4.1}$$

$$H(t)=(0,\ \vec{F}(t),\ G(t)),$$

where

$$B_1(t)\theta=\vec{B}^i(t)\partial_i\theta\,, \tag{4.2}$$

$$A_2(t)\theta(t)=Q^{ij}(t)\partial_i\partial_j\theta\,, \tag{4.3}$$

$$B_2(t)(\vec{u}(t),\ \vec{v}(t))=\vec{R}^{ij}(t)\cdot\partial_i\partial_j\vec{u}-\vec{N}^j(t)\cdot\partial_j\vec{v}\,. \tag{4.4}$$

Then, in terms of the present $U(t)$, U_0, $A(t)$ and $H(t)$ we can rewrite (1.1), (1.2) and (1.3) in the form (L).

In the present case, we put

$$X=\{U=(\vec{u},\ \vec{v},\ \theta);\ \vec{u}\in H^{r+1},\ \vec{v}\in H^{r},\ \theta\in H^{r}\},$$

$$Y=\{U=(\vec{u},\ \vec{v},\ \theta);\ \vec{u}\in H^{r+2},\ \vec{v}\in H^{r+1},\ \theta\in H^{r+2}\}.$$

If we put

$$\|U\|_X^2=\|\vec{u}\|_{r+1}^2+\|\vec{v}\|_r^2+\|\theta\|_r^2 \qquad \text{for} \quad U=(\vec{u},\ \vec{v},\ \theta)\in X,$$

$$\|U\|_Y^2=\|\vec{u}\|_{r+2}^2+\|\vec{v}\|_{r+1}^2+\|\theta\|_{r+2}^2 \qquad \text{for} \quad U=(\vec{u},\ \vec{v},\ \theta)\in Y,$$

then X and Y are Banach spaces and Y is densely and continuously embedded in X. By Lemma 2.2 and the definition of λ_0 (c.f. (3.6)), let us define the innerproduct $(U, U')_t$ $(t\in I)$ of X by

$$(U,\ U')_t=\langle\vec{u},\ \vec{u}'\rangle_t+(\vec{v},\ \vec{v}')_r+(\theta,\ \theta')_r, \tag{4.5}$$

where

$$\langle\vec{u},\ \vec{u}'\rangle_t=\frac{1}{2}(A^{ij}(t)\partial_i\vec{u},\ \partial_j\vec{u}')_r+\frac{1}{2}(A^{ij}(t)\partial_i\vec{u}',\ \partial_j\vec{u})_r+\lambda_0(\vec{u},\ \vec{u}')_r.$$

In particular, the norms $\|U\|_t=(U,\ U)_t^{1/2}$ and $\|U\|_X$ are equivalent for all $t\in I$, i.e.,

$$(\delta_3/2)^{1/2}\|U\|_X\leqq\|U\|_t\leqq c\|U\|_X$$

for $U\in X$ and $t\in I$.

Obviously, the present $A(t)$, $H(t)$ and U_0 satisfy the conditions (2)-(5) in Theorem 4.2. Our task is to show that (1) is satisfied. To do this we shall prove that

(i) the range of $\lambda-A(t)$ equals X for large λ,

(ii) $|(A(t)U,\ U)_t|\leqq c\|U\|_t^2$ for $U\in Y$ and $t\in I$,

(iii) $\|U\|_t\leqq\|U\|_{t'}\exp(c|t-t'|)$ for $U\in X$ and $t,\ t'\in I$,

where c is a constant.

If we prove (i), (ii) and (iii), then by Prop. 3.4 of [4] we can see easily that $A(t)$ is a stable family.

To prove (i), (ii) and (iii) we use the following lemmas.

LEMMA 4.3. *Let A^{ij} be the same as in Lemma 4.1. For $0\leqq r\leqq s$, put*

$$X_r=\{V\in(\vec{u},\ \vec{v});\ \vec{u}\in H^{r+1},\ \vec{v}\in H^{r}\},$$

$$\|V\|_r^2=\|\vec{u}\|_{r+1}^2+\|\vec{v}\|_r^2,$$

$$(V,\ V')_{r,t}=\langle\vec{u},\ \vec{u}'\rangle_t+(\vec{v},\ \vec{v}')_r,$$

for $V=(\vec{u},\ \vec{v})$, $V'=(\vec{u}',\ \vec{v}')\in X_r$.

Put $A_1(t)V=(\vec{v},\ A^{ij}(t)\partial_i\partial_j\vec{u})$ and let $\|\cdot\|_{r:r}$ denote the norm of operators of X_r into itself. Then the, following two assertions are valid.

