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EXPONENTIAL AND SUPER-EXPONENTIAL LOCALIZATIONS FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH LÉVY NOISE POTENTIALS

By

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§ 0. Introduction.

Let us consider the following random Schrödinger operator in $L^2(\boldsymbol{R}; dt)$:

$$H_{\omega}=-d^2/dt^2+Q'_{\omega}(t), \tag{0-1}$$

where $\{Q'_{\omega}(t); -\infty<t<+\infty\}$ is a temporally homogeneous Lévy process and $Q'_{\omega}(t)$ is the "derivative" of its sample function. Intuitively speaking, $\{Q'_{\omega}(t)\}_{t\in\boldsymbol{R}}$ is a continuous parameter family of i. i. d. random variables, which we will call "Lévy noise", so that the above H_{ω} can be viewed as an idealization of the Schrödinger operator with random potential, and it may be of some interest to analyze in detail such an idealized model of disordered system.

On the other hand, it is well known that almost every sample function of a Lévy process is not differentiable (except the case of a trivial Lévy process $Q_{\omega}(t)=ct$, with a real constant c). Hence the expression (0-1) has only a symbolical meaning. The precise definition of H_{ω} was given by the present author ([25]), and it was shown that H_{ω} can actually be realized as a random self-adjoint operator in $L^2(\boldsymbol{R}; dt)$. Moreover, the exact location of the spectrum of H_{ω} was determined.

The purpose of this paper is to study the properties of spectrum and eigenfunctions of H_{ω} in more detail than in [25]. It will be shown that under some condition on $\{Q_{\omega}(t)\}$, almost every realization of H_{ω} has pure point spectrum with exponentially decaying eigenfunctions (exponential localization—see Theorem 5). A remarkable fact is that in some other cases, the eigenfunctions decay faster than exponentially (Theorem 6). We would like to refer to this phenomenon as "super-exponential localization". Moreover, it will be shown that under some conditions on the Lévy measure of $\{Q_{\omega}(t)\}$, the eigenfunctions, in a rough sense, behave like $\exp[-|t|^{\alpha}]$ with $\alpha>1$, or even like $\exp[-\exp[\exp[\cdots\exp[|t|^{\alpha}]\cdots]]$ with $\alpha>0$, as $|t|\to\infty$ (Theorem 7).

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In the proof of localization, we follow the well-known idea of Carmona and Kotani ([3], [16]), which relates in an elegant way the exponential decay of eigenfunctions with the Ljapounov behavior of the transfer matrices. Thus, if the transfer matrix associated to H_ω has the usual Ljapounov behavior as in Theorem 1, then we have exponential localization. The super-exponential localization as in Theorem 6 occurs simply because the Ljapounov exponent of the transfer matrix becomes infinite (Theorems 2 and 3). Hence, in order to obtain a more detailed estimates of super-exponential localization, it suffices to replace Theorem 3 by Theorem 4.

Recently, Lévy and Souillard ([21]) conjectured that the discrete Laplacian on the incipient percolation cluster should have eigenfunctions which decay like $\exp[-|x|^\alpha]$, $\alpha>1$, and called this "superlocalization". The physical basis of their conjecture is that the incipient percolation cluster is fractal within length scale smaller than correlation length. Although Lévy and Souillard consider a different situation from ours, our Theorems 6 and 7 may be considered as first rigorous examples of superlocalization.

The outline of the present paper is as follows. In the first half of § 1, we summarize the results of the author's previous paper [25], preparing at the same time the necessary notation. In the rest of § 1, we state the main results of this paper concerning respectively the Ljapounov behavior of the transfer matrices and the localization. Theorems 1 and 2 show that the Ljapounov exponent is finite or infinite according as the integral $\int_{|x|>1} \log|x|\nu(dx)$ is convergent or divergent, where $\nu(dx)$ is the Lévy measure of $\{Q_\omega(t)\}$. In the finite case, we can apply the well known theorem of Oseledec, to obtain a subspace of $\boldsymbol{R}^2$ which is exponentially stable under the action of the transfer matrix, whereas that general theorem does not seem to have a straightforward extension to the infinite case. Therefore by an explicit analysis, we first prove a theorem of Furstenberg-Kesten type (Theorem 2), and then using this, we obtain the corresponding Oseledec type theorem (Theorem 3). The same line of reasoning was recently exploited by Kotani and Ushiroya [18] in a different problem. One gets Theorem 4 by replacing the use of Theorem 2 in the proof of Theorem 3 by better estimates. In § 2, we prove theorems on localization assuming Theorems 1 to 4 for a while. §§ 3, 4, 5 and 6 are devoted to the proofs of Theorems 1, 2, 3 and 4 respectively.

As was already mentioned, we follow the idea of Carmona and Kotani in § 2. This idea gives us a quite transparent proof of localization in some cases, as one sees on comparing the original works of the Russian school ([10], [26]) with recent papers such as [3], [16], [17], and [6]. But in order that this is so, we must impose some kind of regularity condition on the probability distribution of the random potential in a finite box. This is the reason of the rather technical conditions of Theorems 5, 6, and 7. On the other hand, Carmona, Klein, and Martinelli ([5]), refining the method of Fröhlich et al. ([7], [8]), recently obtained a localization result for the one-dimensional difference Schrödinger operator whose potential is a sequence of i.i.d. random variables with *singular* distribution. An extension of their method to continuum systems might enable us to drop most of the technical conditions of our results, though we have not yet examined this possibility.

The mathematical study of Schrödinger operators with Lévy noise potentials was begun by Fukushima-Nakao [9] and Kotani [14]. They treated respectively the cases where $\{Q_\omega(t)\}$ is the standard Brownian motion or Lévy processes whose sample functions are of bounded variation. At that time, their main interest consisted in estimating the integrated density of states of H_ω, but later Kotani [15] investigated the Ljapounov exponent associated to his former model, and proved its absence of absolutely continuous spectrum. Our Theorems 1 to 4 and Proposition 3 in § 2 are extensions of Kotani's result [15].

§ 1. Preliminaries and the statement of the results. Examples.

1-1. Résumé of the previous paper [25].

Let $Q(t)$, $-\infty<t<+\infty$, be a real valued function which is right continuous and has left-hand limits. For this Q, let us define the (non-random) Schrödinger operator H_Q, which is formally expressed as

$$H_Q=-d^2/dt^2+Q'(t).$$

First let $\mathcal{C}_Q$ be the totality of complex valued functions $u(t)$ on $\boldsymbol{R}$ which satisfy the following two conditions:

(i) $u(t)$ is absolutely continuous and differentiable from the right. We denote the right derivative of $u(t)$ by $u^+(t)$;

(ii) there exists a $v\in L^1_{\mathrm{loc}}(\boldsymbol{R})$ such that the following equation holds:

$$u^+(t)-u^+(s)=Q(t)u(t)-Q(s)u(s)-\int_s^t\{Q(y)u^+(y)+v(y)\}\,dy.$$

It is clear that $v(\cdot)$ is uniquely determined from $u(\cdot)$ up to on a set of Lebesgue measure zero, and we will denote this v by $H_Q u$ for each $u\in\mathcal{C}_Q$. We then set

$$\mathcal{D}_Q=\{u\in\mathcal{C}_Q\cap L^2(\boldsymbol{R};dt);\ H_Q u\in L^2(\boldsymbol{R};dt)\}.$$

Under these definitions, we can give an exact meaning to the initial value problem $H_Q u=\lambda u$, $u(s)=\alpha$, $u^+(s)=\beta$ by the following pair of integral equations:

$$(1\text{-}1)\qquad \begin{aligned} u(t)&=\alpha+\int_s^t u^+(y)dy \\ u^+(t)&=\beta+Q(t)u(t)-Q(s)u(s)-\int_s^t \{Q(y)u^+(y)+\lambda u(y)\}dy\,. \end{aligned}$$

Now let $D(\boldsymbol{R}:\boldsymbol{R})$ be the totality of real functions which are right continuous and have left-hand limits. We endow $D(\boldsymbol{R}:\boldsymbol{R})$ with the Skorohod topology. Consider $\Omega=\{\omega\in D(\boldsymbol{R}:\boldsymbol{R});\ \omega(0)=0\}$ with relative topology and let $\mathcal{F}$ be the topological σ-field of Ω. If we set $Q_\omega(t)=\omega(t)$ (the coordinate map) then $\mathcal{F}$ coincides with the smallest σ-field with respect to which $\omega\to Q_\omega(t)$ is measurable for all $t\in\boldsymbol{R}$.

On the measurable space $(\Omega, \mathcal{F})$, we define a flow $\{T_t;\ t\in\boldsymbol{R}\}$ by

$$(T_t\omega)(\cdot)=\omega(\cdot+t)-\omega(t), \quad t\in\boldsymbol{R},\ \omega\in\Omega\,.$$

Under this setting, we can prove the following result:

PROPOSITION 1. ([25]) *Let P be a probability measure on $(\Omega, \mathcal{F})$ which is invariant under the flow $\{T_t\}$ and is ergodic. Then for P-a.a. $\omega\in\Omega$, H_ω with domain $\mathcal{D}_\omega$ is self-adjoint in $L^2(\boldsymbol{R}; dt)$. Moreover there exists a closed subset $\Sigma=\Sigma(P)$ of $\boldsymbol{R}$ such that the spectrum of H_ω is equal to Σ for a.a. ω.*

In the sequel, we assume that P is the measure of a temporally homogeneous Lévy process. Then the conclusions of Proposition 1 hold for this P.

As is well known, every temporally homogeneous Lévy process is decomposed into a superposition of Brownian motion and Poisson processes in the following way (Lévy's canonical form):

$$(1\text{-}2)\qquad \begin{aligned} Q_\omega(t)-Q_\omega(s)=&\ b(t-s)+v(B_\omega(t)-B_\omega(s)) \\ &+\lim_{n\to\infty}\int_{|x|>1/n}\{xN_\omega((s,t]\times dx)-(t-s)a(x)\nu(dx)\}\,, \end{aligned}$$

where $b\in\boldsymbol{R}$ and $v\geqq 0$ are constants, $\{B_\omega(t);\ -\infty<t<+\infty\}$ is a standard Brownian motion with $B_\omega(0)=0$, $N_\omega(dtdx)$ is a Poisson random measure on $\boldsymbol{R}\times(\boldsymbol{R}\setminus\{0\})$ with intensity measure $dt\nu(dx)$, and $a(x)=(x\wedge 1)\vee(-1)$. The measure $\nu(dx)$, which is called the Lévy measure of $\{Q_\omega(t)\}$, satisfies

$$\int_{\boldsymbol{R}\setminus\{0\}} a(x)^2\nu(dx)<\infty\,.$$

Note that sample paths of $\{Q_\omega(t)-vB_\omega(t)\}$ are locally of bounded variation if and only if $\int_{|x|\leq 1}|x|\nu(dx)<\infty$, and that $\{Q_\omega(t)\}$ has only a finite number of jumps in a finite interval if and only if $\nu(dx)$ is a finite measure. In the following, we assume that v and $\nu(dx)$ do not vanish simultaneously, i.e. that true randomness exists.

For the location of the spectrum $\Sigma=\Sigma(P)$, we have the following results:

PROPOSITION 2. ([25])

(i) *If* $v=0$, $\nu((-\infty, 0))=0$, *and if* $\int_0^1 x\nu(dx)<\infty$, *then* $\Sigma=[c, \infty)$ *with*

$$c=b-\int_0^\infty a(x)\nu(dx).$$

(ii) *In all the other cases, we have* $\Sigma=(-\infty, \infty)$.

1-2. Ljapounov behavior of the transfer matrices.

Let $\varphi(t)=\varphi_\lambda(t, \omega)$ and $\psi(t)=\psi_\lambda(t, \omega)$ be the solutions of $H_\omega u=\lambda u$ such that $\varphi(0)=\psi^+(0)=1$, $\varphi^+(0)=\psi(0)=0$, and consider the following random matrix:

$$U(t)=U_\lambda(t;\omega)=\begin{pmatrix}\varphi(t) & \psi(t)\\ \varphi^+(t) & \psi^+(t)\end{pmatrix}$$

This is called the transfer matrix, since any solution u of $H_\omega u=\lambda u$ is given by

$$\begin{pmatrix}u(t)\\ u^+(t)\end{pmatrix}=U_\lambda(t;\omega)\begin{pmatrix}u(0)\\ u^+(0)\end{pmatrix}. \tag{1-3}$$

Moreover, it is a multiplicative cocycle in the sense that

$$U_\lambda(t+s;\omega)=U_\lambda(t;T_s\omega)U_\lambda(s;\omega),\quad t, s\in\boldsymbol{R},\ \omega\in\Omega, \tag{1-4}$$

and we have $\det U_\lambda(t;\omega)=1$ from the constancy of the Wronskian. Concerning the asymptotic behavior of this transfer matrix, we have the following four results, which we will prove in later sections. Below, for a vector $x={}^t(x_1, x_2)$, $\|x\|$ denotes the Euclidean norm, and for a 2×2-matrix A, $\|A\|$ denotes its operator norm:

$$\|A\|=\sup\{\|Ax\|;\ \|x\|\leq1\},$$

which is equal to the maximum eigenvalue of $(A^*A)^{1/2}$.

THEOREM 1. *If* $\int_{|x|>1}\log|x|\nu(dx)<\infty$, *then for each fixed* $\lambda\in\boldsymbol{C}$, *there exists a strictly positive number* $\gamma(\lambda)$ *such that for P-a.a.* $\omega\in\Omega$,

$$\lim_{n\to\pm\infty}\frac{1}{|t|}\log\|U_\lambda(t;\omega)\|=\gamma(\lambda).$$

Moreover, for P-a.a. $\omega\in\Omega$, *there exists one-dimensional subspaces* $V_\lambda^+(\omega)$ *and* $V_\lambda^-(\omega)$

of $\boldsymbol{C}^2$ *such that if* $v \in V_\lambda^+(\omega)\backslash\{0\}$ [resp. $\in V_\lambda^-(\omega)\backslash\{0\}$], *then*

$$\lim_{t\to+\infty\,[\text{resp. } t\to-\infty]} \frac{1}{|t|}\log\|U_\lambda(t;\omega)v\| = -\gamma(\lambda), \tag{1-5}$$

and if $v \notin V_\lambda^+(\omega)$ [resp. $\notin V_\lambda^-(\omega)$], *then*

$$\lim_{t\to+\infty\,[\text{resp. } t\to-\infty]} \frac{1}{|t|}\log\|U_\lambda(t;\omega)v\| = +\gamma(\lambda). \tag{1-6}$$

The number $\gamma(\lambda)$ is called the Ljapounov exponent of $U_\lambda(t;\omega)$, and we will say that $U_\lambda(t;\omega)$ has Ljapounov behavior at $\pm\infty$ if $V_\lambda^\pm(\omega)$ exist and satisfy (1-5) and (1-6).

In the following three theorems, we fix a $\lambda > \inf \Sigma$, where Σ is the spectrum of H_ω. In case (i) of Proposition 2, this means $\lambda > c$. Otherwise λ can be any real number.

THEOREM 2. *Suppose* $\int_{|x|>1} \log|x|\nu(dx) = +\infty$ *and* $\lambda > \inf \Sigma$. *Then for each fixed* $v \in \boldsymbol{R}^2\backslash\{0\}$, *we have*

$$\lim_{t\to\pm\infty} \frac{1}{|t|}\log\|U_\lambda(t;\omega)v\| = +\infty$$

with probability one.

THEOREM 3. *In addition to the condition of Theorem* 2, *assume that* $\int_{|x|<1} |x|\nu(dx) < \infty$. *Then for P-a.a.* $\omega \in \Omega$, *there exist one-dimensional subspaces* $V_\lambda^+(\omega)$, $V_\lambda^-(\omega)$ *of* $\boldsymbol{R}^2$ *such that* (1-5) *and* (1-6) *hold with* $\gamma(\lambda)$ *replaced by* $+\infty$, *i.e. for P-a.a.* $\omega \in \Omega$, $U_\lambda(t;\omega)$ *has Ljapounov behavior at* $\pm\infty$ *with infinite Ljapounov exponent.*

If we impose some stronger condition on the tail of $\nu(dx)$, then it is possible to obtain more precise estimation of the asymptotic behavior of $U_\lambda(t;\omega)$. To this end, set

$$M(x) = \int_{|y|>e^x-1} \nu(dy), \quad x > 0,$$

and let $\varepsilon(t) = e^t - 1$, $\lambda(t) = \log(1+t)$ for $t \geqq 0$. The k-th iteration of $\varepsilon(\cdot)$ [resp. $\lambda(\cdot)$] is denoted by $\varepsilon_{(k)}(t)$ [resp. $\lambda_{(k)}(t)$]. Of course, $\varepsilon_{(0)}(t) = \lambda_{(0)}(t) = t$. Recall that a real valued function $L(t)$ on $[0, \infty)$ is said to be slowly varying if for all $c > 0$,

$$\lim_{t\to+\infty} \frac{L(ct)}{L(t)} = 1.$$

(for details, see Seneta [30]).

THEOREM 4. *In addition to the condition of Theorems* 2 *and* 3, *assume that there exist an integer* $k \geqq 0$, *a real number* $\beta \geqq 0$, *and a slowly varying function* $L(t)$ *such that*

$$M(\varepsilon_{(k)}(t)) = t^{-\beta} L(t),$$

where in case $k=0$, *we further assume that* $0 \leqq \beta \leqq 1$. (*Otherwise* $\int_{|x|>1} \log|x|\nu(dx)$ *would be finite.*)

Then for each $\lambda > \inf \Sigma$, *and for P*-a. a. ω, $V_\lambda^\pm(\omega)$ *exist and satisfy the following:*
if $v \in V_\lambda^+(\omega) \setminus \{0\}$ [resp. $\in V_\lambda^-(\omega) \setminus \{0\}$], *then*

$$\lim_{t \to +\infty\,[\text{resp. } t \to -\infty]} \frac{1}{|t|^\alpha} \lambda_{(k)}(\|U_\lambda(t;\omega)v\|^{-1}) = 0, \qquad \textit{for } \alpha > \beta^{-1},$$

and

$$\lim_{t \to +\infty\,[\text{resp. } t \to -\infty]} \frac{1}{|t|^\alpha} \lambda_{(k)}(\|U_\lambda(t;\omega)v\|^{-1}) = +\infty, \qquad \textit{for } \alpha < \beta^{-1};$$

and if $v \notin V_\lambda^+(\omega)$ [resp. $\notin V_\lambda^-(\omega)$], *then*

$$\lim_{t \to +\infty\,[\text{resp. } t \to -\infty]} \frac{1}{|t|^\alpha} \lambda_{(k)}(\|U_\lambda(t;\omega)v\|) = 0, \qquad \textit{for } \alpha > \beta^{-1},$$

and

$$\lim_{t \to +\infty\,[\text{resp. } t \to -\infty]} \frac{1}{|t|^\alpha} \lambda_{(k)}(\|U_\lambda(t;\omega)v\|) = \infty, \qquad \textit{for } \alpha < \beta^{-1}.$$

1-3. Exponential and super-exponential localization. Examples.

Using the four theorems in the previous subsection, we can prove the following results on localization.

THEOREM 5. (exponential localization) *Suppose* $\int_{|x|>1} \log|x|\nu(dx) < \infty$, *and assume that one of the following three conditions holds:*

a) $v \neq 0$, *i.e.* $\{Q_\omega(t)\}$ *has a non-trivial Gaussian part, and* $\nu(\boldsymbol{R}) < \infty$

b) $v = \nu((-\infty, 0)) = 0$, *and* $\nu((0, \infty)) < \infty$;

c) $\nu(dx)$ *has a non-trivial component which is absolutely continuous with respect to Lebesgue measure.*

Then for P-a. a. $\omega \in \Omega$, H_ω *has only pure point spectrum, and if* $u(t)$ *is an eigenfunction of* H_ω *with eigenvalue* λ, *then it satisfies*

$$\lim_{t \to \pm\infty} \frac{1}{|t|} \log[|u(t)|^2 + |u^+(t)|^2]^{1/2} = -\gamma(\lambda) < 0.$$

THEOREM 6. (super-exponential localization) *Suppose* $\int_{|x|>1} \log|x|\nu(dx) = +\infty$,

$v=\nu((-\infty, 0))=0$, *and* $\int_0^1 x\nu(dx)<\infty$, *and assume further that either* $\nu((0, \infty))<\infty$ *or* $\nu(dx)$ *has a non-trivial component absolutely continuous with respect to Lebesgue measure. Then for P-a.a.* $\omega\in\Omega$, H_ω *has only pure point spectrum, and each of its eifenfunctions* $u(t)$ *satisfies*

$$\lim_{t\to\pm\infty}\frac{1}{|t|}\log[|u(t)|^2+|u^+(t)|^2]^{1/2}=-\infty .$$

THEOREM 7. (super-exponential localization) *In addition to the conditions of Theorem 6, assume, as in Theorem 4, that there exist an integer* $k\geqq 0$, *a real number* $\beta\geqq 0$, *and a slowly varying function* $L(t)$ *such that*

$$M(\varepsilon_{(k)}(t))=t^{-\beta}L(t),$$

where $0\leqq\beta\leqq 1$ *if* $k=0$.

Then for P-a.a. $\omega\in\Omega$, H_ω *has only pure point spectrum, and each of its eigenfunctions* $u(t)$ *satisfies the following:*

$$\lim_{t\to\pm\infty}\frac{1}{|t|^\alpha}\lambda_{(k+1)}([|u(t)|^2+|u^+(t)|^2]^{-1/2})=0, \qquad \text{for } \alpha>\beta^{-1},$$

$$\lim_{t\to\pm\infty}\frac{1}{|t|^\alpha}\lambda_{(k+1)}([|u(t)|^2+|u^+(t)|^2]^{-1/2})=+\infty, \quad \text{for } \alpha<\beta^{-1}.$$

Finally let us discuss some examples. First of all, we remark that most of the well known Lévy processes satisfy the conditions of Theorem 5. For example, the standard Brownian motion, Poisson process, and stable processes satisfy the conditions a), b), and c) respectively, and they all satisfy the condition $\int_{|x|>1}\log|x|\nu(dx)<\infty$. The case of the Poisson process is a little bit delicate: a Poisson process $\{Q_\omega(t)\}$ satisfies the condition b) but $\{-Q_\omega(t)\}$ does not. Proposition 2 tells us that the spectra of

$$H_\omega^+=-d^2/dt^2+Q'_\omega(t)$$

and

$$H_\omega^-=-d^2/dt^2-Q'_\omega(t)$$

are $[0, \infty)$ and $(-\infty, \infty)$ respectively. In the latter case, we know the absence of the absolutely continuous spectrum of H_ω^- from Proposition 3, and we can show that H_ω^- has pure point spectrum in $[0, \infty)$, in the same manner as in Theorem 5. Unfortunately, we were not able to determine the spectral nature of H_ω^- in $(-\infty, 0)$. Such a problem does not arise in the case of H_ω^+, because it does not have any spectrum in $(-\infty, 0)$ at all.

Next we give some examples for super-exponential localization. Let $\xi(\omega) =\{\cdots <x_{-1}(\omega)<x_0(\omega)\leqq 0<x_1(\omega)<x_2(\omega)< \cdots\}$ be a Poisson point process on $\boldsymbol{R}$, and let $v_n(\omega)\geqq 0$, $n\in\boldsymbol{Z}$, be a sequence of i.i.d. random variables with distribution $\mu(dv)$ which satisfy: (1) $E[\log(1+v_n)]=+\infty$; and (2) $\{v_n(\omega)\}$ is independent of $\xi(\omega)$. The random Schrödinger operator

$$-d^2/dt^2+\sum_{n=-\infty}^{\infty} v_n(\omega)\delta(t-x_n(\omega))$$

can be realized as

$$H_\omega=-d^2/dt^2+Q'_\omega(t),$$

through the Lévy process

$$Q_\omega(t)-Q_\omega(s)=\int_0^\infty xN((s, t\}\times dx),$$

where $N_\omega(\cdot)$ is the Poisson random measure on $\boldsymbol{R}\times(0, \infty)$ such that $E[N_\omega(dtdx)] =dt\mu(dx)$. Then we have super-exponential localization by Theorem 6. If we consider the special case in which

$$\mu(dx)=\{x(\log x)^\gamma\}^{-1}dx, \quad \text{for } x \text{ large},$$

with $1<\gamma<2$, then H_ω satisfies the condition of Theorem 7 with $k=0$, $\beta=\gamma-1$, and hence the eigenfunctions of H_ω decay like $\exp[-|t|^{1/(\gamma-1)}]$. In the same way, if for some $\gamma>1$,

$$\mu(dx)=\{x(\log x)(\log\log x)\cdots(\log\log\cdots\log x)^\gamma]^{-1},$$

for x large, where $\log\log\cdots\log x$ is the $(n+1)$-th iteration of "log", then the condition of Theorem 7 holds with $k=n$ and $\beta=\gamma-1$, and we have eigenfunctions decaying like $\exp[-\exp[\exp[\cdots\exp[|t|^{1/(\gamma-1)}]\cdots]]]$, where "exp" is iterated n-times.

§ 2. Proof of the theorems on localization.

2-1. Some notions and facts from the spectral theory of H_Q.

In this subsection, we will give a brief summary of the spectral theory of singular Sturm-Liouville operators, which is often referred to as the theory of Weyl, Stone, Titchmarsh, and Kodaira (W-S-T-K theory). The subject of this theory is the investigation of the differential operator

$$L=-d^2/dt^2+q(t)$$

on a finite or infinite interval, and it is usually assumed that $q(t)$ satisfies some mild regularity condition such as piecewise continuity, local integrability or the like. However, if one examines the detail of the theory (see e.g. [13], [20], or references therein), it is easily understood that such kind of an assumption does not play any essential role, and that the W-S-T-K theory extends to our H_Q defined in § 1-1 as well. Indeed, the initial value problem $H_Q u=\lambda u$, $u(0)=\alpha$, $u^+(0)=\beta$, which is defined by (1-1), is uniquely solvable by successive approximation, its solution $u(t, \lambda)$ and its derivative $u^+(t, \lambda)$ are entire functions of λ for each fixed t, and we have the Green's formula: for each $u, v\in C_Q$,

$$\int_a^b \{(H_Q u)v-u(H_Q v)\}dt=[u, v](b)-[u, v](a),$$

where $[u, v](t)=u(t)v^+(t)-u^+(t)v(t)$ is the Wronskian. Given these basic facts, W-S-T-K theory can be reconstructed word for word. Hence, we will quote its results without any proof.

To begin with, H_Q is said to be in the limit point case at $+\infty$ $[-\infty]$ if for some $\lambda\in \boldsymbol{C}\backslash \boldsymbol{R}$ (and hence for all $\lambda\in \boldsymbol{C}$), $H_Q u=\lambda u$ has a solution which is not square integrable near $+\infty$ $[-\infty]$. H_Q with domain $\mathcal{D}_Q$ is self-adjoint if and only if H_Q is in the limit point case both at $\pm\infty$. We restrict ourselves to this case.

Let $\{E_Q(\lambda)\}_{\lambda\in \boldsymbol{R}}$ be the resolution of the identity associated with the self-adjoint operator H_Q. It is known that for each bounded interval $\Delta=(\lambda, \mu]$, $E_Q(\Delta)=E_Q(\mu)-E_Q(\lambda)$ has a continuous kernel $E_Q(\Delta; x, y)$, and that there exists a measure $\{\sigma_{ij}(Q; d\xi)\}_{i,j=1}^2$ taking its values in the space of non-negative, symmetric 2×2-matrices such that $E_Q(\Delta; x, y)$ is represented as

$$E_Q(\Delta; x, y)=\int_{(\lambda,\mu]}\sum_{i,j=1}^2 \varphi_i(x; Q, \xi)\varphi_j(y; Q, \xi)\sigma_{ij}(d\xi; Q), \tag{2-1}$$

where φ_1, φ_2 are the solutions of $H_Q u=\xi u$ with $\varphi_1(0)=\varphi_2^+(0)=1$, $\varphi_1^+(0)=\varphi_2(0)=0$. (i.e., $\varphi_1=\varphi$, $\varphi_2=\psi$ in the notation of § 1-2.) We set $\sigma(d\xi; Q)=\sigma_{11}(d\xi; Q)+\sigma_{22}(d\xi; Q)$, and call this the spectral measure of H_Q. Further let $\tau_{ij}(\xi; Q)=\sigma_{ij}(d\xi; Q)/\sigma(d\xi; Q)$ be the density of σ_{ij} with respect to σ.

These measure, σ_{ij}'s, are obtained by taking the limit of the eigenvalue problem on a finite interval as the interval expands to the whole line. More precisely, let $I=[-a, b]$, $a, b>0$, and consider the eigenvalue problem;

$$(H_Q u)(t)=\lambda u(t), \quad t\in I, \tag{2-2}$$

$$u(-a)\cos\alpha-u^+(-a)\sin\alpha=0, \tag{2-3}$$

$$u(b)\cos\beta+u^+(b)\sin\beta=0, \tag{2-4}$$

where α and β are arbitrarily fixed real numbers. Let

$$\lambda_1(Q\,;I)<\lambda_2(Q\,;I)<\cdots<\lambda_n(Q\,;I)<\cdots$$

be its eigenvalues and $v_n(t\,;Q,I)$, $n=1,2,\cdots$ its normalized eigenfunctions. We may assume that v_n's are real valued. Set

$$\sigma_{11}(A\,;Q,I)=\sum_{\lambda_n\in A}v_n(0)^2, \tag{2-5}$$

$$\sigma_{12}(A\,;Q,I)=\sigma_{21}(A\,;Q,I)=\sum_{\lambda_n\in A}v_n(0)v_n^+(0), \tag{2-6}$$

and

$$\sigma_{22}(A\,;Q,I)=\sum_{\lambda_n\in A}v_n^+(0)^2. \tag{2-7}$$

Then as we let $a\to+\infty$, $b\to+\infty$ indenpendently, $\sigma_{ij}(d\xi\,;Q,I)$ converges to $\sigma_{ij}(d\xi\,;Q)$ vaguely.

By the way, in our limit point case,

$$h_+(\lambda\,;Q)=\lim_{t\to+\infty}-\frac{\varphi_\lambda(t\,;Q)}{\psi_\lambda(t\,;Q)},\quad\text{and}\quad h_-(\lambda\,;Q)=\lim_{t\to-\infty}\frac{\varphi_\lambda(t\,;Q)}{\psi_\lambda(t\,;Q)}$$

exist for each $\lambda\in\boldsymbol{C}_+=\{\operatorname{Im}z>0\}$. They are holomorphic functions of $\lambda\in\boldsymbol{C}_+$, with values in $\boldsymbol{C}_+$. If we set for $\lambda\in\boldsymbol{C}_+$,

$$w_\pm(t\,;\lambda,Q)=\varphi_\lambda(t\,;Q)\pm h_\pm(\lambda\,;Q)\psi_\lambda(t\,;Q),$$

then w_+ [resp. w_-] is the unique (up to multiplicative constants) solution of $H_Q u=\lambda u$ which is square integrable near $+\infty$ [resp. $-\infty$], and

$$g_\lambda(t,s\,;Q)=-\{h_+(\lambda\,;Q)+h_-(\lambda\,;Q)\}^{-1}w_+(t\vee s\,;\lambda,Q)w_-(t\wedge s\,;\lambda,Q)$$

is the Green function of H_Q, i.e. the integral kernel of $(H_Q-\lambda)^{-1}$. These are related to the above mentioned σ_{ij}'s in the following manner: Let us define

$$H_{11}(\lambda\,;Q)=-\{h_+(\lambda\,;Q)+h_-(\lambda\,;Q)\}^{-1}=g_\lambda(0,0\,;Q),$$

$$H_{12}(\lambda\,;Q)=H_{21}(\lambda\,;Q)=h_+(\lambda\,;Q)\{h_+(\lambda\,;Q)+h_-(\lambda\,;Q)\}^{-1},$$

and

$$H_{22}(\lambda\,;Q)=h_+(\lambda\,;Q)h_-(\lambda\,;Q)\{h_+(\lambda\,;Q)+h_-(\lambda\,;Q)\}^{-1},$$

then for each finite interval $\Delta=(\lambda,\mu]$, we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_\lambda^\mu \operatorname{Im}H_{ij}(\xi+i\varepsilon\,;Q)d\xi=\sigma_{ij}((\lambda,\mu)\,;Q)+\frac{1}{2}\{\sigma_{ij}(\{\lambda\}\,;Q)+\sigma_{ij}(\{\mu\}\,;Q)\}. \tag{2-8}$$

Finally let us see what changes the objects defined above undergo by the translation $Q\to T_tQ$. First of all, we have

$$h_\pm(\lambda\,;T_tQ)=\pm\frac{w_\pm^+(t\,;\lambda,Q)}{w_\pm(t\,;\lambda,Q)}, \tag{2-9}$$

whence we get

(2-10) $$H_{11}(\lambda\,;T_tQ)=g(0,\,0\,;T_tQ)=g_\lambda(t,\,t\,;Q),$$

and

(2-11) $$H_{22}(\lambda\,;T_tQ)=g_\lambda(0,\,0\,;Q)w_+^\dagger(t\,;\lambda,\,Q)w_-^\dagger(t\,;\lambda,\,Q).$$

Here we have set $w_\pm^\dagger(t)=(\partial^+/\partial t)w_\pm(t\,;\lambda,\,Q)$.

On the other hand, from $\sigma_{ij}(d\xi\,;Q)=\lim_{I\uparrow R}\sigma_{ij}(d\xi\,;Q,\,I)$, it is easily seen that

(2-12) $$\sigma_{11}(d\xi\,;T_tQ)=\sum_{i,j=1}^{2}\varphi_i(t\,;\xi,\,Q)\varphi_j(t\,;\xi,\,Q)\sigma_{ij}(d\xi\,;Q),$$

and

(2-13) $$\sigma_{22}(d\xi\,;T_tQ)=\sum_{i,j=1}^{2}\varphi_i^\dagger(t\,;\xi,\,Q)\varphi_j^\dagger(t\,;\xi,\,Q)\sigma_{ij}(d\xi\,;Q).$$

2-2. A priori estimates of generalized eigenfunctions. Absence of absolutely continuous spectrum.

LEMMA 1. *Suppose* $\int_{|x|>1}\log|x|\nu(dx)<\infty$. *Then for P-a.a.* ω, *and* $\sigma(\cdot\,;\omega)$-*a.a.* $\lambda\in\boldsymbol{R}$, *there exists a solution* $u(\cdot)$ *of* $H_\omega u=\lambda u$ *such that for all* $\varepsilon>0$,

$$\int_{-\infty}^{\infty}e^{-\varepsilon|t|}(|u(t)|^2+|u^+(t)|^2)^{1/2}dt<\infty.$$

PROOF. In general, if $h(\lambda)$ is holomorphic on $\boldsymbol{C}_+$ and has positive imaginary part, then there exists a measure $\sigma(d\xi)$ and a constant $\beta\geqq 0$ such that

$$\operatorname{Im}h(\lambda)=\beta\operatorname{Im}\lambda+\int_{-\infty}^{\infty}\frac{\operatorname{Im}\lambda}{|\xi-\lambda|^2}\sigma(d\xi).$$

This σ is unique and is recovered from $h(\lambda)$ by

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_\lambda^\mu\operatorname{Im}h(\xi+i\varepsilon)d\xi=\sigma((\lambda,\,\mu))+\frac{1}{2}\{\sigma(\{\lambda\})+\sigma(\{\mu\})\}.$$

(See [12].)

From this fact and the results of the previous subsection, we get the following two estimates:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{1+\xi^2}\Big(\sum_{j,k=1}^{2}\tau_{jk}(\xi\,;\omega)\varphi_j(t\,;\xi,\,\omega)\varphi_k(t\,;\xi,\,\omega)\Big)\sigma(d\xi\,;\omega)\\
&=\int_{-\infty}^{\infty}\frac{1}{1+\xi^2}\sigma_{11}(d\xi\,;T_t\omega)\\
&\leqq\operatorname{Im}g_i(0,\,0\,;T_t\omega)\\
&=\operatorname{Im}g_i(t,\,t\,;\omega)\\
&\leqq|g_i(0,\,0\,;\omega)|\,|w_+(t\,;i,\,\omega)|\,|w_-(t\,;i,\,\omega)|,
\end{aligned}$$

and

$$\int_{-\infty}^{\infty}\frac{1}{1+\xi^2}\Big(\sum_{j,k=1}^{2}\tau_{jk}(\xi;\omega)\varphi_j^{+}(t;\xi,\omega)\varphi_k^{+}(t;\xi,\omega)\Big)\sigma(d\xi;\omega)$$
$$=\int_{-\infty}^{\infty}\frac{1}{1+\xi^2}\sigma_{22}(d\xi;T_t\omega)$$
$$\leqq \operatorname{Im} H_{22}(i;T_t\omega)$$
$$\leqq |g_i(0,0;\omega)|\,|w_+^{+}(t;i,\omega)|\,|w_-^{+}(t;i,\omega)|.$$

At this stage, we apply Theorem 1 for $\lambda=i$. Then for P-a. a. ω, there exists a solution $u(t)$ of $H_\omega u=iu$ such that $|u(t)|^2+|u^+(t)|^2$ decays expontentially fast. This solution is square integrable near $+\infty$ and hence coincides with $w_+(t;i,\omega)$. The same thing can be said about $w_-(t;i,\omega)$. Therefore we have

$$\overline{\lim_{t\to\pm\infty}}\frac{1}{|t|}\log\Big(|w_+(t;i,\omega)|\,|w_-(t;i,\omega)|\Big)\leqq\gamma(i)-\gamma(i)=0,$$

and

$$\overline{\lim_{t\to\pm\infty}}\frac{1}{|t|}\log\Big(|w_+^{+}(t;i,\omega)|\,|w_-^{+}(t;i,\omega)|\Big)\leqq\gamma(i)-\gamma(i)=0,$$

for P-a. a. ω. These, combined with the above two estimates, show that for P-a. a. ω,

$$\int_{-\infty}^{\infty}e^{-\varepsilon|t|}dt\int_{-\infty}^{\infty}\frac{1}{1+\xi^2}\Big[\sum_{j,k=1}^{2}\tau_{jk}(\xi;\omega)\{\varphi_j(t;\xi,\omega)\varphi_k(t;\xi,\omega)$$
$$+\varphi_j^{+}(t;\xi,\omega)\varphi_k^{+}(t;\xi,\omega)\}\Big]\sigma(d\xi;\omega)<\infty,$$

for any $\varepsilon>0$.