(1) *For $0\leqq r\leqq s$ and $V\in X_{r+1}$, it follows that*

$$(A_1(t)V, V)_{r,t} \leqq c\|V\|_r^2, \quad (t \in I).$$

(2) *Let* $0 \leqq r \leqq s$. *There exist some constants* $M>0$ *and* $\beta \in R$ *such that for* $\lambda>\beta$ *operators* $\lambda - A_1(t): X_{r+1} \to X_r$ $(t \in I)$ *are surjective and for* $\lambda > \beta$

$$\|(\lambda - A_1(t))^{-1}\|_{r:r} \leqq \frac{M}{\lambda-\beta} \quad (t \in I).$$

LEMMA 4.4. *Let* $t \in I$ *be fixed and let* $Q^{ij} \in B^\infty + H^{s+1}$ *satisfy* (1.5) *and* (1.6). *Let* λ *is sufficiently large then*

$$\lambda - A_2 : H^{r+2} \longrightarrow H^r,$$

is surjective and

$$\|(\lambda - A_2)^{-1}\|_{r:r+2} \leqq C,$$

where C *is some constant depending on* s, m, r, δ_3 *and* $|Q^{ij}|_{s+1}$.

Defering the proofs of lemmas above, we shall prove (i), (ii) and (iii). To prove (i), it suffices that for given $\Phi = (\vec{f}, \vec{g}, \phi) \in X$ we find $U = (\vec{u}, \vec{v}, \theta)$ such that $(\lambda - A(t))U = \Phi$, provided that λ is sufficiently large. Put $V = (\vec{u}, \vec{v})$ and $\Psi = (\vec{f}, \vec{g})$. Then what $(\lambda - A(t))U = \Phi$ implies that

$$\lambda V - A_1(t)V - \tilde{B}_1(t)\theta = \Psi, \tag{4.6}$$

$$\lambda\theta - A_2(t)\theta - B_2(t)V = \phi, \tag{4.7}$$

where $\tilde{B}_1(t) = (0, B_1(t))$.

Let $t \in I$ be fixed and omitted from now. If $V \in X_{r+1}$, then $B_2 U \in H^r$. Hence by Lemma 4.4 we see that (4.7) can be solved in θ as follows,

$$\theta = (\lambda - A_2)^{-1}\{\phi + B_2 V\} \in H^{r+2}, \tag{4.8}$$

if λ is sufficiently large. Substituting (4.8) into (4.6), then we have the equation:

$$(\lambda - A_1 - B_1(\lambda - A_2)^{-1}B_2)V = \Psi + B_1(\lambda - A_2)^{-1}\phi, \tag{4.9}$$

for unknown V. Put $B = B_1(\lambda - A_2)^{-1}B_2$. By Lemma 4.4 we see that B is a bounded linear operator of X_r into itself and that there exists a $K>0$ independent of λ such that $\|B\|_{r:r} \leqq K$ for any large λ. Since

$$\lambda - A_1 - B = (\lambda - A_1)\{1 - (\lambda - A_1)^{-1}B\},$$

and since

$$\|(\lambda - A_1)^{-1}B\|_{r:r} \leqq \frac{MK}{\lambda - \beta} \quad (\lambda > \beta),$$

for some $M>0$ and $\beta \in R^1$ as follows from Lemma 4.3, we see that the inverse operator of the $\{1 - (\lambda - A_1)^{-1}B\}$ exists for $\lambda > MK + \beta$ by Neumann series. Since

$\lambda-A_1$ is surjective mapping on X_{r+1} to X_r for large λ, (4.9) has a unique solution $V\in X_{r+1}$. If we define θ by (4.8) and put $U=(V, \theta)$ then $U=X$ and $(\lambda-A(t))U=\Phi$, which shows (i).

Next we shall prove (ii). Let $U=(\vec{u}, \vec{v}, \theta)\in Y$ and $V=(\vec{u}, \vec{v})$. By Lemma 4.3 (1) and the definition of $A(t)$ (c.f. (4.1)), we have

$$(A(t)U, U)_t\leqq c/\varepsilon(\|V\|_{r,t}^2+\|\theta\|_r^2) \\ +(c\varepsilon-\delta_3/2)\|\theta\|_{r+1}^2,$$

for any $\varepsilon>0$. If we take $\varepsilon>0$ sufficiently small, then (ii) is proved.

Next we shall show (iii). Let $U\in X$ and $t, t'\in I$. Then it follows easily that

$$|(U, U)_t-(U, U)_{t'}|\leqq|t-t'|c\sum|||\partial_t A^{ij}|||_s\|U\|_X^2.$$

Hence we have

$$\|U\|_t^2\leqq\|U\|_{t'}^2+c|t-t'|\,\|U\|_{t'}^2\leqq\|U\|_{t'}^2\exp(c|t-t'|),$$

and the proof is completed.

Finally we shall give proofs of Lemma 4.3 and 4.4.

PROOF OF LEMMA 4.3.

(1) Let $W=(\vec{u}, \vec{v})\in X_{r+1}$. By definition of $A_1(t)$, we have

$$(A_1(t)W, W)_{r,t}=\frac{1}{2}(A^{ij}(t)\partial_i\vec{v}, \partial_j\vec{u})_r+\frac{1}{2}(A^{ij}(t)\partial_j\vec{u}, \partial_j\vec{v})_r \\ +\lambda_0(\vec{v}, \vec{u})_r+(A^{ij}(t)\partial_i\partial_j\vec{u}, \vec{v})_r.$$

Applying integration by parts to the last term in right side of the above inequality and using equations:

$$(A^{ij}(t)\partial^\alpha\partial_i\vec{v}, \partial^\alpha\partial_j\vec{u})_0=(A^{ij}(t)\partial^\alpha\partial_i\vec{u}, \partial^\alpha\partial_j\vec{v})_0$$

for $|\alpha|\leqq r$, we obtain (1).