Now if we set

$$v_{\xi,t}^{\omega}(x)=\sum_{j,k=1}^{2}\tau_{jk}(\xi;\omega)\varphi_j(t;\xi,\omega)\varphi_k(x;\xi,\omega),$$

then $v_{\xi,t}^{\omega}(\cdot)$ is a solution of $H_\omega u=\lambda u$ for each $t\in\boldsymbol{R}$. From the positive semi-definiteness of the matrix $\{\tau_{jk}(\xi;\omega)\}$ and Schwarz' inequality, it is easily seen that

$$\int_{-\infty}^{\infty}e^{-\varepsilon|t|}dt\int_{-\infty}^{\infty}e^{-\varepsilon|x|}dx\int_{-\infty}^{\infty}\frac{1}{1+\xi^2}\Big(|v_{\xi,t}^{\omega}(x)|+\Big|\frac{\partial^+}{\partial x}v_{\xi,t}^{\omega}(x)\Big|\Big)\sigma(d\xi;\omega)<\infty,$$

for all $\varepsilon>0$. Therefore for $\sigma(\cdot;\omega)$-a. a. ξ, we can choose a $t\in\boldsymbol{R}$ so that

$$\int_{-\infty}^{\infty}e^{-\varepsilon|x|}\Big(|v_{t,\xi}^{\omega}(x)|^2+\Big|\frac{\partial^+}{\partial x}v_{t,\xi}^{\omega}(x)\Big|^2\Big)^{1/2}dx<\infty,$$

for all $\varepsilon>0$. This $v_{t,\xi}^{\omega}(\cdot)$ satisfies the desired condition.

LEMMA 2. *If $Q(t)\in\Omega$ is such that $Q(t)$-ct is non-decreasing for some $c\in\boldsymbol{R}$, then H_Q is self-adjoint and for $\sigma(\cdot;Q)$-a. a. λ, there exists a solution of $H_Q u=\lambda u$*

such that

$$\int_{-\infty}^{\infty}\frac{1}{1+|t|^{\alpha}}\left(|u(t)|^2+|u^+(t)|^2\right)dt<\infty,$$

for all $\alpha>1$.

PROOF. We may suppose $c=0$. It is easy to see that for $\lambda\leqq 0$, $\varphi_\lambda(t;Q)$ is not square integrable near $\pm\infty$, hence H_Q is in the limit point case both at $\pm\infty$, whence follows the self-adjointness. It is also easy to see that the spectrum of H_Q is contained in $[0, \infty)$.

In order to prove the second half of the lemma, let us fix a $\lambda<0$. A simple comparison argument with $H_0=-d^2/dt^2$ show that $g_\lambda(t, t; Q)$ is bounded in t. On the other hand, if $dQ(t)\geqq 0$, and $\lambda<0$, then $g_\lambda(t, s; Q)$ itself has an eigenfunction expansion (see [24]):

$$g_\lambda(t, s; Q)=\int_{0-}^{\infty}\frac{1}{\xi-\lambda}\left(\sum_{j,k=1}^{2}\tau_{jk}(\xi; Q)\varphi_j(t; \xi, Q)\varphi_k(s; \xi, Q)\right)\sigma(d\xi; Q).$$

Therefore for any $\alpha>1$,

$$\int_{0-}^{\infty}\frac{1}{\xi-\lambda}\sigma(d\xi; Q)\int_{-\infty}^{\infty}\frac{1}{1+|t|^{\alpha}}v_{\xi,t}^{Q}(t)dt=\int_{-\infty}^{\infty}\frac{1}{1+|t|^{\alpha}}g_\lambda(t, t; Q)<\infty.$$

Then by using the positive semi-definiteness of $\{\tau_{ij}(\xi; Q)\}$ as before, we can show that for $\sigma(\cdot; Q)$-a.a. ξ, there is $x\in\boldsymbol{R}$ such that

$$\int_{-\infty}^{\infty}\frac{1}{1+|t|^{\alpha}}|v_{\xi,x}^{Q}(t)|^2dt<\infty,\quad \text{for all } \alpha>1.$$

In the same manner as in [15] (Lemma 2.1.), one obtains from this,

$$\int_{-\infty}^{\infty}\frac{1}{1+|t|^{\alpha}}\left|\frac{\partial^+}{\partial t}v_{\xi,x}^{Q}(t)\right|^2dt<\infty$$

as well. $u(\cdot)=v_{\xi,x}^{Q}(\cdot)$ satisfies the desired condition.

If we combine Theorem 1 [resp. 2] with Lemma 1 [resp. 2], we get the following result by virtue of a well known argument (see e.g. Pastur [29]).

PROPOSITION 3. *Assume that one of the following two conditions holds for our Lévy process* $\{Q_\omega(t)\}$:

(i) $\int_{|x|>1}\log|x|\nu(dx)<\infty$;

(ii) $\int_{|x|>1}\log|x|\nu(dx)=\infty$, *but* $v=\nu((-\infty, 0))=0$, *and* $\int_0^1 x\nu(dx)<\infty$.

Then for P-a.a. ω, H_ω *has no absolutely continuous spectrum.*

2-3. Proof of Theorems 5, 6, and 7.

In this subsection, we prove our main theorems on localization, assuming Theorems 1, 2, 3, and 4. In fact, it suffices to prove Theorem 5 only, because in order to prove Theorem 6, we have only to replace the use of Theorem 1 and Lemma 1 by that of Theorem 3 and Lemma 2 respectively. We get Theorem 7 if we further replace Theorem 3 by Theorem 4. No alternation is necessary in the other parts of the proof.

Before proceeding to the proof of Theorem 5, let us quote some notions and facts from the deterministic part of the theory of Carmona and Kotani. For details, we refer the reader to [4], [16], and [17].

Suppose that $H_Q=-d^2/dt^2+Q'(t)$ is in the limit point case both at $\pm\infty$. If we define for each $\theta\in[0, \pi)$,

$$\mathscr{D}_Q^\theta=\{u|_{(-\infty,0]}\,;\ u\in\mathscr{D}_Q,\ \text{and}\ u(0)\cos\theta-u^+(0)\sin\theta=0\},$$

and $H_Q^\theta v=(H_Q u)|_{(-\infty,0]}$ if $v=u|_{(-\infty,0]}\in\mathscr{D}_Q^\theta$, then the operator H_Q^θ with domain $\mathscr{D}_Q^\theta$ is self-adjoint in $L^2((-\infty, 0]\,;dt)$. If $\{E_Q^\theta(\lambda)\}_{\lambda\in\boldsymbol{R}}$ is its corresponding resolution of identity, then as in § 2-1, for each finite interval $\Delta=(\lambda, \mu]$, $E_Q^\theta(\Delta)=E_Q^\theta(\mu)-E_Q^\theta(\lambda)$ has a continuous kernel $E_Q^\theta(\Delta\,;x, y)$, $x, y\leqq 0$, which has the following representation:

$$E_Q^\theta(\Delta\,;x, y)=\int_\Delta\varphi_\xi^\theta(x\,;Q)\varphi_\xi^\theta(y\,;Q)\sigma^\theta(d\xi\,;Q),$$

where $\varphi_\xi^\theta(x\,;Q)=\sin\theta\varphi_\xi(x\,;Q)+\cos\theta\psi_\xi(x\,;Q)$. We shall call the measure $\sigma^\theta(d\xi\,;Q)$ the spectral measure of H_Q^θ.

The following two facts are essential ([17], Proposition 2.5.):

(2-14) $\displaystyle\int_0^\pi d\theta\int_{-\infty}^\infty f(\lambda)\sigma^\theta(d\lambda\,;Q)=\int_{-\infty}^\infty f(\lambda)d\lambda$, for any $f\in C_0(\boldsymbol{R})$;

(2-15) if $\sigma(d\lambda\,;Q)$ is the spectral measure of H_Q (see § 2-1),

then

$$\int_{-\infty}^\infty f(\lambda)\sigma(d\lambda\,;Q)=\frac{1}{\pi}\lim_{t\to+\infty}\int_0^\pi d\theta\int_{-\infty}^\infty\|U_\lambda(t\,;Q)\hat\theta\|^{-2}f(\lambda)\sigma^\theta(d\lambda\,;Q),$$

for each $f\in C_0(\boldsymbol{R})$, where $U_\lambda(t\,;Q)$ was introduced in § 1-2, and

$$\hat\theta=\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}.$$

We also use the following change of variable formula ([17], Lemma 2.4.): for $U\in Sl(2\,;\boldsymbol{R})$ and $\theta\in[0, \pi)$, define a new angle $\varphi\in[0, \pi)$ by

$$U\hat\theta=\pm\|U\hat\theta\|\hat\varphi,$$

and denote this by $U\cdot\theta=\varphi$. Then

$$\int_0^{\pi} f(U\cdot\theta)d\theta=\int_0^{\pi} f(\theta)\|U^{-1}\hat{\theta}\|^{-2}d\theta. \tag{2-16}$$

Now let us turn to the proof of Theorem 5. We divide our argument into three cases each of which corresponds to conditions a), b), and c) respectively.

CASE a). Suppose $v\neq 0$ and $\nu(\boldsymbol{R})<\infty$. Then Lévy's canonical form of $\{Q_\omega(t)\}$ takes the following form:

$$Q_\omega(t)-Q_\omega(s)=b(t-s)+v(B_\omega(t)-B_\omega(s))+(\tilde{Q}_\omega(t)-\tilde{Q}_\omega(s)),$$

where

$$\tilde{Q}_\omega(t)-\tilde{Q}_\omega(s)=\int_{-\infty}^{\infty} xN_\omega((s,t]\times dx)$$

is a "step process", i.e. has only finitely many jumps in a finite interval, and is constant between jumps.

If we set

$$\Omega_\varepsilon=\{\omega\in\Omega\,;\ \sup_{0\leq t\leq 1}|bt+vB_\omega(t)|\leqq\varepsilon,\ \text{and}\ \tilde{Q}_\omega(t)=0\ \text{for}\ t\in[0,1]\},$$

then it is clear that $P(\Omega_\varepsilon)>0$ for any $\varepsilon>0$. For each finite open interval I, we choose an $\varepsilon>0$ so small that

$$C_0=\inf\{\|U_\lambda(1\,;\omega)\hat{\theta}\|\,;\ 0\leqq\theta<\pi,\ \lambda\in I,\ \omega\in\Omega_\varepsilon\}>0.$$

Further let us define

$$\mathcal{G}=\sigma[Q_\omega(t)-Q_\omega(s)\,;\ t,\ s\leqq 0\ \text{or}\ t,\ s\geqq 1].$$

$\mathcal{G}$ is the sub σ-field of $\mathcal{F}$ generated by the random potential outside the interval $[0, 1]$. We denote respectively by $P_\omega^{\mathcal{G}}(d\omega')$ and $E_\omega^{\mathcal{G}}[\cdot]$ the conditional probability and the conditional expectation given $\mathcal{G}$.

At this stage, we claim that for P-a.a. $\omega\in\Omega$, the conditional expectation taken on the set Ω_ε of the random measure $\sigma(d\lambda\,;\omega)$ is absolutely continuous on I, i.e.,

$$E_{\mathcal{G}}^{\omega}[1_{\Omega_\varepsilon}(\omega')\sigma(d\lambda\,;\omega')]\ll d\lambda.$$

Indeed, if we set

$$\theta_\lambda(t\,;\omega,\theta)=U_\lambda(t\,;\omega)\cdot\theta,$$

then for each non-negative $f\in C_0$, we have from (2-15) and (1-4),

$$\int_I f(\lambda)\sigma(d\lambda;\omega)$$

$$=\frac{1}{\pi}\lim_{t\to+\infty}\int_0^\pi d\theta\int_I f(\lambda)\|U_\lambda(t;T_1\omega)\hat{\theta}_\lambda(1;\omega,\theta)\|^{-2}\|U_\lambda(1;\omega)\hat{\theta}\|^{-2}\sigma^\theta(d\lambda;\omega).$$

Therefore by Fatou's lemma,

$$E_\omega^{\mathcal{G}}\Big[1_{\Omega_\varepsilon}(\omega')\int_I f(\lambda)\sigma(d\lambda;\omega')\Big]$$

$$\leqq\frac{1}{\pi}\varliminf_{t\to+\infty}E_\omega^{\mathcal{G}}\Big[1_{\Omega_\varepsilon}(\omega')\int_0^\pi d\theta\int_I f(\lambda)\|U_\lambda(t;T_1\omega')\hat{\theta}_\lambda(1;\omega',\theta)\|^{-2}$$

$$\times\|U_\lambda(1;\omega')\hat{\theta}\|^{-2}\sigma^\theta(d\lambda;\omega')\Big]$$

$$\leqq\frac{1}{\pi C_0^2}\varliminf_{t\to+\infty}\int_0^\pi d\theta\int_I f(\lambda)E_\omega^{\mathcal{G}}[1_{\Omega_\varepsilon}(\omega')\|U_\lambda(t;T_1\omega')\hat{\theta}_\lambda(1;\omega',\theta)\|^{-2}]\sigma^\theta(d\lambda;\omega).$$

Note that $\sigma^\theta(d\lambda;\omega')$ and $U_\lambda(t;T_1\omega')$ are $\mathcal{G}$-measurable, and that $1_{\Omega_\varepsilon}(\omega')$ and $\theta_\lambda(1;\omega';\theta)$ are independent of $\mathcal{G}$. Hence our claim will follow from (2-14) as soon as we have shown

$$\text{(2-17)}\qquad \sup_{t\geq 0,\,\lambda\in I,\,0\leq\theta<\pi}E_\omega^{\mathcal{G}}[1_{\Omega_\varepsilon}(\omega')\|U_\lambda(t;T_1\omega')\hat{\theta}_\lambda(1;\omega',\theta)\|^{-2}]<\infty.$$

For this purpose, let

$$\tilde{\Omega}=\{\omega\in\Omega;\ \tilde{Q}_\omega(t)=0,\ \text{for } t\in[0,1]\}.$$

Then $\Omega_\varepsilon\subset\tilde{\Omega}$, and it is easy to see that the process

$$\{\theta_\lambda(t;\omega',\theta);\ 0\leqq t\leqq 1\}$$

is a nice diffusion process on the circle $\boldsymbol{R}/\pi\boldsymbol{Z}$ under the measure $P_\omega^{\mathcal{G}}(d\omega'|\tilde{\Omega})$ (see [9]), and by a standard method, one shows that it has a transition density $p_\lambda(t;x,y)$ which is jointly continuous in (x,y,λ). Let C_1 be its bound as (x,y,λ) varies in $[0,\pi)^2\times I$:

$$C_1=\sup_{0\leq x,\,y<\pi,\,\lambda\in I}p_\lambda(1,x,y).$$

Then we obtain from (2-16),

$$E_\omega^{\mathcal{G}}[1_{\Omega_\varepsilon}(\omega')\|U_\lambda(t;T_1\omega')\hat{\theta}_\lambda(1;\omega',\theta)\|^{-2}]$$

$$\leqq E_\omega^{\mathcal{G}}[1_{\tilde{\Omega}}(\omega')\|U_\lambda(t;T_1\omega')\hat{\theta}_\lambda(1;\omega',\theta)\|^{-2}]$$

$$\leqq C_1\int_0^\pi\|U_\lambda(t;T_1\omega')\hat{\theta}\|^{-2}d\theta$$

$$=\pi C_1,$$

proving (2-17), and consequently our claim.

Now let

$$A=\{(\omega, \lambda)\in\Omega\times\boldsymbol{R};\ U_{\lambda}(t;\omega) \text{ has Ljapounov behavior at } \pm\infty\},$$

i.e. let A be the totality of the pair (ω, λ) for which there exist $V_{\lambda}^{\pm}(\omega)$ satisfying the conclusion of Theorem 3. By examining the proof of Oseledec's theorem (see e.g. [19]), it is easily seen that A belongs to $\mathcal{G}\times\mathcal{B}(\boldsymbol{R})$, where $\mathcal{B}(\boldsymbol{R})$ is the Borel field of $\boldsymbol{R}$. The assertion of Theorem 1 is that for each $\lambda\in\boldsymbol{R}$, the set

$$A(\lambda)=\{\omega\in\Omega;\ (\omega, \lambda)\in A\}$$

has full probability measure. Hence by Fubini's theorem, the set

$$A(\omega)=\{\lambda\in\boldsymbol{R};\ (\omega, \lambda)\in A\}$$

has full Lebesgue measure for P-a.a. ω. Therefore from what has been proven above, follows

$$E\Big[1_{\Omega_{\varepsilon}}(\omega)\int_I 1_{A^c}(\omega, \lambda)\sigma(d\lambda;\omega)\Big]$$

$$=E\Big[\int_I 1_{A(\omega)^c}(\lambda)E_{\omega}^{\mathcal{G}}[1_{\Omega_{\varepsilon}}(\omega')\sigma(d\lambda;\omega')]\Big]=0,$$

which shows that for P-a.a. $\omega\in\Omega_{\varepsilon}$, $\sigma_{\omega}(I\setminus A(\omega))=0$. But since $P(\Omega_{\varepsilon})>0$, the ergodic argument in the Appendix of [17] shows that actually $\sigma_{\omega}(I\setminus A(\omega))=0$ with probability one. Finally fix an ω satisfying this and the conclusion of Lemma 1. Then we must have $V_{\lambda}^{+}(\omega)=V_{\lambda}^{-}(\omega)$ for $\sigma(\cdot;\omega)$-a.a. $\lambda\in I$. Therefore the solution of $H_{\omega}u=\lambda u$ whose initial condition ${}^t(u(0), u^{+}(0))$ belongs to $V_{\lambda}^{+}(\omega)$ $=V_{\lambda}^{-}(\omega)$ satisfies

$$\lim_{t\to\pm\infty}\frac{1}{|t|}\log[|u(t)|^2+|u^{+}(t)|^2]^{1/2}=-\gamma(\lambda).$$

In particular it belongs to $L^2(\boldsymbol{R};dt)$ and we have shown that $\sigma(\cdot;\omega)$-a.e. $\lambda\in I$ is an eigenvalue of H_{ω}. Letting $I\uparrow\boldsymbol{R}$, we finish the proof.

CASE b). Suppose $v=\nu((-\infty, 0))=0$, and $\nu((0, \infty))<\infty$. In this case, without loss of generality, we can assume that $\{Q_{\omega}(t)\}$ is a step process, i.e. is constant between its jumps. Then the spectrum Σ of H_{ω} is $[0, \infty)$ almost surely. Set

$$\tau(\omega)=\inf\{t>0;\ \Delta Q_{\omega}(t)>0\},$$

where $\Delta Q_{\omega}(t)=Q_{\omega}(t)-Q_{\omega}(t-)$. Then $\tau(\omega)>0$ almost surely.