(2) First we will show the surjectiveness of $\lambda-A_1(t)$. Let $F=(\vec{f}, \vec{g})\in X_r$. That $(\lambda-A_1(t))W=F$ implies that

$$\lambda\vec{u}-\vec{v}=\vec{f}, \tag{4.10}$$

$$\lambda\vec{v}-A^{ij}(t)\partial_i\partial_j u=\vec{g}, \tag{4.11}$$

Substituting (4.10) into (4.11), we obtain

$$\lambda^2\vec{u}-A^{ij}(t)\partial_i\partial_j\vec{u}=\vec{g}+\lambda\vec{f} \tag{4.12}$$

Since $\vec{g}+\lambda\vec{f}\in H^r$, (4.12) has a unique solution $\vec{u}\in H^{r+2}$ if λ is sufficiently large. If we define $\vec{v}$ by (4.10), then $\vec{v}\in H^{r+1}$ and, $\vec{u}$ and $\vec{v}$ satisfy (4.10) and (4.11).

Next we shall estimate the resolvent of $A_1(t)$. Let $W\in X_{r+1}$. By the result of (1),

$$((\lambda-A_1(t))W,\ (\lambda-A_1(t))W)_{r,t} \tag{4.13}$$
$$\geqq\lambda^2(W,\ W)_{r,t}-2\lambda|(A_1(t)W,\ W)_{r,t}|$$
$$\geqq(\lambda-C')^2\|W\|_{r,t}^2,\qquad \text{if}\quad \lambda>C'$$

where $C'=4c/\delta_3$. Since norms $\|\cdot\|_r$ and $\|\cdot\|_{r,t}$ are equivalent, there are some constants $M>0$ and $\beta>0$ such that

$$\|(\lambda-A_1(t))^{-1}\|_{r:r}\leqq\frac{M}{\lambda-\beta},\quad (t\in I)$$

for $\lambda>\beta$. This completes the proof.

PROOF OF LEMMA 4.4. Let $f\in H^{r+2}$. Then, the equation:

$$\lambda\theta-Q^{ij}\partial_i\partial_j\theta=f$$

has a unique solution $\theta\in H^{r+2}$ if λ is sufficiently large. By a priori estimate for the elliptic equations we have,

$$\|\theta\|_{r+2}\leqq C\|f\|_r$$

where C depends on $s,\ m,\ r,\ \delta_3,\ |Q_{ij}|_{s+1}$. This completes the proof.

5. A proof of linear theorem (Theorem 1.3).

Let $\phi\in C_0^\infty(\boldsymbol{R}^m)$ such that $\operatorname{supp}\phi\subset\{x\in\boldsymbol{R}^m;\ |x|\leqq1\}$, $\phi(x)\geqq0$ and $\int_{\boldsymbol{R}^m}\phi(x)dx=1$ and let $\rho\in C_0^\infty(\boldsymbol{R}^1)$ such that $\operatorname{supp}\rho\subset[-1,0]$, $\rho(t)\geqq0$ and $\int_{-\infty}^{+\infty}\rho(t)dt=1$. For $\sigma>0$ put

$$\phi_\sigma(x)=\sigma^{-m}\phi(x\sigma^{-1}),\quad (x\in\boldsymbol{R}^m)$$
$$\rho_\sigma(t)=\sigma^{-1}\rho(t\sigma^{-}{}_1),\quad (t\in\boldsymbol{R}^1).$$

Let T' of any $0<T'<T$ be fixed, and for any σ of $0<\sigma<T-T'$ put

$$P_\sigma(t,\ x)=(P\underset{(x)}{*}\phi_\sigma)\underset{(t)}{*}\rho_\sigma.$$

In the same manner we define A_σ^{ij}, $\vec{B}_\sigma^i$, Q_σ^{ij}, $\vec{R}_\sigma^{ij}$ and $\vec{N}_\sigma^j$. In the case of regularizing only in x, we put

$$F_a^\sigma(t,\ x)=F_a\underset{(x)}{*}\phi_\sigma,\quad (a=1,\ \cdots,\ n).$$

We define g^σ and θ_0^σ in the same manner. On the other hand, put $\tilde{Q}_\sigma^{ij}=P_\delta^{-1}Q_\delta^{ij}$, $\tilde{R}_\delta^{ij}=P_\sigma^{-1}\vec{R}_\sigma^{ij}$, $\tilde{N}_\sigma^j=P_\sigma^{-1}\vec{N}_\sigma^j$ and $\tilde{G}^\sigma=P_\sigma^{-1}G^\sigma$. Then by Lemma 4.1, the following linear problem:

(5.1) $$\partial_t^2\vec{u}-A_\sigma^{ij}\partial_i\partial_j\vec{u}-\vec{B}_\sigma^i\cdot\partial_i\theta=\vec{F}^\sigma,$$

(5.2) $$\partial_t\theta-\tilde{Q}_\sigma^{ij}\partial_i\partial_j\theta-\tilde{R}_\sigma^{ij}\cdot\partial_i\partial_j\vec{u}+\tilde{N}_\sigma^j\cdot\partial_t\partial_j\vec{u}=\tilde{G}^\sigma,$$

(5.3) $$\vec{u}(0)=\vec{u}_0,\quad \partial_t\vec{u}(0)=\vec{u}_1,\quad \theta(0)=\theta_0^\sigma$$

has a unique pair of solutions

$$\vec{u}\in C(I';H^{s+3})\cap C_1(I';H^{s+2})\cap C^2(I';H^{s+1}),$$

$$\theta\in C(I';H^{s+3})\cap C^1(I';H^{s+1}),$$

where $I'=[0, T']$. We denote these solutions by $\vec{u}_\sigma$, θ_σ. From (3.10), if follows that

(5.4) $$\|U_\sigma(t)\|_{s+2}^2+2e^{cT'}\int_0^{T'}\|\theta_\sigma\|_{s+3}^2dt$$

$$\leqq 2/\delta_3\left\{E_{s+2,\sigma}(0)+\int_0^{T'}(\|\vec{F}^\sigma\|_{s+2}^2+c\|G^\sigma\|_{s+1}^2)dt\right\}e^{cT'},$$

for $t\in I'$ where $\|U_\sigma(t)\|_{s+2}^2$ and $E_{s+2,\sigma}$ are defined by replacing $\vec{u}$ and θ by $\vec{u}_\sigma$ and θ_σ in (3.5) and (3.6). Since the constants c in (5.4) and λ_0 in (3.6) are independent of t and σ, from (5.4) it follows that there exists a constant $M>0$ such that

(5.5) $$\|U_\sigma(t)\|_{s+2}^2+2e^{cT'}\int_0^{T'}\|\theta_\sigma\|_{s+3}^2dt\leqq M^2,$$

for $t\in I'$ and $0<\sigma\leqq T-T'$.

Let $0<\sigma, \sigma'<T-T'$. Then $\vec{u}_\sigma-\vec{u}_{\sigma'}$ and $\theta_\sigma-\theta_{\sigma'}$ satisfy the following equation:

$$\partial_t^2(\vec{u}_\sigma-\vec{u}_{\sigma'})-A_\sigma^{ij}\partial_i\partial_j(\vec{u}_\sigma-\vec{u}_{\sigma'})-\vec{B}_\sigma^i\cdot\partial_i(\theta_\sigma-\theta_{\sigma'})=\vec{F}_{\sigma,\sigma'},$$

$$\partial_t(\theta_\sigma-\theta_{\sigma'})-\tilde{Q}_\sigma^{ij}\partial_i\partial_j(\theta_\sigma-\theta_{\sigma'})-\tilde{R}_\sigma^{ij}\cdot\partial_i\partial_j(\vec{u}_\sigma-\vec{u}_{\sigma'})+\tilde{N}_\sigma^j\cdot\partial_t\partial_j(\vec{u}_\sigma-\vec{u}_{\sigma'})=\tilde{G}_{\sigma,\sigma'},$$

with initial conditions

$$(\vec{u}_\sigma-\vec{u}_{\sigma'})(0)=0,\quad \partial_t(\vec{u}_\sigma-\vec{u}_{\sigma'})(0)=0,\quad (\theta_\sigma-\theta_{\sigma'})(0)=\theta_0^\sigma-\theta_0^{\sigma'},$$

where

$$\vec{F}_{\sigma,\sigma'}=\vec{F}^\sigma-\vec{F}^{\sigma'}+(A_\sigma^{ij}-A_{\sigma'}^{ij})\partial_i\partial_j\vec{u}_{\sigma'}+(\vec{B}_\sigma^i-\vec{B}_{\sigma'}^i)\cdot\partial_i\theta_{\sigma'},$$

$$\tilde{G}_{\sigma,\sigma'}=\tilde{G}^\sigma-\tilde{G}^{\sigma'}+(\tilde{Q}_\sigma^{ij}-\tilde{Q}_{\sigma'}^{ij})\partial_i\partial_j\theta_{\sigma'}+(\tilde{R}_\sigma^{ij}-\tilde{R}_{\sigma'}^{ij})\cdot\partial_i\partial_j\vec{u}_{\sigma'}-(\tilde{N}_\sigma^j-\tilde{N}_{\sigma'}^j)\cdot\partial_t\partial_j\vec{u}_{\sigma'}.$$

Applying (3.10) with $p=s+1$ and letting $\sigma, \sigma'\downarrow 0$, we can see that $\{\partial_t\vec{u}(t)\}$, $\{\vec{u}_\sigma(t)\}$ and $\{\theta_\sigma(t)\}$ are Cauchy sequences in H^{s+1}, H^{s+2} and H^{s+1}, respectively and their convergences are uniform in $t\in I'$. Let $\vec{u}'$, $\vec{u}$, θ be the limits of