Now let us fix an arbitrary finite open interval I such that $\inf I>0$. Corresponding to Ω_{ε}, $\mathcal{G}$, and $\theta_{\lambda}(t;\omega, \theta)$ in the preceding case a), we introduce the following objects:

$$\Omega'=\{\omega\in\Omega\,;\ 0<\tau(\omega)\leqq 1\}\,;$$

$$\mathcal{G}'=\sigma[Q_\omega(t),\ t\leqq 0\,;\ Q_\omega(\tau(\omega)+t)-Q_\omega(\tau(\omega)),\ t\geqq 0\,;\ \Delta Q_\omega(\tau(\omega))]\,;$$

$$\theta'_\lambda(\omega\,;\,\theta)=U_\lambda(\tau(\omega)-\,;\,\theta)\cdot\theta\,.$$

Then noting that

$$\|U_\lambda(t+\tau(\omega)\,;\,\omega)\hat\theta\|=\|U_\lambda(t\,;\,T_{\tau(\omega)}(\omega)\Delta U_\lambda(\tau(\omega)\,;\,\omega)\hat\theta'_\lambda(\omega\,;\,\theta)\|\,\|U_\lambda(\tau(\omega)-\,;\,\omega)\hat\theta\|\,,$$

where

$$\Delta U_\lambda(\tau(\omega)\,;\,\omega)=U_\lambda(\tau(\omega)\,;\,\omega)U_\lambda(\tau(\omega)-\,;\,\omega)^{-1}$$
$$=\begin{pmatrix}1 & 0\\ \Delta Q_\omega(\tau(\omega)) & 1\end{pmatrix},$$

and that

$$C_0=\inf\{\|U_\lambda(\tau(\omega)-\,;\,\omega)\hat\theta\|\,;\ 0\leqq\theta<\pi,\ \lambda\in I,\ \omega\in\Omega'\}>0\,,$$

we get as in case a),

$$E_\omega^{\mathcal{G}'}\Big[1_{\Omega'}(\omega')\int_I f(\lambda)\sigma(d\lambda\,;\,\omega)\Big]$$
$$\leqq\frac{1}{\pi}\varlimsup_{t\to+\infty}E_\omega^{\mathcal{G}'}\Big[1_{\Omega'}(\omega')\int_0^\pi d\theta\int_I f(\lambda)\|W_\lambda(t\,;\,\omega')\hat\theta'_\lambda(\omega'\,;\,\theta)\|^{-2}$$
$$\times\|U_\lambda(\tau(\omega)-\,;\,\omega)\hat\theta\|^{-2}\sigma^\theta(d\lambda\,;\,\omega')\Big]$$
$$\leqq\frac{1}{C_0^2\pi}\varlimsup_{t\to+\infty}\int_0^\pi d\theta\int_I f(\lambda)E_\omega^{\mathcal{G}'}[\|W_\lambda(t\,;\,\omega)\hat\theta'_\lambda(\omega'\,;\,\theta)\|^{-2}]\sigma^\theta(d\lambda\,;\,\omega)\,,$$

where we have set

$$W_\lambda(t\,;\,\omega)=U_\lambda(t\,;\,T_{\tau(\omega)}\omega)\Delta U_\lambda(\tau(\omega)\,;\,\omega)\,.$$

Note that this and $\sigma^\theta(d\lambda\,;\,\omega)$ are $\mathcal{G}'$-measurable.

It remains only to prove that the random variable $\theta'_\lambda(\omega'\,;\,\theta)$ has, under the probability measure $P_\omega^{\mathcal{G}'}(d\omega')$, a distribution density which is uniformly bounded in $(\theta,\lambda)\in[0,\pi)\times I$. Indeed, if this is true, then in the same manner as in case a), we will obtain

$$E_\omega^{\mathcal{G}'}[\sigma(d\lambda\,;\,\omega')]\ll d\lambda,\quad\text{on } I,$$

and from this, the conclusion will follow.

Now $\tau(\omega)$ is independent from $\mathcal{G}'$, and it obeys the exponential distribution with parameter $\beta\equiv\nu((0,\infty))>0$. Hence by the definition of $\theta'_\lambda(\omega\,;\,\theta)$, we have for any Borel function $F\geqq 0$,

$$E_\omega^{\mathcal{G}'}[F(\theta'_\lambda(\omega'\,;\,\theta))]=\beta\int_0^\infty F(U_\lambda(\tau\,;\,\omega_0)\cdot\theta)e^{-\beta\tau}d\tau\,,$$

where $\omega_0(t)\equiv 0$. By a direct calculation, we can show without difficulty that the right-hand side is bounded by

$$C_I\int_0^\pi F(\theta)d\theta\,,$$

C_I being a constant which depends only on I. (The assumption $\inf I>0$ is important in this respect.) Therefore the distribution density of $\theta_\lambda'(\omega';\theta)$ is uniformly bounded, as was claimed.

CASE c). Let $\nu(dx)=\nu_s(dx)+\nu_{ac}'(x)dx$ be the Lebesgue decomposition of the Lévy measure $\nu(dx)$, where we assume $\nu_{ac}'(x)\neq 0$, and let S be a Borel set of zero Lebesgue measure which supports $\nu_s(dx)$. If we set

$$J=J(\delta,M)=\{x\in S^c\,;\,|x|>\delta,\ 0<\nu_{ac}'(x)\leqq M\}\,,$$

then J has positive Lebesgue measure for δ sufficiently small and M sufficiently large. Fixing such δ and M, let us decompose $\{Q_\omega(t)\}$ as

$$Q_\omega(t)=Q_\omega^1(t)+Q_\omega^2(t)\,,\quad Q_\omega^2(t)=\int_J xN_\omega((0,t]\times dx)\,.$$

Then $\{Q_\omega^2(t)\}$ is a step process. Let

$$\cdots<\tau_{-1}(\omega)<\tau_0(\omega)\leqq 0<\tau_1(\omega)<\tau_2(\omega)<\cdots$$

be the points at which $Q_\omega^2(t)$ jumps, and set

$$\mathcal{G}_j=\sigma[Q_\omega^1(t),\ t\in\boldsymbol{R}\,;\,\{\tau_n(\omega)\}_{n=-\infty}^\infty\,;\,\{\Delta Q_\omega^2(\tau_n(\omega))\}_{n\neq j}]\,,\quad j\in\boldsymbol{Z}.$$

Then under the conditional probability $P_\omega^{\mathcal{G}_j}(d\omega')$, only $\Delta Q_\omega^2(\tau_j(\omega'))$ is random, and its distribution is proportional to $1_J(x)\nu_{ac}'(x)dx$. Now let us define the mapping $\Phi_j\colon\Omega\to\Omega$ by

$$Q_{\Phi_j}(t)=\begin{cases}Q_\omega(t)\,, & \text{for } t<\tau_j(\omega)\\ Q_\omega(t)-\Delta Q_\omega(\tau_j(\omega))\,, & \text{for } t\geqq\tau_j(\omega)\,,\end{cases}$$

i.e. $\Phi_j(\omega)$ is obtained from ω by removing its j-th jump. Then from the construction of the Green function of H_ω and $H_{\Phi_j(\omega)}$ (see § 2-1), we see that

$$g_\lambda(\tau_j(\omega),\tau_j(\omega)\,;\,\omega)=-\{-g_\lambda(\tau_j(\omega),\tau_j(\omega)\,;\,\Phi_j(\omega))^{-1}-\Delta Q_\omega(\tau_j(\omega))\}^{-1}$$

holds for $\lambda\in\boldsymbol{C}_+$. If we set

$$\zeta(\omega)=\xi(\omega)+i\eta(\omega)=-g_\lambda(\tau_j(\omega),\tau_j(\omega)\,;\,\Phi_j(\omega))^{-1}\in\boldsymbol{C}_+\,,$$

then ζ is a $\mathcal{G}_j$-measurable random variable. Therefore, with some constant M,

$$\begin{aligned}&E_\omega^{\mathcal{G}_j}[\operatorname{Im} g_\lambda(\tau_j(\omega'),\tau_j(\omega')\,;\,\omega)]\\ &\leqq\int\operatorname{Im}\Big(\frac{1}{\zeta(\omega)-x}\Big)P(\Delta Q_\omega^2(\tau_j(\omega')\in dx)\\ &\leqq M\int_{-\infty}^\infty\frac{\eta(\omega)}{(\xi(\omega)-x)^2+\eta(\omega)^2}\,dx=M\pi\,.\end{aligned}$$

Hence from (2-8),

$$E^{g_j}[\sigma_{11}(I\,;T_{\tau_j(\omega')}\omega')]$$
$$\leqq \frac{1}{\pi}\varlimsup_{\varepsilon\downarrow 0} E^{g_j}_{\omega'}\Big[\int_I \operatorname{Im} g_{\xi+i\varepsilon}(\tau_j(\omega'),\, \tau_j(\omega')\,;\omega')\Big] \leqq M|I|\,,$$

for any bounded open interval I, namely

$$E^{g_j}_{\omega'}[\sigma_{11}(d\lambda\,;T_{\tau_j(\omega')}\omega')] \ll d\lambda\,.$$

This implies, as in case a), that for P-a. a. $\omega\in\Omega$,

$$\sigma_{11}(A(\omega)^c\,;T_{\tau_j(\omega)}\omega)=0\,,\quad j\in\boldsymbol{Z}. \tag{2-18}$$

The proof will be finished if we have further shown that $\sigma(A(\omega)^c\,;\omega)=0$. To this end, fix an ω for which (2-18) holds. Then from (2-12),

$$\int_{A(\omega)^c}\Big\{\sum_{i,j=1}\tau_{ij}(\lambda\,;\omega)\varphi_i(t\,;\lambda,\omega)\varphi_j(t\,;\lambda,\omega)\Big\}\sigma(d\lambda\,;\omega)=0\,, \tag{2-19}$$

for $t=\tau_j(\omega)$, $j\in\boldsymbol{Z}$. Let $\mu_k(\lambda\,;\omega)$, $k=1, 2$, be the eigenvalue of the matrix $(\tau_{ij}(\lambda\,;\omega))$, $K(\lambda\,;\omega)$ the orthogonal matrix which diagonalize (τ_{ij}), and set

$$\begin{pmatrix} f_1(t\,;\lambda,\omega)\\ f_2(t\,;\lambda,\omega)\end{pmatrix}=K(\lambda\,;\omega)\begin{pmatrix}\varphi_1(t\,;\lambda,\omega)\\ \varphi_2(t\,;\lambda,\omega)\end{pmatrix}.$$

(2-19) means that for $\sigma(\cdot\,;\omega)$-a. a. $\lambda\in A(\omega)^c$, one has

$$\sum_{k=1}^{2}\mu_k(\lambda\,;\omega)f_k(\tau_j(\omega)\,;\lambda,\omega)=0\,,\quad j\in\boldsymbol{Z}.$$

Then for such λ, one and only one of the following two cases is possible:

(i) for every $j\in\boldsymbol{Z}$, $F(t)\equiv\sum_{k=1}^{2}\mu_k(\lambda\,;\omega)f_k(t\,;\lambda,\omega)$ does not vanish identically on $(\tau_j(\omega),\tau_{j+1}(\omega))$;

(ii) $F(t)\equiv 0$ on $\boldsymbol{R}$.

For suppose (i) does not hold. Then $F(t)$ vanishes identically on some open interval. But f_k is a solution of $H_\omega u=\lambda u$. Hence $F(t)$ must vanish entirely on $\boldsymbol{R}$.

Now let $\Lambda_j(\omega)$ be the totality of eigenvalues of $H_\omega|_{[\tau_j(\omega),\tau_{j+1}(\omega)]}$ with Dirichlet boundary conditions. It is clear that if (i) holds, then $\lambda\in\Lambda(\omega)\equiv\bigcap_j\Lambda_j(\omega)$. On the other hand, if (ii) holds, it is also clear that $\mu_k(\lambda,\omega)f_k(t\,;\lambda,\omega)$, $k=1, 2$, vanish for all t. In particular,

$$\mu_k(\lambda\,;\omega)f_k(0\,;\lambda,\omega)=\mu_k(\lambda,\omega)f_k^+(0\,;\lambda,\omega)=0\,,\quad k=1, 2,$$

and we get

$$\int_{A(\omega)^c\setminus\Lambda(\omega)}\sum_{k=1}^{2}\mu_k(\lambda,\omega)\{f_k(0;\lambda,\omega)^2+f_k^+(0;\lambda,\omega)^2\}\sigma(d\lambda;\omega)$$

$$=\int_{A(\omega)^c\setminus\Lambda(\omega)}\sum_{i,j=1}^{2}\tau_{ij}(\lambda;\omega)\{\varphi_i(0;\lambda,\omega)\varphi_j(0;\lambda,\omega)+\varphi_i^+(0;\lambda,\omega)\varphi_j^+(0;\lambda,\omega)\}\sigma(d\lambda;\omega)$$

$$=\int_{A(\omega)^c\setminus\Lambda(\omega)}\{\tau_{11}(\lambda;\omega)+\tau_{22}(\lambda,\omega)\}\sigma(d\lambda;\omega)$$

$$=\sigma(A(\omega)^c\setminus\Lambda(\omega);\omega)=0.$$

Thus we have proved that the spectral measure $\sigma(d\lambda;\omega)$ is concentrated on $A(\omega)\cup\Lambda(\omega)$. It should be noted that this already proves that H_ω has only point spectrum because $\Lambda(\omega)$ is at most countable, and because for $\sigma(\cdot;\omega)$-a. a. $\lambda\in A(\omega)$, one can show the existence of exponentially decaying eigenfunctions as before. However some additional probabilistic considerations show that we have actually $\Lambda(\omega)=\emptyset$ for P-a. a. ω. We will show in fact that $\Lambda_1(\omega)\cap\Lambda_2(\omega)=\emptyset$ almost surely.

First note that $\Lambda_j(\omega)$ is determined from $\{Q_\omega(t)-Q_\omega(\tau_j(\omega));t\in[\tau_j(\omega),\tau_{j+1}(\omega)]\}$. Therefore $\Lambda_1(\omega)$ and $\Lambda_2(\omega)$ are independent (set-valued) random variable because of the strong Markov property of $\{Q_\omega(t)\}$. Hence it suffices to show $P(\lambda\in\Lambda_1(\omega))=0$ for each fixed λ. Now let $0<s_1(\omega)<s_2(\omega)<\cdots$ be the positive zero's of the solution of $H_\omega u=\lambda u$, $u(0)=0$. Then $\lambda\in\Lambda_1(\omega)$ if and only if $s_n(T_{\tau_1(\omega)}\omega)=\tau_1(T_{\tau_1(\omega)}\omega)$ for some $n\geqq 1$. On the other hand, $s_n(\omega)=\tau_1(\omega)$ is equivalent to $s_n(\omega)=s_n(\Psi(\omega))=\tau_1(\omega)$, where we define $\Psi(\omega)$ by $Q_{\Psi(\omega)}(t)=Q_\omega^1(t)$. From the statistical independence of $\{Q_\omega^1(t)\}$ and $\{Q_\omega^2(t)\}$, we see that $s_n(\Psi(\omega))$ and $\tau_1(\omega)$ are independent. Since $\tau_1(\omega)$ has a continuous distribution, we finally obtain

$$P(\lambda\in\Lambda_1(\omega))\leqq\sum_{n=1}^{\infty}P\Big(s_n(T_{\tau_1(\omega)}\omega)=\tau_1(T_{\tau_1(\omega)}\omega)\Big)$$

$$=\sum_{n=1}^{\infty}P(s_n(\omega)=\tau_1(\omega))$$

$$\leqq\sum_{n=1}^{\infty}P(s_n(\Psi(\omega))=\tau_1(\omega))=0,$$

completing the proof.

§ 3. Proof of Theorem 1.

Except the assertion $\gamma(\lambda)>0$, the theorem follows from the well known theorem of Oseledec (see e. g. [19] and [28]) as soon as the condition

$$E\Big[\sup_{0\leq t\leq T}\log\|U_\lambda(t;\omega)\|\Big]<\infty \tag{3-1}$$

is verified for some $T>0$. But if $\{e_1, e_2\}$ is the standard basis of $\boldsymbol{C}^2$,

$$\|U\| \leqq \|Ue_1\| + \|Ue_2\|,$$

so that in order to prove (3-1), it suffices to prove

$$E\Big[\sup_{0 \leqq t \leqq T} \log \|U_\lambda(t;\omega)v\|\Big] < \infty \tag{3-2}$$

for each fixed $v = {}^t(\alpha, \beta) \in \boldsymbol{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.

Now let $u(t)$ be the solution of $H_\omega u = \lambda u$, $u(0) = \alpha$, $u^+(0) = \beta$. Then we have $\|U_\lambda(t;\omega)v\|^2 = |u(t)|^2 + |u^+(t)|^2$. On the other hand, Lévy's canonical form can be rewritten as (see [11])

$$Q_\omega(t) = c_\delta t + vB_\omega(t) + \int_{|x|>\delta} xN_\omega((0, t] \times dx) + \int_0^{t+}\int_{|x|\leqq\delta} x\tilde{N}_\omega(dsdx),$$

for each $\delta > 0$. Here we have set $c_\delta = b - \int_{|x|>\delta} a(x)\nu(dx)$. Then for each fixed α and β, the random equation $H_\omega u = \lambda u$, $u(0) = \alpha$, $u^+(0) = \beta$ can be considered as the following pair of stochastic integral equation:

$$u(t) = \alpha + \int_0^t u^+(s)ds$$

$$\begin{aligned} u^+(t) = \beta &+ (c_\delta - \lambda)\int_0^t u(s)ds + v\int_0^t u(s)dB_\omega(s) \\ &+ \int_0^{t+}\int_{|x|>\delta} xu(s)N_\omega(dsdx) + \int_0^{t+}\int_{|x|\leqq\delta} xu(s)\tilde{N}_\omega(dsdx). \end{aligned}$$

Hence from the generalized Ito's formula ([11], Chapter II, §5), we obtain

$$\log(|u(t)|^2 + |u^+(t)|^2) = \int_0^t p(z(s))ds + M(t) + S(t), \tag{3-3}$$

where we set

$$z(t) = u^+(t)/u(t) \in \boldsymbol{C} \cup \{\infty\}, \tag{3-4}$$

$$\begin{aligned} p(z) = {} & 2(1 + c_\delta)\frac{\mathrm{Re}\, z}{1 + |z|^2} - 2\frac{(\mathrm{Re}\,\lambda)(\mathrm{Re}\, z) + (\mathrm{Im}\,\lambda)(\mathrm{Im}\, z)}{1 + |z|^2} \\ & + v^2\frac{1 - (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}{1 + |z|^2} \\ & + \int_{|x|\leqq\delta}\Big\{\log\frac{1 + (x + \mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}{1 + |z|^2} - \frac{2x\,\mathrm{Re}\, z}{1 + |z|^2}\Big\}\nu(dx), \end{aligned} \tag{3-5}$$

$$\begin{aligned} M(t) = {} & 2v\int_0^t \frac{\mathrm{Re}\, z(s)}{1 + |z(s)|^2} dB_\omega(s) \\ & + \int_0^{t+}\int_{|x|\leqq\delta} \log\Big\{\frac{1 + (x + \mathrm{Re}\, z(s-))^2}{1 + |z(s-)|^2}\Big\}\tilde{N}_\omega(dsdx), \end{aligned} \tag{3-6}$$

$$S(t) = \int_0^{t+}\int_{|x|>\delta} \log\Big\{\frac{1 + (x + \mathrm{Re}\, z(s-))^2 + (\mathrm{Im}\, z(s))^2}{1 + |z(s-)|^2}\Big\}N(dsdx). \tag{3-7}$$

For each $\delta>0$, it is easily seen that $p(z)$ is a bounded continuous function on $C\cup\{\infty\}$, and that $\{M(t)\}$ is a square integrable martingale with right continuous paths. Therefore

$$E\Big[\sup_{0\leq t\leq T}\Big|\int_0^T p(z(s))ds\Big|\Big]\leqq T\|p\|_\infty<\infty, \tag{3-8}$$

and

$$E\Big[\sup_{0\leq t\leq T}|M(t)|\Big]<\infty, \tag{3-9}$$

by martingale inequality.