$\{\partial_t \vec{u}_\sigma\}$, $\{\vec{u}_\sigma\}$, $\{\theta_\sigma\}$, respectively. Then we can see easily

$$\vec{u}' \in C(I'; H^{s+1}), \quad \vec{u} \in C(I'; H^{s+2}), \quad \theta \in C(I'; H^{s+1})$$

and that $\partial_t \vec{u} = \vec{u}'$. By (5.5), $\{\theta_\sigma\}$ is bounded in $L^2(I'; H^{s+3})$. We can choose a subsequence $\{\theta_{\sigma_i}\}$ of $\{\theta_\sigma\}$ which converges weekly in $L^2(I'; H^{s+3})$. Let us put its limit in $\tilde{\theta}$. For $\{\theta_\sigma\}$ converges to θ in $C(I'; H^{s+1})$, we can see easily $\tilde{\theta} = \theta$ a.e. $t \in I'$, which implies $\theta \in L^2(I'; H^{s+3})$.

By Lemma 3.5 and (5.1)-(5.3), we see that u, θ is a solution of (1.1), (1.2), (1.3) and satisfies

$$\vec{u} \in L^\infty(I'; H^{s+3}) \cap (C(I'; H^{s+2}) \cap C^1(I'; H^{s+1}) \cap C^2(I'; H^s),$$

$$\partial_t \vec{u} \in L^\infty(I'; H^{s+2}),$$

$$\theta \in L^2(I'; H^{s+3}) \cap L^\infty(I'; H^{s+2}) \cap C(I'; H^{s+1}) \cap C^1(I'; H^{s-1}).$$

Thus by Corollary 3.4 we get

$$\vec{u} \in C(I'; H^{s+3}) \cap C^1(I'; H^{s+2}) \cap C^2(I'; H^{s+1}),$$

$$\theta \in L^2(I'; H^{s+3}) \cap C(I'; H^{s+2}) \cap C^1(I'; H^s).$$

Uniquness of solution is seen clearly by energy inequality (3.10).

6. A proof of main theorem

For preparation we list two lemmas.

LEMMA 6.1 (Shibata [13] Theorem Ap. 6). *Let I be a closed interval and*

$$u \in L^\infty(I; H^{[m/2]+1}(\boldsymbol{R}^m)) \cap Lip(I; H^{[m/2]}(\boldsymbol{R}^m)).$$

Then for any $\varepsilon \in (0, [m/2]+1-m/2)$, $u \in B^\varepsilon(I \times \boldsymbol{R}^m)$. Furthermore

$$|u(t, x) - u(s, y)| \leqq C|(t, x(-(s, y)|^\varepsilon |u|_{1,[m/2],I,\boldsymbol{R}^m}$$

where C is a constant depending on n and ε, and

$$|u|_{1,[m/2],I,\boldsymbol{R}^m} = \operatorname*{ess\text{-}sup}_{t \in I} \|u(t)\|_{[m/2]+1} + \sup_{\substack{t,s \in I \\ t \neq s}} \frac{\|u(t) - u(s)\|_{[m/2]}}{t-s}.$$

LEMMA 6.2. *Let $\vec{a} = (a_1, \cdots, a_k) \in \boldsymbol{R}^k$, $\Omega = \{y \in \boldsymbol{R}^k; |y_i - a_i| \leqq \sigma_i, \sigma_i \geqq 0$, for $i = 1, \cdots, k\}$, $A \in B^\infty(\boldsymbol{R}^m \times \Omega)$, s be an integer $\geqq [m/2]+1$ and $0 \leqq K \leqq C_0^{-1} \min\{\sigma_i\}$ is a constant. For any $\vec{u} = (u_1, \cdots, u_k)$, $\vec{v} = (v_1, \cdots, v_k) \in H^s(\boldsymbol{R}^m)$ of $\|u_i\|_s$, $\|v_i\|_s \leqq K$ $(i = 1, \cdots, k)$, it follows that*

$$A(\cdot, \vec{u}(\cdot) + \vec{a}) - A(\cdot, \vec{v}(\cdot) + \vec{a}) \in H^s,$$

$$\|A(\cdot, \vec{u}(\cdot) + \vec{a}) - A(\cdot, \vec{v}(\cdot) + \vec{a})\|_s \leqq C\|\vec{u} - \vec{v}\|_s,$$

where C is some constant depending on s, m, k, K and $\|A\|_{B^{s+1}}$.

PROOF.

$$A(x, \vec{u}(x)+\vec{d})-A(x, \vec{v}(x)+\vec{d}) \tag{6.1}$$
$$=\int_0^1 \sum_i \frac{\partial A}{\partial y_i}(x, \tau\vec{u}(x)+(1-\tau)\vec{v}(x)+\vec{d})(u_i(x)-v_i(x))d\tau.$$

Let τ and i be fixed and put $F=\frac{\partial A}{\partial y_i}$, $\phi_i=\tau\vec{u}+(1-\tau)\vec{v}$ and $\psi=u_i-v_i$. For $|\alpha|\leqq s$, by direct calculation we have

$$\partial^\alpha F(\cdot, \phi+\vec{d})\psi=\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\{\sum C_{\gamma_1\cdots\gamma_j}B(\cdot, \phi+\vec{d})\partial^{\gamma_1}\omega_1\cdots\partial^{\gamma_j}\omega_j\}\partial^{\alpha-\beta}\psi, \tag{6.2}$$

where B is some derivative of F of order$\leqq j$ and for each l, $\omega_l=\phi_i$ for some $i=1, \cdots, k$. The summation in braces above is taken over all combinations of multi-indices $\gamma_1>0, \cdots, \gamma_j>0$ such that $\gamma_1+\cdots+\gamma_j=\beta$. By Sobolev embedding theorem, each term of right side of (6.2) belongs to L^2, so we have

$$A(\cdot, \vec{u}(\cdot)+\vec{d})-A(\cdot, \vec{v}(\cdot)+\vec{d})\in H^s,$$

and

$$\|A(\cdot, \vec{u}(\cdot)+\vec{d})-A(\cdot, \vec{v}(\cdot)+\vec{d})\|_s\leqq C\|A\|_{B^{s+1}}\sum_{j=1}^{s}K^j\|\vec{u}-\vec{v}\|_s.$$

where C is a constant depending on s, m and k.

Let us proceed to the main issue. The method of the proof is the successive approximation.

Let $s\geqq[m/2]+1$. For any initial data such that

$$\vec{u}_0\in H^{s+3},\quad \vec{u}_1\in H^{s+2},\quad \theta_0\in H^{s+2},$$

$$\|\vec{u}_0\|_{L^\infty}, \|\nabla\vec{u}_0\|_{L^\infty}, \|\vec{u}_1\|_{L^\infty}, \|\nabla\theta_0\|_{L^\infty}<L, \tag{6.3}$$

$$\|\theta_0\|_{L^\infty}<\min\{T_0, L\}, \tag{6.4}$$

we choose positive constants L', δ, γ such that

$$\max\{\|\vec{u}_0\|_{L^\infty}, \|\nabla\vec{u}_0\|_{L^\infty}, \|\vec{u}_1\|_{L^\infty}, \|\nabla\theta_0\|_{L^\infty}\}<L'<L, \tag{6.5}$$

$$\|\theta_0\|_{L^\infty}<\delta<\min\{T_0, L\}, \tag{6.6}$$

$$\|\vec{u}_0\|_{B^2}<\gamma. \tag{6.7}$$

In the definition of E_r (3.6), λ_0 can be chosen as some constant depending on s, m, n, δ_1, γ, L' and $\|A^i\|_{B^2}$ such that

$$\lambda_0(\vec{u}, \vec{u})_r+\sum_{|\alpha|\leq r}(A^{ij}\partial^\alpha\partial_i\vec{u},\ \partial^\alpha\partial_j\vec{u})_0\geqq\delta_1/2(\vec{u}, \vec{u})_{r+1}, \tag{6.8}$$

is satisfied for $0\leqq r\leqq s+2$ and $t\in[0, T]=I$ when $\vec{u}$ and θ satisfy (6.10), (6.12), (6.17) and (6.18) below. The coefficients A^{ij} in (6.8) are functions of $(t, x)\in I\times\boldsymbol{R}^m$ substituted by $\theta(t, x)+T_0$, $\nabla\vec{u}(t, x)$.

Let M be some positive constant such that

$$(2/\delta_1)E_{s+2}(0)<M^2, \tag{6.9}$$

where

$$E_{s+2}(0)=\|\vec{u}_1\|_{s+2}^2+\lambda_0\|\vec{u}_0\|_{s+2}^2+\|\theta_0\|_{s+2}^2 + \sum_{|\alpha|\leqq s+2}(A^{ij}(0, \cdot, \theta_0+T_0, \nabla u_0)\partial^\alpha\partial_i\vec{u}_0, \partial^\alpha\partial_j\vec{u}_0)_0.$$

From now, we denote by c various constants depending on some of the following: $s, m, n, \delta_1, \delta_2, T_0, L$, which appear in assumptions of Theorem 1.2; δ, L', γ, M appearing above; B^d-norms $(d\in\boldsymbol{N})$ and values of special points of $A^i, \vec{f}, N, Q^i$ and g.

Now let us define $\vec{u}^p$ and θ^p for $p\in\boldsymbol{N}$ by induction on p with fixed δ, L', γ, M above. Let T' be a positive number determined below. For any $p\in\boldsymbol{N}$, put $T_p=T''+1/p(T'-T'')$, $I_p=[0, T_p]$ and $I''=[0, T'']$ where T'' is an arbitrary number$\in(0, T')$. Assume that $\vec{u}^k$ and θ^k for $1\leqq k\leqq p-1$ are already defined and satisfy the following conditions:

$$\vec{u}^k\in L^\infty(I_k; H^{s+3})\cap C(I_k; H^{s+2})\cap C^1(I_k; H^{s+1}), \tag{6.10}$$

$$\partial_t\vec{u}^k\in L^\infty(I_k; H^{s+2}), \tag{6.11}$$

$$\theta^k\in L^2(I_k; H^{s+3})\cap L^\infty(I_k; H^{s+2})\cap C(I_k; H^{s+1}), \tag{6.12}$$