On the other hand, it is elementary to show

$$\sup_{z\in C\cup\{\infty\}}\left|\log\left\{\frac{1+(x+\mathrm{Re}\,z)^2+(\mathrm{Im}\,z)^2}{1+|z|^2}\right\}\right|\leqq 3\log(1+|x|),$$

so that

$$\begin{aligned} E\Big[\sup_{0\leq t\leq T}|S(t)|\Big] &\leqq E\Big[\int_0^{T+}\int_{|x|>\delta}3\log(1+|x|)N(dsdx)\Big] \\ &=3T\int_{|x|>\delta}\log(1+|x|)\nu(dx)<\infty, \end{aligned} \tag{3-10}$$

from the assumption.

Combining (3-8), (3-9), and (3-10) with (3-3), we arrive at (3-2).

It remains to prove $\gamma(\lambda)>0$. Having established the almost sure existence of the limit, it suffices to let $t\to\infty$ through some discrete set, namely it suffices to prove that for some $a>0$, we have

$$\lim_{n\to\infty}\frac{1}{n}\log\|U_\lambda(na\,;\omega)v\|>0, \quad \text{a. s.}$$

for each $v\in C^2\backslash\{0\}$. But $U_\lambda(na\,;\omega)$ is a product of independent random matrices;

$$U_\lambda(na\,;\omega)=U_\lambda(a\,;T_{(n-1)a}\omega)U_\lambda(a\,;T_{(n-2)a}\omega)\cdots U_\lambda(a\,;\omega).$$

Hence it suffices to verify that for some $a>0$, $U_\lambda(a\,;\omega)$ satisfies the condition of the following Furstenberg's theorem:

Furstenberg's theorem—two dimensional version ([1] Part A, Chapter II, Theorem 3.6, Theorem 4.1, and Proposition 4.3.)

Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent, identically distributed random variables in $Sl(2, C)$, and let $\mu(dY)$ be their distribution. Further, let G_μ be the closed subgroup of $Sl(2, C)$ generated by the topological support of μ. If μ and G_μ satisfy the following three conditions;

(i) $E^\mu[\log\|Y_1\|]<\infty$;

(ii) G_μ is not compact;

(iii) for any $\bar{x}\in P(C^2)$, $\{M\cdot\bar{x}\,;M\in G_\mu\}$ contains at least three different points, then for each $v \in \boldsymbol{C}^2 \setminus \{0\}$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|Y_n Y_{n-1} \cdots Y_1 v\| > 0, \quad \text{a.s.}$$

Here the definitions of the notation in (iii) are the following: We identify two elements x, y of $\boldsymbol{C}^2 \setminus \{0\}$ if $x = \lambda y$ for some $\lambda \in \boldsymbol{C}$. This defines an equivalence relation $\sim$ in $\boldsymbol{C}^2 \setminus \{0\}$. We set $P(\boldsymbol{C}^2) = [\boldsymbol{C}^2 \setminus \{0\}]/\sim$. The equivalence class to which $x \in \boldsymbol{C}^2 \setminus \{0\}$ belongs is denoted by $\bar{x}$, and we define $M \cdot \bar{x} = \overline{Mx}$ for $M \in Sl(2, \boldsymbol{C})$.

Now for each $a > 0$ and $\lambda \in \boldsymbol{C}$, set $Y_n = U_\lambda(a; T_{(n-1)a}\omega)$, and let $\mu_\lambda(a)$ be the distribution of Y_1 in $Sl(2, \boldsymbol{C})$. We have already seen that $E[\log \|Y_1\|] < \infty$ holds for all $a > 0$ and $\lambda \in \boldsymbol{C}$. Let us show that for each fixed $\lambda \in \boldsymbol{C}$, we can find an $a > 0$ such that $G_{\mu_\lambda(a)}$ satisfies the conditions (ii) and (iii).

Verification of (ii). First consider the case $\lambda > \inf \Sigma$, which is the most important.

Let Supp(P) be the topological support of the probability measure P on Ω. As we already noted, $\Omega = \{\omega \in D(\boldsymbol{R} \to \boldsymbol{R}); \omega(0) = 0\}$ is endowed with the Skorohod topology. In order that condition (ii) holds for a given $a > 0$, it is sufficient that there exists an $\omega_0 \in \text{Supp}(P)$ satisfying the following two conditions:

(1) $\omega_0(t)$ is continuous both at $t = 0$ and $t = a$;

(2) $|\operatorname{tr} U_\lambda(a; \omega_0)| > 2$.

Indeed (1) implies that the correspondence $\omega \to U_\lambda(a; \omega)$ is continuous at $\omega = \omega_0$ (see [25], Lemma 2). Hence $U_\lambda(a; \omega_0)$ belongs to Supp($\mu_\lambda(a)$). On the other hand, (2) implies that $\|U_\lambda(a; \omega_0)^n\| \to \infty$ as $n \to \infty$, so that $G_{\mu_\lambda(a)}$ cannot be compact (see the argument of Matsuda and Ishii [23], p. 67).

In order to show the existence of such ω_0, we divide our argument into four cases.

Case 1. $v = \nu((-\infty, 0)) = 0$, and $\int_0^1 x\nu(dx) < \infty$ in the Lévy's canonical form, in which case we may assume that $\{Q_\omega(t)\}$ is of the following form:

$$Q_\omega(t) = \int_0^\infty x N_\omega((0, t] \times dx).$$

Then $\Sigma = [0, \infty)$ and we consider only $\lambda > 0$. Fix an $\alpha \in (0, \infty) \cap \text{Supp}(\nu)$, and set

$$\omega_\beta(t) = \alpha 1_{[\beta, \infty)}(t),$$

for each $\beta > 0$. Then $\omega_\beta \in \text{Supp}(P)$, and an elementary calculation gives

$$g(\beta) \equiv \operatorname{tr} U_\lambda(\beta; \omega_\beta) = 2\cos(\sqrt{\lambda}\beta) + \frac{\alpha}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\beta); \tag{3-10}$$

in particular $g(0)=2$, $g'(0)=\alpha>0$. Hence for a sufficiently small $\beta>0$, $\mathrm{tr}\,U_\lambda(\beta;\omega_\beta)>2$. Since $t\to\mathrm{tr}\,U_\lambda(t;\omega_\beta)$ is right-continuous, $\mathrm{tr}\,U_\lambda(a;\omega_\beta)>2$ still holds for $a>\beta$ which is sufficiently close to β. These ω_β and a satisfy the desired conditions.

In the next three cases, we have $\Sigma=(-\infty, \infty)$, so that we shall consider an arbitrarily fixed $\lambda>0$.

CASE 2. $v=\nu((-\infty, 0))=0$, but $\int_0^1 x\nu(dx)=+\infty$. In this case, if we rewrite Lévy's canonical form as

$$Q_\omega(t)=c_\eta t+\int_\eta^\infty xN_\omega((0, t]\times dx)+Q_\omega^\eta(t),$$

where

$$c_\eta=b-\int_\eta^\infty a(x)\nu(dx),$$

then $c_\eta\downarrow-\infty$ $(\eta\downarrow 0)$ from the assumption. Choose $\eta>0$ so small that $c_\eta<\lambda$ and $\nu((\eta, \infty))>0$, and fix an $\alpha\in(\eta, \infty)\cap\mathrm{Supp}(\nu)$. If we set for each $\beta>0$,

$$\omega_\beta(t)=c_\eta t+\alpha 1_{[\beta,\infty)}(t),$$

then as in [25], one shows $\omega_\beta\in\mathrm{Supp}(P)$. The rest is the same as in case 1.

CASE 3. $v\neq0$. In this case, $\omega_\gamma(t)\equiv\gamma t$ belongs to Supp(P) for any $\gamma\in\boldsymbol{R}$. It suffices to take $\gamma>\lambda$, because then we have

$$\mathrm{tr}\,U_\lambda(a;\omega_\gamma)=2\cosh(\sqrt{\gamma-\lambda}\,a)>2,$$

for all $a>0$.

CASE 4. $v=0$ and $\nu((-\infty, 0))>0$. As in [25], it can be shown that there exists a $\mu\in\boldsymbol{R}$ and an $\alpha<0$ such that ω_β, defined by

$$\omega_\beta(t)=\mu t+\alpha 1_{[\beta,\infty)}(t),$$

belongs to Supp(P) for any $\beta>0$. We shall assume $\mu=0$ for simplicity.

Consider first the case $\lambda>0$. Then $g(\beta)\equiv\mathrm{tr}\,U_\lambda(\beta;\omega_\beta)$ is given by (3-10) with $\alpha<0$. Then $g(2\pi/\sqrt{\lambda})=2$, and $g'(2\pi/\sqrt{\lambda})=\alpha<0$. Hence we have $g(\beta)>2$ for $\beta<2\pi/\sqrt{\lambda}$ sufficiently close to $2\pi/\sqrt{\lambda}$. As before, we choose an $a\in(\beta, 2\pi/\sqrt{\lambda})$ sufficiently close to β, to obtain $\mathrm{tr}\,U_\lambda(a;\omega_\beta)>2$.

Secondly let $\lambda\in(-\infty, -\alpha^2/4)\cup(-\alpha^2/4, 0]$. Then

$$g(\beta)\equiv\mathrm{tr}\,U_\lambda(\beta;\omega_\beta)=2\cosh(\beta\sqrt{|\lambda|})+\alpha/\sqrt{|\lambda|}\sinh(\beta\sqrt{|\lambda|}).$$

In this case, we have $\lim_{\beta\to\infty}|g(\beta)|=+\infty$, so that we have to choose a $\beta>0$ sufficiently large and then an $a>\beta$ sufficiently close to β.

Finally in case $\lambda=-\alpha^2/4$, we consider $\omega_{\beta,\gamma}$, defined by

$$\omega_{\beta,\gamma}(t)=\{1_{[\beta,\infty)}(t)+1_{[\beta+\gamma,\infty)}(t)\}$$

with β, $\gamma>0$. Then

$$\mathrm{tr}\, U^2_{-\alpha/4}(\beta+\gamma\,;\,\omega_{\beta,\gamma})=(\cosh b-2\sinh b)(\cosh c-2\sinh c)-\sinh b\sinh c\,,$$

where $b=|\alpha|\beta/2$, $c=|\alpha|\gamma/2$. Let $\beta>0$ be such that $\cosh\beta-2\sinh\beta=0$. Then

$$\lim_{\gamma\to\infty}\mathrm{tr}\, U_\lambda(\beta+\gamma\,;\,\omega_{\beta,\gamma})=\lim_{\gamma\to\infty}(-\sinh b\sinh c)=-\infty\,,$$

so that it suffices to pick a sufficiently large $\gamma>0$ and an $a>\beta+\gamma$ which is sufficiently close to $\beta+\gamma$.

Now that we have verified condition (ii) in the case $\lambda>\inf\Sigma$, let us consider the case $\lambda\in\boldsymbol{C}\backslash\boldsymbol{R}$ or $\lambda\leqq\inf\Sigma$. For this, it suffices to show that for P-a.a. ω, there exists a solution $u(t)$ of $H_\omega u=\lambda u$ such that

(3-11) $$\overline{\lim_{n\to\infty}}(|u(na)|^2+|u^+(na)|^2)=+\infty\,.$$

Indeed, since we have

$$\binom{u(na)}{u^+(na)}=Y_nY_{n-1}\cdots Y_1\binom{u(0)}{u^+(0)},$$

(3-11) implies that $Y_nY_{n-1}\cdots Y_1\in G_{\mu_\lambda(a)}$, $n=1, 2, \cdots$ are not bounded as $n\to\infty$, contradicting the compactness of $G_{\mu_\lambda(a)}$.

Now suppose $\lambda\in\boldsymbol{C}\backslash\boldsymbol{R}$, and $a>0$. For P-a.a. ω, H_ω is in the limit point case at $+\infty$. Hence there is a solution u of $H_\omega u=\lambda u$ such that

$$\int_0^\infty |u(t)|^2dt=+\infty\,.$$

But from the Green's formula (§2-1),

$$(2i\,\mathrm{Im}\,\lambda)\int_0^{na}|u(t)|^2dt=[u,\bar{u}](na)-[u,\bar{u}](0),$$

so that $\{(u(na), u^+(na))\}_{n\geqq 0}$ cannot be bounded.

If $\lambda\leqq\inf\Sigma\equiv c$, then $H_\omega\geqq -d^2/dt^2+c$, and by a simple comparison argument, we can show that for the solution ϕ of $H_\omega u=\lambda u$ with $\phi(0)=0$, $\phi^+(0)=1$, $\{\phi(na)\}_{n\geqq 0}$ is unbounded for any $a>0$.

Let us pass to the verification of the condition (iii). In fact, we will prove that for all $a>0$, $\lambda\in\boldsymbol{C}$, and $\bar{x}\in P(\boldsymbol{C}^2)$, $\{M\cdot\bar{x}\,;\,M\in\mathrm{Supp}(\mu_\lambda(a))\}$ already contains more than two points. To this end, we will find, each time we fix a, λ, and $\bar{x}$, three elements ω_1, ω_2, and ω_3 of Supp(P) of which $t=0$ and $t=a$ are continuity points, and such that $U_\lambda(a\,;\,\omega_j)\cdot\bar{x}$, $j=1, 2, 3$ are different from each other.

But if $x={}^t(\alpha, \beta)$, $U_\lambda(t;\omega_j)x={}^t(u_j(t), u_j^+(t))$, and if we set $z=\beta/\alpha$ and $z_\lambda(t;\omega_j, z)=u_j^+(t)/u_j(t)$, then it is clear that the last condition is equivalent to saying that $z_\lambda(a;\omega_j, z)$, $j=1, 2, 3$ are different from each other.

In order to show the existence of such ω_j's, we divide our argument into two cases. First fix $a>0$, $\lambda\in\boldsymbol{C}$, and $z\in\boldsymbol{C}\cup\{\infty\}$.

CASE 1. $v\neq0$, Let us define $\omega_{\gamma,\tau,\delta}$ by

$$\omega_{\gamma,\tau,\delta}(t)=\begin{cases} 0 & ;\ t\leqq\tau \\ \gamma(t-\tau) & ;\ \tau\leqq t\leqq\tau+\delta, \\ \gamma\cdot\delta & ;\ \tau+\delta\leqq t \end{cases}$$

where $\tau, \delta>0$ and $\tau+\delta<a$. Clearly $\omega_{\gamma,\tau,\delta}\in\mathrm{Supp}(P)$. First choose a $\tau\in(0, a)$ such that $\zeta\equiv z_\lambda(\tau;\omega_{\gamma,\tau,\delta}, z)\neq\infty$. This choice does not depend on δ and γ. At this stage, we note that $z_\lambda(t;\omega, z)$ satisfies the equation

$$z(t)=z(s)+Q_\omega(t)-Q_\omega(s)-\int_s^t(\lambda+z(\sigma)^2)d\sigma \tag{3-12}$$

provided $z_\lambda(\sigma;\omega, z)\neq\infty$ for all $\sigma\in[s, t]$ (see §4-1). Hence

$$z_\lambda(t;\omega_{\gamma,\tau,\delta}, z)=\zeta+\gamma(t-\tau)-\int_\tau^t\{\lambda+z_\lambda(s;\omega_{\gamma,\tau,\delta}, z)^2\}ds,$$

for $t\geqq\tau$ sufficiently close to τ. By differentiating this with respect to γ, we can show without difficulty that for each compact interval Γ, there is a $\delta\in(0, a-\tau)$ such that

$$\frac{\partial}{\partial\gamma}z_\lambda(\tau+t;\omega_{\gamma,\tau,\delta}, z)\neq0,$$

for all $(\gamma, t)\in\Gamma\times[0, \delta]$. In particular $z_\lambda(\tau+\delta;\omega_{\gamma,\tau,\delta}, z)$ takes various values as we vary $\gamma\in\Gamma$. Finally, if we note that $z\to z_\lambda(t;\omega, z)$ is a bijection from $\boldsymbol{C}\cup\{\infty\}$ onto itself for any $t\geqq0$ and $\omega\in\boldsymbol{\Omega}$, we see that

$$z_\lambda(a;\omega_{\gamma,\tau,\delta}, z)=z_\lambda(a-\tau-\delta;T_{\tau+\delta}\omega_{\gamma,\tau,\delta}, z_\lambda(\tau+\delta;\omega_{\gamma,\tau,\delta}, z))$$

takes different values as one varies $\gamma\in\Gamma$. It suffices to choose suitable three values of γ.

CASE 2. $v=0$. In this case, we choose a $\mu\in\boldsymbol{R}$ and an $\alpha\in\mathrm{Supp}(\nu)\setminus\{0\}$ so that for every sequence $S=\{\sigma_j\}_{j=1}^n$ with $0<\sigma_1<\cdots<\sigma_n$,

$$\omega_S(t)\equiv\mu t+\alpha\sum_{j=1}^n 1_{[\sigma_j,\infty)}(t)$$

belongs to $\mathrm{Supp}(P)$. Again we may assume $\mu=0$. First let $S_1=\emptyset$ so that $\omega_{S_1}(t)\equiv0$. Choose a $\sigma\in(0, a)$ such that $\zeta\equiv z_\lambda(\sigma;\omega_{S_1}, z)\neq\infty$. Let

$K=\{z\in \boldsymbol{C}\,;\,|z-\zeta|\leqq 3|\alpha|\}$. Then it is easy to see that one can choose a $\delta\in(0, a-\sigma)$ such that for any $z\in K$, $z_\lambda(t\,;\omega_{S_1}, z)$ remains in the disk of radius $|\alpha|/6$ centered at z for $t\in[0, \delta]$. If $S_2=\{\sigma\}$, $S_3=\{\sigma, \sigma'\}$, $\sigma<\sigma'<\sigma+\delta$, then from (3-12) (note that $\Delta z(t)=\Delta Q_\omega(t)$ when $z(t)\neq\infty$),

$$z_\lambda(\sigma\,;\omega_{S_2}, z)=z_\lambda(\sigma-\,;\omega_{S_1}, z)+\alpha=\zeta+\alpha\,,$$

$$z_\lambda(\sigma'\,;\omega_{S_3}, z)=z_\lambda(\sigma'-\,;\omega_{S_2}, z)=\alpha$$

$$=z_\lambda(\sigma'-\sigma\,;\omega_{S_1}, \zeta+\alpha)+\alpha\,,$$

and consequently it follows that

$$|z_\lambda(\sigma+\delta\,;\omega_{S_i}, z)-z_\lambda(\sigma+\delta\,;\omega_{S_j}, z)|\geqq|\alpha|/3, \quad \text{for } i\neq j.$$

Therefore $z_\lambda(a\,;\omega_{S_j}, z)$, $j=1, 2, 3$ take different values as well. The proof of Theorem 1 is now complete.