$$\partial_t\theta^k\in L^\infty(I_k; H^s), \tag{6.13}$$

$$\vec{u}^k(0)=\vec{u}_0,\quad \partial_t\vec{u}^k(0)=\vec{u}_1,\quad \theta^k(0)=\theta_0, \tag{6.14}$$

$$\sup_{t\in I_k}\|U_k(t)\|_{s+2}^2+2\exp(cT_k)\int_{I_k}\|\theta^k\|_{s+3}^2dt\leqq M^2, \tag{6.15}$$

$$\sup_{t\in I_k}\|\theta^k(t)\|_{L^\infty}\leqq\delta, \tag{6.16}$$

$$\sup_{t\in I_k}(\|\vec{u}^k(t)\|_{L^\infty}, \|\nabla\vec{u}^k(t)\|_{L^\infty}, \|\partial_t\vec{u}^k(t)\|_{L^\infty}, \|\nabla\theta^k(t)\|_{L^\infty})\leqq L', \tag{6.17}$$

$$\sup_{t\in I_k}\|\vec{u}^k(t)\|_{B^2}\leqq\gamma. \tag{6.18}$$

Then, we shall define $\vec{u}^p$ and θ^p as follows. Let $A^{ij}(k)$ be a function of (t, x) substituting $\vec{u}^k$ and θ^k in A^{ij}. Similarly we define $\vec{B}^i(k)$, $\vec{F}(k)$, $P(k)$, $Q^{ij}(k)$, $\vec{R}^{ij}(k)$, $\vec{N}^j(k)$, $G(k)$. Then they satisfy all the hypotheses of the linear theorem (Theorem 1.3) with $\delta_3=\delta_1$, $\delta_4=(T_0-\delta)\delta_2$ and $\delta_5=\delta_2$ by Lemma 6.2. Thus the following linear problem:

$$(*)\quad\begin{cases} \partial_t^2\vec{u}-A^{ij}(p-1)\partial_i\partial_j\vec{u}-\vec{B}^i(p-1)\cdot\partial_i\theta=\vec{F}(p-1) \\ P(p-1)\partial_t\theta-Q^{ij}(p-1)\partial_i\partial_j\theta-\vec{R}^{ij}(p-1)\cdot\partial_i\partial_j\vec{u}+\vec{N}^i(p-1)\cdot\partial_t\partial_j\vec{u}=G(p-1) \\ \vec{u}(0)=\vec{u}_0,\quad \partial_t\vec{u}(0)=\vec{u}_1,\quad \theta(0)=\theta_0 \end{cases}$$

has a unique pair of solutions $\vec{u}$ and θ on I_p which satisfies the conclusions of Theorem 1.3. We denote these solutions by $\vec{u}^p$ and θ^p.

On the other hand, let $\vec{u}^1$, θ^1 be the solution of (*) on $[0, T']$ with $\vec{u}^0\equiv 0$, $\theta^0\equiv 0$. Then $\vec{u}^1$ and θ^1 satisfy the conditions (6.15)-(6.18) if $T'>0$ is sufficiently small.

Applying (3.10) to (*) by Lemma 6.2 and (6.10)-(6.18) we see that

$$\|U^p(t)\|_{s+2}^2+2\exp(cT_p)\int_{I_p}\|\theta^p\|_{s+3}^2dt \leqq 2/\delta_1\{E_{s+2}(0)+cT_p\}\exp(cT_p),\quad (t\in I_p). \tag{6.19}$$

Thus, by the definition of M, we see that if $T'>0$ is sufficiently small then

$$\sup_{t\in I_p}\|U^p(t)\|_{s+2}^2+2\exp(cT_p)\int_{I_p}\|\theta^p\|_{s+3}^2dt\leqq M^2. \tag{6.20}$$

By the system of equations (*), (6.20) and Lemma 6.2 we can see easily that

$$\|\partial_t^2\vec{u}^p(t)\|_s\leqq c,\quad (t\in I_p), \tag{6.21}$$

$$\|\partial_t\theta^p(t)\|_s\leqq c,\quad (t\in I_p). \tag{6.22}$$

Thus by Sobolev's embedding theorem,

$$\|\theta^p(t)-\theta_0\|_{L^\infty}\leqq C_0\|\theta^p(t)-\theta_0\|_s\leqq C_0ct,$$

for $t\in I_p$. In the same manner it follows that

$$\|\partial_t\vec{u}^p(t)-\vec{u}_1\|_{L^\infty}\leqq C_0ct,$$

$$\|\vec{u}^p(t)-\vec{u}_0\|_{L^\infty}\leqq C_0Mt,$$

$$\|\nabla\vec{u}^p(t)-\nabla\vec{u}_0\|_{L^\infty}\leqq C_0Mt,$$

$$\|\vec{u}^p(t)-\vec{u}_0\|_{B^2}\leqq C_0Mt.$$

Also by Lemma 6.1, put $\varepsilon=1/2([m/2]+1-m/2)$ then we get

$$|\nabla\theta^p(t,x)-\nabla\theta^p(0,x)|\leqq ct^\varepsilon(\sup_{t\in I_p}\|\nabla\theta^p(t)\|_s+\sup_{t\in I_p}\|\partial_t\nabla\theta^p(t)\|_{s-1})$$

$$\leqq ct^\varepsilon(M+b).$$

Thus $\vec{u}^p$ and θ^p satisfy (6.16)-(6.18) if $T'>0$ is sufficiently small.

Now, we shall show $\vec{u}^p$, θ^p converge as $p\to\infty$ if $T''>0$ is sufficiently small. Note that $T_p>T''>0$ for all $p\in N$.

$\vec{u}^{p+1}+\vec{u}^{p}$ and $\theta^{p+1}-\theta^{p}$ satisfy the following equation:

$$\partial_t^2(\vec{u}^{p+1}-\vec{u}^{p})-A^{ij}(p)\partial_i\partial_j(\vec{u}^{p+1}-\vec{u}^{p})-\vec{B}^{i}(p)\cdot\partial_i(\theta^{p+1}-\theta^{p})=\vec{F}_p,$$

$$P(p)\partial_t(\theta^{p+1}-\theta^{p})-Q^{ij}(p)\partial_i\partial_j(\theta^{p+1}-\theta^{p})-\vec{R}^{ij}(p)\cdot\partial_i\partial_j(\vec{u}^{p+1}-\vec{u}^{p})$$
$$+\vec{N}^{j}(p)\cdot\partial_t\partial_j(\vec{u}^{p+1}-\vec{u}^{p})=G_p,$$

with initial conditions

$$(\vec{u}^{p+1}-\vec{u}^{p})(0)=0,\ \partial_t(\vec{u}^{p+1}-\vec{u}^{p})(0)=0,\quad (\theta^{p+1}-\theta^{p})(0)=0,$$

where

$$\vec{F}_p=\vec{F}(p)-\vec{F}(p-1)+(A^{ij}(p)-A^{ij}(p-1))\partial_i\partial_j\vec{u}^{p}+(\vec{B}^{i}(p)-\vec{B}^{i}(p-1))\cdot\partial_i\theta^{p},$$

$$G_p=G(p)-G(p-1)-(P(p)-P(p-1))\partial_t\theta^{p}+(Q^{ij}(p)-Q^{ij}(p-1))\partial_i\partial_j\theta^{p}$$
$$+(\vec{R}^{ij}(p)-\vec{R}^{ij}(p-1))\cdot\partial_i\partial_j\vec{u}^{p}-(\vec{N}^{j}(p)-\vec{N}^{j}(p-1))\cdot\partial_t\partial_j\vec{u}^{p}.$$

By (3.10) and Lemma 6.2 we get

$$\sup_{t\in I''}(\|\vec{u}^{p+1}-\vec{u}^{p}\|_{s+2}^2+\|\partial_t\vec{u}^{p+1}-\partial_t\vec{u}^{p}\|_{s+1}^2+\|\theta^{p+1}-\theta^{p}\|_{s+1}^2)\tag{6.23}$$
$$\leqq 2/\delta_1\cdot cT''\exp(cT'')\sup_{t\in I''}(\|\vec{u}^{p}-\vec{u}^{p-1}\|_{s+2}^2+\|\partial_t\vec{u}^{p}-\partial_t\vec{u}^{p-1}\|_{s+1}^2$$
$$+\|\theta^{p}-\theta^{p-1}\|_{s+1}^2).$$

Thus if $T''>0$ is sufficiently small then we have

$$(2/\delta_1)cT''\exp(cT'')<1.\tag{6.24}$$

Then $\{\vec{u}^{p}(t)\}$, $\{\partial_t\vec{u}^{p}(t)\}$ and $\{\theta^{p}(t)\}$ are Cauchy sequences on $p\in N$ in H^{s+2}, H^{s+1} and H^{s+1}, respectively and they converge uniformly in $t\in I''$. Let $\vec{u}$, $\vec{u}'$, θ be the limits of $\{\vec{u}^{p}\}$, $\{\partial_t\vec{u}^{p}\}$, $\{u^{p}\}$. Then from the definition of $\vec{u}^{p}$ and θ^{p} and the boundedness of $\{\theta^{p}\}$ in $L^2(I''; H^{s+3})$ and Lemmas 5.1 and 6.2 it follows that $\partial_t\vec{u}=\vec{u}'$ and that $\vec{u}$ and θ satisfy (6.10)-(6.18) with $\vec{u}^{k}$, θ^{k} and I^{k} replaced by $\vec{u}$, θ and I''. Since $\vec{u}$ and θ satisfy (*) with $\vec{u}^{p-1}$ and θ^{p-1} replaced by $\vec{u}$ and θ, by Corollary 3.4 we obtain

$$\vec{u}\in C(I''; H^{s+3})\cap C^1(I''; H^{s+2})\cap C^2(I''; H^{s+1}),$$
$$\theta\in L^2(I''\ H^{s+3})\cap C(I''; H^{s+2})\cap C^1(I''; H^{s}).$$

Thus $\vec{u}$ and θ are so smooth in t and x that $\vec{u}$ and θ are solutions to (0.1), (0.2) and (0.3).

The uniqueness of solutions follows easily from the same arguments as the calculations of (6.23) and (6.24).

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