§4. Proof of Theorem 2.

4-1. Outline.

Our problem is to show, under the condition $\int_{|x|>1}\log|x|\nu(dx)=\infty$ and $\lambda>\inf\Sigma$, that the solution $u(t)$ of $H_\omega u=\lambda u$, $u(0)=\alpha$, $u^+(0)=\beta$, $(\alpha^2+\beta^2=1)$ grows up faster than exponentially, i.e. that

(4-1)
$$\lim_{t\to\infty}\frac{1}{t}\log(u(t)^2+u^+(t)^2)$$
$$=\lim_{t\to\infty}\frac{1}{t}\left\{\int_0^t p(z(s))ds+M(t)+S(t)\right\}=+\infty\,,\quad \text{a.s.}$$

(see (3-2)). Since everything is real-valued in this case, p, M, and S above take the following forms:

(4-2)
$$z(t)=u^+(t)/u(t)\in\bar{\boldsymbol{R}}\equiv\boldsymbol{R}\cup\{\infty\}\,;$$

(4-3)
$$p(z)=2(1+c_\delta-\lambda)\frac{z}{1+z^2}+v^2\frac{1-z^2}{(1+z^2)^2}$$
$$+\int_{|x|\leq\delta}\left\{\log\frac{1+(z+x)^2}{1+z^2}-\frac{2xz}{1+z^2}\right\}\nu(dx)\,;$$

(4-4)
$$M(t)=2v\int_0^t\frac{z(s)}{1+z(s)^2}dB(s)$$
$$+\int_0^{t+}\int_{|x|\leq\delta}\log\left\{\frac{1+(x+z(s-))^2}{1+z(s-)^2}\right\}\tilde{N}(dsdx)\,;$$

and

$$S(t)=\int_0^{t+}\int_{|x|>\delta}\log\left\{\frac{1+(x+z(s-))^2}{1+z(s-)^2}\right\}N(dsdx). \tag{4-5}$$

As we noted in § 3, for any $\delta>0$, $p(z)$ is bounded continuous on $\bar{\boldsymbol{R}}$, and $\{M(t)\}$ is a square integrable martingale. In fact, it can be shown at the same time that

$$E[M(t)^2]=O(t), \quad \text{as } t\to\infty. \tag{4-6}$$

Therefore we have, first of all,

$$\overline{\lim_{t\to\infty}}\left|\frac{1}{t}\int_0^t p(z(s))ds\right|\leqq\|p\|_\infty<\infty, \tag{4-7}$$

and next, from (4-6) and Lemma 1.2 of [15],

$$\lim_{t\to\infty}\frac{1}{t}M(t)=0, \quad \text{a.s.}$$

Thus, the sole thing which is not trivial is to prove that for some suitable choice of $\delta>0$,

$$\lim_{t\to\infty}\frac{1}{t}S(t)=+\infty, \quad \text{a.s.} \tag{4-8}$$

We do this by analysing in detail the asymptotic behavior of the process $\{z(t)\}$ defined by (4-2) and the Markov chain associated to it.

First suppose that $u(\tau)\neq 0$ for $\tau\in[s, t]$. Then from (1-1), it is easily seen that

$$\frac{d^+}{d\tau}\left(\frac{u^+(\tau)}{u(\tau)}-Q_\omega(\tau)\right)=-\left(\frac{u^+(\tau)}{u(\tau)}\right)^2-\lambda.$$

In other words $z(t)$ satisfies

$$z(t)-z(s)=Q_\omega(t)-Q(s)-\int_s^t(\lambda+z(\tau)^2)d\tau, \tag{4-9}$$

provided $z(\tau)\neq\infty$ for $\tau\in[s, t]$. In particular, $\Delta z(t)=\Delta Q_\omega(t)$ whenever $z(t)\neq\infty$. Moreover it is clear from (1-1) that $u^+(t)=u^+(t-)$ whenever $u(t)=0$. Hence $z(t)=z(t-)=\infty$ whenever $z(t)=\infty$.

Keeping these in mind, we proceed as follows. Let us define a sequence of random variables $\sigma_0(\omega)\equiv 0<\sigma_1(\omega)<\sigma_2(\omega)<\cdots$ by

$$\sigma_{n+1}(\omega)=\inf\{t>\sigma_n(\omega);\ |\Delta Q_\omega(t)|>\delta\}, \quad n\geqq 0,$$

and set

$$S_-(t)=\sum_{\sigma_n\leqq t}\log(1+z(\sigma_n-)^2),$$

$$S_+(t)=\sum_{\sigma_n\leqq t}\log(1+z(\sigma_n)^2),$$

with the convention that $\log(1+\infty^2)\equiv 0$. Then from (4-5) and $z(\sigma_n)=z(\sigma_n-)+\Delta Q_\omega(\sigma_n)$, we have the decomposition

$$S(t)=S_+(t)-S_-(t).$$

Hence it is sufficient to show that

$$\lim_{t\to\infty}\frac{1}{t}S_-(t)<\infty, \quad \text{a.s.},$$

and

$$\lim_{t\to\infty}\frac{1}{t}S_+(t)=+\infty, \quad \text{a.s.}$$

Our plan is the following.

In § 4-2, we prove that

1) $\{z(t)\}$ is a strong Markov process, and if $\lambda>\inf\Sigma$, it has an invariant distribution $\pi(dz)$;

2) $\{z(t)\}$ satisfies the individual ergodic theorem, i.e. for any bounded Borel function f on $\bar{\boldsymbol{R}}$, and for any starting point $z=z(0)$, one has

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(z(s))ds=\int_{\bar{\boldsymbol{R}}} f(z)\pi(dz), \quad \text{a.s.}$$

In particular, (4-7) can be strengthened so that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t p(z(s))ds=\int_{\bar{\boldsymbol{R}}} p(z)\pi(dz), \quad \text{a.s.}$$

In the next § 4-3, we prove the following;

3) if we set $(\zeta_n, q_n)=(z(\sigma_n-), \Delta Q(\sigma_n))$, then it is a Markov chain in $\bar{\boldsymbol{R}}\times\boldsymbol{R}_\delta$, where $\boldsymbol{R}_\delta=\boldsymbol{R}\setminus[-\delta, \delta]$;

4) if $\lambda>\inf\Sigma$, then $\{(\zeta_n, q_n)\}$ has an invariant distribution $\mu(d\zeta dq)$. Moreover, the individual ergodic theorem holds as follows: for any Borel function $0\leqq F(\zeta, q)<\infty$ on $\bar{\boldsymbol{R}}\times\boldsymbol{R}_\delta$, and for any starting point (ζ, q),

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n F(\zeta_j, q_j)=\iint_{\bar{\boldsymbol{R}}\times\boldsymbol{R}_\delta} F(\zeta, q)\mu(d\zeta dq), \quad \text{a.s.}$$

5) μ is a product measure: $\mu(d\zeta dq)=m(d\zeta)\nu(dq)$.

Concerning this $m(d\zeta)$, we prove in § 4-4, that for a suitable choice of $\delta>0$,

6) $$\int_{\bar{\boldsymbol{R}}}|\zeta|^\beta m(d\zeta)<\infty \quad \text{for any } 0<\beta<1,$$

in particular

$$\int_{\bar{\boldsymbol{R}}}\log(1+\zeta^2)m(d\zeta)<\infty.$$

From (1)-(6), (4-8) can be deduced without difficulty. Indeed, if we set

$$n(t)=\max\{n\,;\ \sigma_n\leqq t\},$$

then noting that $\sigma_n-\sigma_{n-1}$, $n\geqq 1$, are i.i.d. which obey the exponential distribution with parameter $\nu(\boldsymbol{R}_\delta)>0$, we see that

$$\lim_{t\to\infty}\frac{n(t)}{t}=\lim_{t\to\infty}\Big(\frac{1}{n}\sum_{j=1}^{n}(\sigma_j-\sigma_{j-1})\Big)^{-1}=E[\sigma_1]^{-1}=\nu(\boldsymbol{R}_\delta).$$

Therefore from 4) and 6),

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}S_-(t)&=\lim_{t\to\infty}\frac{n(t)}{t}\frac{1}{n(t)}\sum_{j=1}^{n(t)}\log(1+z(\sigma_j-)^2)\\&=\lim_{t\to\infty}\frac{n(t)}{t}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\log(1+\zeta_j^2)\\&=\nu(\boldsymbol{R}_\delta)\int_{\bar{\boldsymbol{R}}}\log(1+\zeta^2)m(d\zeta)<\infty,\quad \text{a.s..}\end{aligned}$$

On the other hand, we have from the assumption,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}S_+(t)&=\lim_{t\to\infty}\frac{n(t)}{t}\frac{1}{n(t)}\sum_{j=1}^{n(t)}\log(1+z(\sigma_j)^2)\\&=\lim_{t\to\infty}\frac{n(t)}{t}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\log(1+(\zeta_j+q_j)^2)\\&=\nu(\boldsymbol{R}_\delta)\int_{\bar{\boldsymbol{R}}}\int_{\boldsymbol{R}_\delta}\log(1+(\zeta+q)^2)\nu(dq)m(d\zeta)\\&=+\infty,\quad \text{a.s.,}\end{aligned}$$

and this completes the proof of Theorem 2.

In fact, the assertions 1)-6) (hence Theorem 2 as well) seem to hold without the assumption $\lambda>\inf\Sigma$, but we did not investigate this because it is not necessary for our final purpose. We also remark that the condition $\int_{|x|>1}\log|x|\nu(dx)=+\infty$ is not used until the very last step of the proof.

4-2. Analysis of $\{z(t)\}$.

To begin with, let us fix our basic notation. On our probability space $(\Omega, \mathcal{F}, P)$, define the increasing family $\{\mathcal{F}_t\}_{t\geq 0}$ of sub σ-fields of $\mathcal{F}$ by $\mathcal{F}_t=\sigma[Q_\omega(s)\,;\ -\infty<s\leqq t]$. It is well known that if $0\leqq\tau(\omega)<\infty$ is an $\{\mathcal{F}_{t+}\}$-stopping time, where $\mathcal{F}_{t+}=\bigcap_{s>t}\mathcal{F}_s$, then the process $\{Q_{T_{\tau(\omega)}\omega}(t)\,;\ t\geqq 0\}$ has the same distribution as $\{Q_\omega(t)\,;\ t\geqq 0\}$, and is independent from $\mathcal{F}_\tau$. This is the strong Markov property of Lévy processes. (See e.g. [2].) Note that T_τ does not preserve the measure P itself in general. We set $W=\bar{\boldsymbol{R}}\times\Omega$, its element being denoted by $w=(z, \omega)$, and $\mathcal{B}=\mathcal{B}(\bar{\boldsymbol{R}})\times\mathcal{F}$, $\mathcal{B}_t=\mathcal{B}(\bar{\boldsymbol{R}})\times\mathcal{F}_{t+}$. For

${}^t(\alpha, \beta)\in \boldsymbol{R}^2\backslash\{0\}$ and for the solution $u(t)$ of $H_\omega u=\lambda u$, $u(0)=\alpha$, $u^+(0)=\beta$, set $z(t)=u^+(t)/u(t)\in\bar{\boldsymbol{R}}$. $z(t)$ is determined from $z=\alpha/\beta$ and $\{Q_\omega(s);\ s\leqq t\}$, hence $\{z(t)\}_{t\geqq 0}$ is a stochastic process on $(W, \mathcal{B})$ adapted to $\{\mathcal{B}_t\}$. We shall also write $z(t;w)$ $(=z(t;z,\omega))$, $z_\lambda(t)$, or $z_\lambda(t;w)$ whenever it is necessary to emphasize the dependence of $z(t)$ on each of its variables. For $w=(z, \omega)\in W$ and for $t\geqq 0$, we define $\theta_t w=\left(z(t;z,\omega), T_t\omega\right)\in W$. Finally, note that from (1-4), we have $z(t+s;z,\omega)=z(t;z(s;\omega), T_s\omega)$ for $t, s\geqq 0$, i.e. $z(t+s;w)=z(t;\theta_s w)$.

PROPOSITION 4. *Let $P_z(dw)=\delta_z\times P$. Then the triple $\left(z(t;w), \{\mathcal{B}_t\}_{t\geqq 0}, \{P_z\}_{z\in\bar{\boldsymbol{R}}}\right)$ is a strong Markov process in the following sense:*
a) *$z(t;\cdot)$ is $\mathcal{B}_t$-measurable for each $t\geqq 0$;*
b) *for any $B\in\mathcal{B}(\bar{\boldsymbol{R}})$, $z\to P_z(z(t;w)\in B)$ is $\mathcal{B}(\bar{\boldsymbol{R}})$-measurable;*
c) *$P_z(z(0;w)=z)=1$;*
d) *let $\tau(w)$ be a finite $\{\mathcal{B}_t\}$-stopping time, $f(w)$ be bounded and $\mathcal{B}_\tau$-measurable, and $g(z)$ be bounded and Borel on $\bar{\boldsymbol{R}}$, then*

$$E_z[f(w)g(z(t;\theta_\tau w))]=E_z[f(w)E_{z(\tau(w);w)}[g(z(t))]].$$

PROOF. Obvious from the construction and the strong Markov property of the Lévy process $\{Q_\omega(t)\}$.

REMARK. Let $\tau(w)=\tau(z,\omega)\geqq 0$ be such that $\tau(z,\cdot)$ is an $\{\mathcal{F}_{t+}\}$-stopping time for each $z\in\bar{\boldsymbol{R}}$. If $\tau(z,\omega)$ is, as a function of $z\in\bar{\boldsymbol{R}}=(-\infty, +\infty]$, non-decreasing [resp. non-increasing] and left-continuous [resp. right-continuous], then $\tau(w)$ is a $\{\mathcal{B}_t\}$-stopping time. Indeed, if we define

$$\tau_n(z,\omega)=\sum_{j=-\infty}^{+\infty} 1_{[j/n,(j+1)/n)}(z)\tau(j/n,\omega)+1_{\{\infty\}}(z)\tau(\infty,\omega),$$

[resp. $\tau_n(z,\omega)=\sum_{j=-\infty}^{+\infty} 1_{(j/n,(j+1)/n]}(z)\tau((j+1)/n,\omega)+1_{\{\infty\}}(z)\tau(\infty,\omega)$], then τ_n's are $\{\mathcal{B}_t\}$-stopping times and $\tau_n(w)\uparrow\tau(w)$ as $n\to\infty$.

Now let us define

$$\tau_0(w)=\tau_0^\lambda(w)=0,$$

$$\tau_{n+1}(w)=\tau_{n+1}^\lambda(w)=\inf\{t>\tau_n(w);\ z_\lambda(t;w)=\infty\},\quad n\geqq 0.$$

Then since τ_n's are zero's of the solution of $H_\omega u=\lambda u$, we have $\tau_n<\tau_{n+1}$ and $z(\tau_n(w);w)=z(\tau_n(w)-;w)=\infty$ whenever $\tau_n(w)<\infty$.

LEMMA 3. *For each $z\in\bar{\boldsymbol{R}}$, $\tau_n(z,\cdot)$'s are $\{\mathcal{F}_t\}$-stopping times. For each ω, $\tau_n(z,\omega)$ is non-decreasing and continuous as a function of $z\in(-\infty,\infty]$. In* *particular, $\tau_n(w)$'s are $\{\mathcal{B}_t\}$-stopping times. Finally, if $\lambda>\mu$ and if $\tau_1^{\mu}(w)<\infty$, then $\tau_1^{\lambda}(w)<\tau_1^{\mu}(w)$.*

PROOF. The first assertion is obvious. The monotonicity and continuity in z, as well as the monotonicity in λ can be shown in the same way as in Kotani [14] (Proposition 1.5 and Proposition 1.3).

LEMMA 4. *Assume $\lambda>\inf\Sigma$. Then for each $z\in\bar{\boldsymbol{R}}$, we have $\tau_1^{\lambda}(z, \omega)<\infty$ for P-a.a. ω. Moreover, we have $E[\tau_1^{\lambda}(z, \omega)^k]<\infty$, for all $k>0$.*

PROOF. Consider the eigenvalue problem (2-3)-(2-4) with $I=I_l=[-l, l]$, $\beta=0$, and $\cot\alpha=z$. Since H_ω is self-adjoint almost surely, we have for P-a.a. ω,

$$\sigma(d\xi\,;\omega, I_l)\equiv(\sigma_{11}+\sigma_{22})(d\xi\,;\omega, I_l)\longrightarrow\sigma(d\xi\,;\omega),$$

vaguely as $l\to\infty$, and

$$\mathrm{Supp}(\sigma(\cdot\,;\omega))=\Sigma.$$

In particular, for $\lambda>\inf\Sigma$,

$$\varliminf_{l\to\infty}\sigma((-\infty, \lambda)\,;\omega, I_l)\geqq\sigma((-\infty, \lambda)\,;\omega)>0, \quad \text{a.s..}$$

Hence

$$\varliminf_{l\to\infty}P(\lambda_1(\omega\,;I_l)<\lambda)\geqq P\Big(\varliminf_{l\to\infty}\{\omega\,|\,\lambda_1(\omega\,;I_l)<\lambda\}\Big)=1,$$

or what is the same, for any $\varepsilon>0$, we can choose l so large that

$$P(\lambda_1(\omega\,;I_l)<\lambda)\geqq1-\varepsilon.$$

Next, consider the eigenvalue problem on the interval $J_l=[0, 2l]$ with the same boundary conditions as above. Then by the stationarity of the random potential $Q'_\omega(t)$, we still have

$$P(\lambda_1(\omega\,;J_l)<\lambda)\geqq1-\varepsilon$$

with the same choice of λ, ε, and l. On the other hand, if $\lambda_1(\omega\,;J_l)<\lambda$, then from Lemma 3,

$$\tau_1^{\lambda}(z\,;\omega)<\tau_1^{\lambda_1(\omega;J_l)}(z\,;\omega)=2l.$$

Therefore for any $\varepsilon>0$, one has

$$P(\tau_1^{\lambda}(z\,;\omega)<2l)\geqq1-\varepsilon,$$

for l large enough. Letting $\varepsilon\downarrow0$, we arrive at the first assertion of the lemma.

Now from the monotonicity of $\tau_1(z\,;\omega)$ in z, there exists a $T>0$ such that

$$0\leqq\rho\equiv\sup_{z\in\bar{\boldsymbol{R}}}P(\tau_1(z\,;\omega)>T)=P(\tau_1(\infty\,;\omega)>T)<1.$$

Using the Markov property of $\{z(t)\}$, we can show inductively that

$$P(\tau_1(\infty\,;\omega)>nT)\leqq\rho^n, \quad \text{for all } n\geqq 1,$$

whence follows the second assertion of the lemma.

PROPOSITION 5. *Assume* $\lambda>\inf\Sigma$. *For any bounded Borel function* f *on* $\bar{\boldsymbol{R}}$ *and any* $z\in\bar{\boldsymbol{R}}$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(z(s\,;w))ds=\int_{\bar{\boldsymbol{R}}} f(z)\pi(dz), \quad P_z\text{-almost surely.}$$

Here, $\pi(dz)$ *is the unique invariant distribution of* $\{z(t)\}$, *which is given by*

$$\int_{\bar{\boldsymbol{R}}} f(z)\pi(dz)=\frac{1}{E_\infty[\tau_1]}E_\infty\left[\int_0^{\tau_1} f(z(t))dt\right]. \tag{4-10}$$

PROOF. By Lemma 4, we see that the right-hand side of (4-10) defines a probability measure on $\bar{\boldsymbol{R}}$. For a bounded Borel function f, set

$$\Phi_f(w)=\int_0^{\tau_1(w)} f(z(t\,;w))dt.$$

Then since $\tau_{n+1}(w)=\tau_n(w)+\tau_1(\theta_{\tau_n}w)$, $n\geqq 0$, one has

$$\int_{\tau_n(w)}^{\tau_{n+1}(w)} f(z(t\,;w))dt=\Phi_f(\theta_{\tau_n}w).$$

Since $z(\tau_n)=\infty$, from the strong Markov property of $\{z(t)\}$, it follows that $\Phi_f(\theta_{\tau_n}w)$, $n\geqq 1$, are i.i.d., and that

$$E_z[\Phi_f(\theta_{\tau_n}w)]=E_\infty[\Phi_f(w)].$$

In particular, $\tau_{n+1}(w)-\tau_n(w)=\Phi_1(\theta_{\tau_n}w)$, $n\geqq 1$, are i.i.d.. Hence from the law of large numbers, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t f(z(s\,;w))ds&=\lim_{n\to\infty}\frac{n}{\tau_n(w)}\frac{1}{n}\sum_{j=1}^n\Phi_f(\tau_j w)\\&=\frac{1}{E_\infty[\tau_1]}E_\infty\left[\int_0^{\tau_1(w)} f(z(t))dt\right],\end{aligned}$$

as desired.

Having shown such an individual ergodic theorem, it is now easy to see by general considerations that the probability measure $\pi(dz)$ is the unique invariant distribution of the Markov process $\{z(t)\}$.

4-3. Analysis of $\{(\zeta_n, q_n)\}$.

For $w=(z,\omega)\in W$, set $\zeta_n(w)=z(\sigma_n(\omega)-\,;w)$, $q_n(w)=\Delta Q_\omega(\sigma_n(\omega))$, $n=1,2,\cdots$, where $\sigma_n(\omega)$'s were defined in §4-1. (We do not define $\zeta_0(w)$ and $q_0(w)$.) Since σ_n's are $\{\mathcal{F}_t\}$-stopping times which do not depend on z, they are also

$\{\mathcal{B}_t\}$-stopping times. Set

$$\Pi(\zeta, q; A\times B)=P(\zeta_1(\zeta+q, \omega)\in A,\ q_1(\omega)\in B)$$
$$=P_{\zeta+q}(z(\sigma_1-)\in A,\ \Delta Q(\sigma_1)\in B),$$

for $A\in\mathcal{B}(\bar{\boldsymbol{R}})$ and $B\in\mathcal{B}(\bar{\boldsymbol{R}}_\delta)$. Finally set $\mathcal{B}_n=\mathcal{B}_{\sigma_n}$.

PROPOSITION 6. *$\{(\zeta_n, q_n)\}_{n\geq 0}$ is a Markov chain in the sense that for any bounded Borel function $F(\zeta, q)$ on $\bar{\boldsymbol{R}}\times\boldsymbol{R}_\delta$ and for each $z\in\bar{\boldsymbol{R}}$,*

$$E_z[F(\zeta_{n+1}, q_{n+1})\mid\mathcal{B}_n](w)=\int_{\bar{\boldsymbol{R}}\times\boldsymbol{R}_\delta}\Pi(\zeta_n(w), q_n(w); d\zeta dq)F(\zeta, q),\quad P_z\text{-a.s.}. \tag{4-11}$$

The transition probability Π is given in the following manner:

$$\Pi(\zeta, q; A\times B)=\Big(\int_0^\infty dte^{-t\nu(R_\delta)}P_t^\delta(\zeta+q; A)\Big)\times\nu(B), \tag{4-12}$$

where (i) $A\in\mathcal{B}(\bar{\boldsymbol{R}})$, $B\in\mathcal{B}(\boldsymbol{R}_\delta)$; (ii) $\{Q_\omega^\delta(t)\}$ is the Lévy process obtained from $\{Q_\omega(t)\}$ by removing all of its jumps such that $|\Delta Q_\omega(t)|>\delta$; (iii) $\{z^\delta(t; z, \omega)\}$ is the Markov process constructed from $\{Q_\omega^\delta(t)\}$ in the same way as $\{z(t; z, \omega)\}$; and (iv) $P_t^\delta(z; A)=P(z^\delta(t; z, \omega)\in A)$.

PROOF. (4-11) follows from Proposition 5. Let us verify (4-12). From the definition of $z^\delta(t)$, it is clear that

$$z^\delta(\sigma_1(\omega); z, \omega)=z(\sigma_1(\omega)-; z, \omega).$$

On the other hand, $(\sigma_1(\omega), \Delta Q_\omega(\sigma_1(\omega)))$ is independent from $\{Q_\omega^\delta(t)\}$, and hence from $\{z^\delta(t; z, \omega)\}$. Moreover, as is well known,

$$P(\sigma_1(\omega)\in dt,\ \Delta Q_\omega(\sigma_1(\omega))\in dq)=e^{-t\nu(R_\delta)}dt\nu(dq).$$

Therefore by the definition of Π,

$$\Pi(\zeta, q; A\times B)=P(z^\delta(\sigma_1(\omega); \zeta+q, \omega)\in A,\ \Delta Q_\omega(\sigma_1(\omega))\in B)$$
$$=\int_B\int_0^\infty P(z^\delta(t; \zeta+q, \omega)\in A)P(\sigma_1(\omega)\in dt,\ \Delta Q_\omega(\sigma_1(\omega))\in dq)$$
$$=\Big(\int_0^\infty dte^{-t\nu(R_\delta)}P_t^\delta(\zeta+q; A)\Big)\times\nu(B).$$

PROPOSITION 7. *Suppose $\lambda>\inf\Sigma$, then the transition probability Π has a unique invariant distribution $\mu(d\zeta dq)$. For any Borel function $0\leqq F(\zeta, q)<\infty$ and for any $z\in\bar{\boldsymbol{R}}$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n F(\zeta_j(\omega), q_j(\omega))=\iint_{\bar{\boldsymbol{R}}\times\boldsymbol{R}_\delta}F(\zeta, q)\mu(d\zeta dq),\quad P_z\text{-a.s.}.$$

The measure μ *is a product measure*:

$$\mu(d\zeta dq)=m(d\zeta)\nu(dq), \tag{4-13}$$

where m *satisfies*

$$m(d\zeta)=\int_{\bar{R}}m(d\zeta')\int_{R_\delta}\nu(dq')\int_0^\infty dte^{-t\nu(R_\delta)}P_t^\delta(\zeta'+q'\,;\,d\zeta). \tag{4-14}$$

PROOF. Since $\sigma_n\to\infty$ and $\tau_n\to\infty$ $(n\to\infty)$, P_z-almost surely, we can define an integer valued function $J(n, w)$, $n\geqq 1$, so that $\sigma_n\in[\tau_{J(n)}, \tau_{J(n)+1})$. Clearly $J(n)\to\infty$ as $n\to\infty$.

If we set

$$\Psi(w)=\Psi(z, \omega)=\sum_{i\,:\,\sigma_i(\omega)<\tau_1(w)}F(z(\sigma_i(\omega)-\,;\,w),\ \Delta Q_\omega(\sigma_i(\omega))),$$

then

$$\frac{1}{n}\sum_{j=1}^{n}F(\zeta_j, q_j)=\frac{1}{n}\sum_{j=1}^{J(n)}\Psi(\theta_{\tau_{j-1}}w)+\frac{1}{n}\sum_{i\le n,\,\sigma_i\ge\tau_{J(n)}}F(z(\sigma_i-),\ \Delta Q(\sigma_i)). \tag{4-15}$$

As in Proposition 5, we see that $\Psi(\theta_{\tau_n}w)$, $n\geqq 1$, are i.i.d. with $E_z[\Psi(\theta_{\tau_n}w)]=E_\infty[\Psi(w)]$, and the last term on the right-hand side of (4-15) is bounded by $\Psi(\theta_{\tau_{J(n)}}w)/n$. Hence by the law of large numbers,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}F(\zeta_j, q_j)&=\lim_{n\to\infty}\frac{\sigma_n}{n}\ \frac{\tau_{J(n)}}{\sigma_n}\ \frac{J(n)}{\tau_{J(n)}}\ \frac{1}{J(n)}\sum_{j=1}^{J(n)}\Psi(\theta_{\tau_{j-1}}w)\\ &=\frac{E[\sigma_1]}{E_\infty[\tau_1]}E_\infty[\Psi]\\ &=\frac{E[\sigma_1]}{E_\infty[\tau_1]}E_\infty\Big[\sum_{\sigma_i<\tau_1}F(z(\sigma_i-),\ \Delta Q(\sigma_i))\Big],\quad P_z\text{-a.s.}.\end{aligned}$$

The right-hand side defines a measure $\mu(d\zeta dq)$ on $\bar{\boldsymbol{R}}\times\boldsymbol{R}_\delta$, and it is a probability measure as one sees on setting $F=1$ on the left-hand side. It is clear that this μ is the unique invariant measure of our Markov chain. In particular, we have

$$\begin{aligned}\mu(A\times B)&=\iint\mu(d\zeta dq)\Pi(\zeta, q\,;\,A\times B)\\ &=\Big(\iint\mu(d\zeta dq)\int_0^\infty dte^{-t\nu(R_\delta)}P_t^\delta(\zeta+q\,;\,A)\Big)\nu(B)\end{aligned}$$

by (4-12), whence follows (4-13) and (4-14).

Finally we remark that from (4-14) and

$$\int_0^\infty 1_{\{\infty\}}(z^\delta(t\,;\,w))dt=0,\quad P_z\text{-a.s.}.$$

it follows $m(\{\infty\})=0$.

4-4. The moment condition for $m(d\zeta)$.

For $\delta>0$, set $H_\omega^\delta=-d^2/dt^2+Q_\omega^{\delta'}(t)$, and let Σ^δ be the spectrum of (P-almost all) H_ω^δ. In this subsection, we choose and fix a $\delta>0$ so that $\Sigma^\delta=\Sigma$. This is possible: When $v=0$ and $\nu((-\infty, 0))>0$, we shall take a $\delta>0$ (possibly large) so that $\nu([-\delta, 0))>0$, and in all the other cases, $\delta>0$ may be chosen arbitrarily. (See Proposition 2.) In any case, we have $\alpha\equiv\nu(\boldsymbol{R}_\delta)>0$, because we are assuming $\int_{|x|>1}\log|x|\nu(dx)=+\infty$.

PROPOSITION 8. *If we chose $\delta>0$ as above, and if $\lambda>\inf\Sigma$, then*

$$\int_{\boldsymbol{R}}|\zeta|^\beta m(d\zeta)<\infty, \quad \textit{for any } 0<\beta<1.$$

PROOF. Define $\tau_0^\delta(w)=0$, and $\tau_{n+1}^\delta(w)=\inf\{t>\tau_n^\delta(w)\,;\,z^\delta(t\,;\,w)=\infty\}$ for $n\geqq 0$. From the assumption and the choice of $\delta>0$, we have $\lambda>\inf\Sigma^\delta$, and hence $\tau_n^\delta(w)<\infty$, a.s. by Lemma 4. It suffices for our purpose to prove

$$J\equiv\sup_{z\in\bar{\boldsymbol{R}}}E_z\left[\int_0^{\tau_1^\delta(w)}|z^\delta(t\,;\,w)|^\beta dt\right]<\infty,$$

for each $\beta\in(0, 1)$. Indeed if this is the case, then using the strong Markov property of $\{z^\delta(t)\}$,

$$\begin{aligned}
&\int_0^\infty dt e^{-\alpha t}\int P_t^\delta(\zeta+q\,;\,d\zeta')|\zeta'|^\beta \\
&=E_{\zeta+q}\left[\int_0^\infty e^{-\alpha t}|z^\delta(t)|^\beta dt\right] \\
&=\sum_{n=0}^\infty E_{\zeta+q}\left[\int_{\tau_n^\delta}^{\tau_{n+1}^\delta} e^{-\alpha t}|z^\delta(t)|^\beta dt\right] \\
&=\sum_{n=0}^\infty E_{\zeta+q}\left[\int_0^{\tau_1^\delta(\theta_{\tau_n}w)} e^{-\alpha\tau_n^\delta(w)}e^{-\alpha t}|z^\delta(t\,;\,\theta_{\tau_n^\delta}w)|^\beta dt\right] \\
&=E_{\zeta+q}\left[\int_0^{\tau_1^\delta}e^{-\alpha t}|z^\delta(t)|^\beta dt\right]+\left(\sum_{n=1}^\infty E_{\zeta+q}\left[e^{-\alpha\tau_n^\delta(w)}\right]\right)E_\infty\left[\int_0^{\tau^\delta}e^{-\alpha t}|z^\delta(t)|^\beta dt\right] \\
&\leqq J\left(1+\sum_{n=1}^\infty E_{\zeta+q}\left[e^{-\alpha\tau_n^\delta}\right]\right).
\end{aligned}$$

But $\tau_n^\delta(w)=\sum_{j=0}^{n-1}\tau_1^\delta(\theta_{\tau_j^\delta}w)$, and $\rho\equiv E_\infty[e^{-\alpha\tau_1^\delta}]<1$, so that we have

$$E_{\zeta+q}[e^{-\alpha\tau_n^\delta}]=E_{\zeta+q}[e^{-\alpha\tau_1^\delta}]\rho^{n-1}\leqq\rho^{n-1}.$$

Therefore from (4-14),

$$\int_R |\zeta|^\beta m(d\zeta)=\int_R m(d\zeta)\int_{R_\delta}\nu(dq)\int_0^\infty dte^{-\alpha t}\int P_t^\delta(\zeta+q\,;\,d\zeta')|\zeta'|^\beta$$
$$\leqq m(\boldsymbol{R})\nu(\boldsymbol{R}_\delta)J\Big(1+\sum_{n=1}^\infty \rho^{n-1}\Big)<\infty\,.$$

Let us fix $a<0<b$. a and b will be suitably chosen in the sequel, but for the moment we assume $\delta<|a|\wedge b$. Then to prove $J<\infty$, it is sufficient to prove the following inequalities:

$$J_1\equiv \sup_{a<z\leqq b} E_z\Big[\int_0^{\sigma_a}|z^\delta(t)|^\beta dt\Big]<\infty\,;$$

$$J_2\equiv \sup_{z\leqq a} E_z\Big[\int_0^{\tau_1^\delta}|z^\delta(t)|^\beta dt\Big]<\infty\,;$$

and

$$J_3\equiv \sup_{z>b} E_z\Big[\int_0^{\sigma_b}|z^\delta(t)|^\beta dt\Big]<\infty\,,$$

where we have set

$$\sigma_\zeta(w)=\inf\{t>0\,;\, z^\delta(t\,;\,w)<\zeta\}\,.$$

Indeed if we note that $\sigma_\zeta(z,\omega)<\tau_1^\delta(z,\omega)$ for any $\zeta<z$, and that $\zeta-\delta\leqq z^\delta(\sigma_\zeta)\leqq\zeta$, then for $a<z\leqq b$,

$$E_z\Big[\int_0^{\tau_1^\delta}|z^\delta(t)|^\beta dt\Big]=E_z\Big[\int_0^{\sigma_a}|z^\delta(t)|^\beta dt\Big]+E_z\Big[E_{z^\delta(\sigma_a)}\Big[\int_0^{\tau_1^\delta}|z^\delta(t)|^\beta dt\Big]\Big]\leqq J_1+J_2\,;$$

for $z>b$,

$$E_z\Big[\int_0^{\tau_1^\delta}|z^\delta(t)|^\beta dt\Big]=E_z\Big[\int_0^{\sigma_b}|z^\delta(t)|^\beta dt\Big]+E_z\Big[E_{z^\delta(\sigma_b)}\Big[\int_0^{\sigma_a}|z^\delta(t)|^\beta dt\Big]\Big]$$
$$+E_z\Big[E_{z^\delta(\sigma_a)}\Big[\int_0^{\tau_1^\delta}|z^\delta(t)|^\beta dt\Big]\Big]\leqq J_3+J_1+J_2\,.$$

and finally

$$E_\infty\Big[\int_0^{\tau_1^\delta}|z^\delta(t)|^\beta dt\Big]=\lim_{\zeta\uparrow\infty}E_\infty\Big[\int_{\sigma_\zeta}^{\tau_1^\delta}|z^\delta(t)|^\beta dt\Big]$$
$$=\lim_{\zeta\uparrow\infty}E_\infty\Big[E_{z^\delta(\sigma_\zeta)}\Big[\int_0^{\tau_1^\delta}|z^\delta(t)|^\beta dt\Big]\Big]\leqq J_3+J_1+J_2\,.$$

Therefore $J\leqq\max\{J_2,\,J_1+J_2,\,J_1+J_2+J_3\}<\infty$.

Now let us proceed to the proof of $J_k<\infty$, $k=1,2,3$.

PROOF OF $J_1<\infty$. For notational convenience, we assume $\delta<1$ in the following. Define

$$s_\zeta(w)=\inf\{t>0\,;\,z^\delta(t\,;\,w)>\zeta\}\,.$$

Then we have

$$(4\text{-}16)\qquad E_z\Big[\int_0^{\sigma_a}|z^\delta(t)|^\beta dt\Big]=E_z\Big[\int_0^{\sigma_a}|z^\delta(t)|^\beta dt\,;\ \sigma_a<s_{z+1}\Big]$$
$$+\sum_{l=1}^{\infty}E_z\Big[\int_0^{\sigma_a}|z^\delta(t)|^\beta dt\,;\ s_{z+l}<\sigma_a<s_{z+l+1}\Big]$$
$$\leqq\Big(\sup_{a-\delta\leqq x\leqq b+1}|x|^\beta\Big)E_z[\sigma_a\,;\ \sigma_a<s_{z+1}]$$
$$+\sum_{l=1}^{\infty}\Big(\sup_{a-\delta\leqq x\leqq b+l+1}|x|^\beta\Big)E_z[\sigma_a\,;\ s_{z+l}<\sigma_a<s_{z+l+1}].$$

But we have

$$E_z[\sigma_a\,;\ s_{z+l}<\sigma_a<s_{z+l+1}]\leqq(E_z[\sigma_a^2])^{1/2}(P_z(s_{z+l}<\sigma_a<s_{z+l+1}))^{1/2}$$

by Schwarz's inequality, and

$$E_z[\sigma_a^2]\leqq E_z[(\tau_1^\delta)^2]\leqq E_\infty[(\tau_1^\delta)^2]<\infty\,,$$
$$E_z[\sigma_a\,;\ \sigma_a<s_{z+l}]\leqq E_\infty[\tau_1^\delta]<\infty$$

by Lemma 4. Hence if we have shown that for some $0<r<1$,

$$(4\text{-}17)\qquad P_z(s_{z+l}<\sigma_a<s_{z+l+1})\leqq P_z(s_{z+l}\leqq\tau_1^\delta)\leqq r^l,\quad \text{all } l\geqq 0,$$

then the series on the right-hand side of (4-16) converges uniformly in $z\in(a, b]$, and $J_1<\infty$ follows from this. On the other hand, (4-17) is a simple consequence of the following

LEMMA 5. $\sup_{z\in\boldsymbol{R}}P_z(s_{z+k}<\tau_1^\delta)<1$, *for any* $k>0$.

PROOF. Let P^δ be the distribution of $\{Q_\omega^\delta(t)\}$ in Ω. We will find an $\omega_0\in\mathrm{Supp}(P^\delta)$ such that for some neighborhood U of ω_0,

$$(4\text{-}18)\qquad s_{z+k}(z, \omega)>\tau_1^\delta(z, \omega)$$

holds for all $z\in\boldsymbol{R}$ and $\omega\in U$. Then

$$\inf_{z\in\boldsymbol{R}}P_z(s_{z+k}(w)>\tau_1^\delta(w))\geqq P^\delta(U)>0\,,$$

which is equivalent to the assertion of the lemma.

We divide our argument into four cases.

CASE 1. Lévy's canonical form reduces to

$$Q_\omega^\delta(t)=\int_0^\delta xN_\omega((0, t]\times dx)\,,$$

and $\lambda>0$. In this case, we choose $\omega_0(t)\equiv 0$. Since $z(t)=z^\delta(t\,;\,z,\,\omega_0)$ satisfies

$$z(t)=z-\int_0^t(\lambda+z(s))^2ds\,,$$

$z(t)$ is monotone decreasing in $t<T\equiv\tau_1^\delta(\infty,\,\omega_0)<\infty$. Let $0<\varepsilon<k$, and $U=\{\omega\in\Omega\,;\,|\omega(t)-\omega_0(t)|\leqq\varepsilon$, for $0\leqq t\leqq T\}$. A simple comparison argument shows that for $\omega\in U$ and $z\in\boldsymbol{R}$, one has $z^\delta(t\,;\,z,\,\omega)\leqq z(t\,;\,z+\varepsilon,\,\omega_0)$ for $t\leqq\tau_1^\delta(z\,\text{`}\omega)$. Hence (4-18) holds for all $z\in\boldsymbol{R}$ and $\omega\in U$.

CASE 2. $v\neq 0$ and $\lambda\in\boldsymbol{R}$ is arbitrary. In this case, it suffices to take $\omega_0(t)=\gamma t$, with $\gamma<\lambda$. The rest is the same as in case 1.

CASE 3. $v=\nu((-\infty,0))=0$ and $\int_0^\delta x\nu(dx)=\infty$. In this case, let $\eta\in(0,\delta)\cap\mathrm{Supp}(\nu)$ be so small that

$$c_\eta\equiv b-\int_\eta^\delta a(x)\nu(dx)<\lambda\,.$$

Then, $\omega_0(t)\equiv c_\eta t$ suffices for our purpose.

CASE 4. $v=0$ and $\nu((-\delta,\,0))>0$. Fix an $\alpha\in[-\delta,\,0)\cap\mathrm{Supp}(\nu)$. Then for some $\mu\in\boldsymbol{R}$,

$$\omega^\beta(t)\equiv\mu t+\alpha[t/\beta]$$

belongs to $\mathrm{Supp}(P^\delta)$ for all $\beta>0$. It is not difficult to see that $z^\delta(t\,;\,\infty,\,\omega^\beta)$ hits ∞ in a finite time for sufficiently small $\beta>0$. Fix such a $\beta>0$ and a $T>\tau_1^\delta(\infty,\omega^\beta)$. Finally let

$$U=\left\{\omega(t)+\sum_{i=1}^{[T/\beta]}a_i 1_{[b_i,\infty)}(t)\,;\,|a_i-\alpha|<\varepsilon,\ |b_i-i\beta|<\varepsilon\right\}$$

with sufficiently small $\varepsilon>0$. This U and $\omega_0=\omega^\beta$ satisfy the desired condition.

Now let us verify (4-17). Set $k=1-\delta$ in Lemma 5, and let

$$r=\sup_{z\in R}P_z(s_{z+1-\delta}<\tau_1^\delta)<1\,.$$

If we assume

$$\sup_{z\in R}P_z(s_{z+l}<\tau_1^\delta)<r^l\,,$$

then by the strong Markov property of $\{z^\delta(t)\}$,

$$\begin{aligned}P_z(s_{z+l+1}<\tau_1^\delta)&=P_z(s_{z+l}<\tau_1^\delta,\ s_{z+l+1}(\theta_{s_{z+l}}w)<\tau_1^\delta(\theta_{s_{z+l}}w))\\&=E_z[s_{z+l}<\tau_1^\delta\,;\,P_{z^\delta(s_{z+l})}[s_{z+l+1}<\tau_1^\delta]]\\&\leqq E_z[s_{z+l}<\tau_1^\delta\,;\,P_{z^\delta(s_{z+l})}[s_{z^\delta(s_{z+l})+1-\delta}<\tau_1^\delta]]\\&\leqq r^{l+1}\,,\end{aligned}$$

where we have used the fact $z+l\leqq z^\delta(s_{z+l})\leqq z+l+\delta$.

PROOF OF $J_2<\infty$. To begin with, note that for each $z\in\boldsymbol{R}$, we have $\sigma_{z-n}(z,\omega)\uparrow\tau_1^\delta(z,\omega)$, $n\to\infty$, P-almost surely. Then by the strong Markov property,

$$(4\text{-}19)\quad E_z\Big[\int_0^{\tau^\delta}|z^\delta(t)|^\beta dt\Big]=E_z\Big[\int_0^{\sigma_{z-1}}|z^\delta(t)|^\beta dt\Big]+\sum_{n=1}^\infty E_z\Big[\int_{\sigma_{z-n}}^{\sigma_{z-n-1}}|z^\delta(t)|^\beta dt\Big]$$
$$=E_z\Big[\int_0^{\sigma_{z-1}}|z^\delta(t)|^\beta dt\Big]+\sum_{n=1}^\infty E_z\Big[E_{z^\delta(\sigma_{z-n})}\Big[\int_0^{\sigma_{z-n-1}}|z^\delta(t)|^\beta dt\Big]\Big].$$

Let $a<-2(\sqrt{(-\lambda)\vee 0}+1)$. We want to show that the series in (4-19) converges uniformly in $z\leqq a$, and to this end, we will estimate

$$E_{z_n}\Big[\int_0^{\sigma_{z-n-1}}|z^\delta(t)|^\beta dt\Big],\quad n\geqq 0,$$

for an arbitrary $z_n\in[z-n-\delta,\ z-n]$.

Now for each n,

$$E_{z_n}\Big[\int_0^{\sigma_{z-n-1}}|z^\delta(t)|^\beta dt\Big]\leqq|z-n-2|^\beta E_{z_n}[\sigma_{z-n-1};\ \sigma_{z-n-1}<s_{(z-n)/2}]$$
$$+|z-n-2|^\beta E_{z_n}[\sigma_{z-n-1};\ s_{(z-n)/2}<\sigma_{z-n-1}<s_{-z+n}]$$
$$+\sum_{j=0}^\infty|-z+n+j+1|^\beta E_{z_n}[\sigma_{z-n-1};\ s_{-z+n+j}<\sigma_{z-n-1}<s_{-z+n+j+1}]$$
$$\equiv I_1(n,\ z)+I_2(n,\ z)+I_3(n,\ z).$$

First let us estimate $I_1(z,\ n)$. To this end, note that on the set $\{\sigma_{z-n-1}<s_{(z-n)/2}\}$, one has for $0\leqq t\leqq\sigma_{z-n-1}$,

$$z^\delta(t)=z_n+Q^\delta(t)-\int_0^t(\lambda+z^\delta(s)^2)ds$$
$$\leqq z_n+Q^\delta(t)-\Big\{\lambda+\frac{1}{4}(z-n)^2\Big\}t.$$

Hence if we define

$$S(A)=\inf\{t>0;\ Q^\delta(t)-At<-1\},$$

then $\sigma_{z-n-1}<S(\lambda+\frac{1}{4}(z-n)^2)$ on the set $\{\sigma_{z-n-1}<s_{(z-n)/2}\}$. Therefore,

$$(4\text{-}20)\qquad I_1(z,\ n)\leqq|z-n-2|^\beta E\Big[S\Big(\lambda+\frac{1}{4}(z-n)^2\Big)\Big].$$

Next consider $I_2(z,\ n)$. By Schwarz's inequality and Lemma 5, we can fino an $r\in(0,\ 1)$, so that

$$E_{z_n}[\sigma_{z-n-1};\ s_{(z-n)/2}<\sigma_{z-n-1}<s_{-z+n}]$$
$$\leqq E_{z_n}[\sigma_{z-n-1};\ s_{z_n+[|z-n|/2]}<\tau_1^\delta]$$
$$\leqq\Big(E_\infty[(\tau_1^\delta)^2]\Big)^{1/2}\Big(P_{z_n}(s_{z_n+[|z-n|/2]}<\tau_1^\delta)\Big)^{1/2}$$

$$\leqq\left(E_\infty[(\tau_1^\delta)^2]\right)^{1/2} r^{[|z-n|/2]}.$$

Consequently

$$I_2(z,\ n)\leqq\text{const.}\,|z-n-2|^\beta r^{[|z-n|/2]}. \tag{4-21}$$

In the same way,

$$I_3(z,\ n)\leqq\sum_{j=0}^{\infty}|-z+n+j+1|^\beta\left(E_\infty[(\tau_1^\delta)^2]\right)^{1/2}\left(P_{z_n}(s_{z_n+2|z-n|+j}<\tau_1^\delta)\right)^{1/2} \tag{4-22}$$

$$\leqq\left(E_\infty[(\tau_1^\delta)^2]\right)^{1/2}\sum_{j=1}^{\infty}|-z+n+j+1|^\beta r^{|z-n|+j/2}.$$

Therefore, if we have shown

$$E\left[S\left(\lambda+\frac{1}{4}(z-n)^2\right)\right]\leqq\text{const.}\,|z-n|^{-2}, \tag{4-23}$$

then the supremum over $z\leqq a$ of the right-hand side of (4-19) is bounded by

$$\sup_{z\leqq a}\sum_{n=0}^{\infty}\left(I_1(z,\ n)+I_2(z,\ n)+I_3(z,\ n)\right)$$

$$\leqq\text{const.}\sup_{z\leqq a}\sum_{n=0}^{\infty}\left(|z-n|^{\beta-2}+|z-n|^\beta r^{|z-n|/2}+|z-n|^\beta r^{|z-n|}\right),$$

which is finite provided $\beta<1$.

Now (4-23) is an immediate consequence of the following lemma.

LEMMA 6. *For any* $q\geqq 1$, $E[S(A)^q]=O(A^{-q})$, *as* $A\to\infty$.

PROOF. Since $|\Delta Q_\omega^\delta(t)|\leqq\delta$, $|Q_\omega^\delta(t)|$ has moments of all order ([2]). Without loss of generality, we shall assume $E[Q_\omega^\delta(t)]=0$, so that $\{Q_\omega^\delta(t)\}$ is a martingale. Noting

$$AT=-(Q_\omega^\delta(T)-AT)+Q_\omega^\delta(T)$$

$$\leqq-\inf_{0\leq t\leq T}(Q_\omega^\delta(t)-At)+\sup_{0\leq t\leq T}|Q_\omega^\delta(t)|\,,$$

we see that

$$P(S(A)>T)=P\left(\inf_{0\leq t\leq T}(Q_\omega^\delta(t)-At)\geqq-1\right)$$

$$=P\left(\sup_{0\leq t\leq T}|Q_\omega^\delta(t)|\geqq AT-1\right)$$

$$\begin{cases}=1 \text{ if } T\leqq A^{-1}\\ \leqq(AT-1)^{-2p}E[|Q_\omega^\delta(T)|^{2p}, \text{ for any } p\geqq 1, \text{ if } T>A^{-1},\end{cases}$$

by martingale inequality. Hence

$$(4\text{-}24)\qquad E[S(A)^q] \leqq (2/A)^q P(S(A) \leqq 2/A) + \sum_{n=2}^{\infty} \left(\frac{n+1}{A}\right)^q P\left(\frac{n}{A} < S(A) \leqq \frac{n+1}{A}\right)$$
$$\leqq A^{-q}\left\{2^q + \sum_{n=1}^{\infty} (n+1)^q \frac{1}{(n-1)^{2p}} E\left[\left|Q_\omega^\delta\left(\frac{n}{A}\right)\right|^{2p}\right]\right\}.$$

Now we claim that for every integer $p \geqq 1$, there is a constant $C_p > 0$ such that

$$(4\text{-}25)\qquad E[|Q_\omega^\delta(t)|^{2p}] \leqq C_p t^p.$$

Indeed, since we are assuming $E[Q_\omega^\delta(t)]=0$, $\{Q_\omega^\delta(t)\}$ has the expression

$$Q_\omega^\delta(t) = vB_\omega(t) + \int_0^{t+}\int_{[-\delta,\delta]} x\tilde{N}_\omega(dsdx).$$

Hence by Ito's formula,

$$Q_\omega^\delta(t)^{2p} = 2pv\int_0^t Q_\omega^\delta(s)^{2p-1} dB(s) + p(2p-1)v^2 \int_0^t Q_\omega^\delta(s)^{2(p-1)} ds$$
$$+ \int_0^{t+}\int_{[-\delta,\delta]} \{(Q_\omega^\delta(s-)+x)^{2p} - Q_\omega^\delta(s-)^{2p}\}\tilde{N}_\omega(dsdx)$$
$$+ \int_0^t \int_{[-\delta,\delta]} \{(Q_\omega^\delta(s)+x)^{2p} - Q_\omega^\delta(s)^{2p} - 2pQ_\omega^\delta(s)^{2p-1}x\} ds\nu(dx),$$

so that

$$(4\text{-}26)\qquad E[Q^\delta(t)^{2p}] = p(2p-1)v^2 \int_0^t E[Q^\delta(s)^{2(p-1)}] ds$$
$$+ \int_0^t ds \int_{[-\delta,\delta]} E[(Q^\delta(s)+x)^{2p} - Q^\delta(s)^{2p} - 2pQ^\delta(s)x]\nu(dx).$$

Setting $p=1$, we first obtain $E[Q^\delta(t)^2] = C_1 t$, with

$$C_1 = v^2 + \int_{[-\delta,\delta]} x^2 \nu(dx).$$

Suppose we have shown $E[|Q^\delta(t)|^{2j}] \leqq C_j t^j$ for $j \leqq p$. Then noting that

$$E[|Q^\delta(t)|^{2j-1}] \leqq \left(E[Q^\delta(t)^{2(j-1)}]\right)^{1/2} \left(E[Q^\delta(t)^{2j}]\right)^{1/2}$$
$$\leqq (C_{j-1}C_j)^{1/2} t^{(2j-1)/2}, \quad j \leqq p,$$

we obtain from (4-26),

$$E[Q^\delta(t)^{2(p+1)}] \leqq (p+1)(2p+1)v^2 \int_0^t C_p s^p ds$$
$$+ \int_0^t ds \int_{[-\delta,\delta]} E\left[\sum_{j=2}^{2(p+1)} \binom{2(p+1)}{j} |Q^\delta(t)|^{2(p+1)-j} |x|^j\right] \nu(dx)$$
$$\leqq C_{p+1} t^{p+1},$$

with some constant C_{p+1}.

Letting $p=q+2$ in (4-25), and substituting this into (4-24), we finally get

$$E[S(A)^q]\leqq A^{-q}\Big\{2^q+C_{q+2}A^{-4}\sum_{n=2}^{\infty}\frac{(n+1)^q n^{q+2}}{(n-1)^{2(q+2)}}\Big\}=O(A^{-q}),$$

completing the proof of the lemma.

PROOF OF $J_3<\infty$. We assume $b>\sqrt{|\lambda\wedge 0|}$. Let $z>b$ and $N=N(z)=[z-b]-1$. Then as before,

$$(4\text{-}27)\qquad E_z\Big[\int_0^{\sigma_b}|z^\delta(t)|^\beta dt\Big]=E_z\Big[\int_0^{\sigma_{z-1}}|z^\delta(t)|^\beta dt\Big]$$
$$+\sum_{n=1}^{N(z)-1}E_z\Big[E_{z^\delta(\sigma_{z-n})}\Big[\int_0^{\sigma_{z-n-1}}|z^\delta(t)|^\beta dt\Big]\Big]$$
$$+E_z\Big[E_{z^\delta(\sigma_{z-N})}\Big[\int_0^{\sigma_b}|z^\delta(t)|^\beta dt\Big]\Big].$$

Since $z^\delta(\sigma_{z-N})\in[b, b+1]$, one can show

$$E_z\Big[E_{z^\delta(\sigma_{z-N})}\Big[\int_0^{\sigma_b}|z^\delta(t)|^\beta dt\Big]\Big]\leqq\sup_{b\leqq z\leqq b+1}E_z\Big[\int_0^{\sigma_b}\Big]<\infty,$$

as in the proof of $J_1<\infty$.

As for the second term of (4-27), we have, for some $\rho\in(0, 1)$,

$$E_{z_n}\Big[\int_0^{\sigma_{z-n-1}}|z^\delta(t)|^\beta dt\Big]$$
$$\leqq|z-n|^\beta E_{z_n}[\sigma_{z-n-1};\ \sigma_{z-n-1}<s_{z-n}]$$
$$+\sum_{j=0}^{\infty}|z-n+j+1|^\beta E_{z_n}[\sigma_{z-n-1};\ s_{z-n+j}<\sigma_{z-n-1}<s_{z-n+j+1}]$$
$$\leqq|z-n|^\beta E[S(\lambda+(z-n-1)^2)]$$
$$+\sum_{j=0}^{\infty}|z-n+j+1|^\beta\rho^j\Big(E[S(\lambda+(z-n-1)^2)^2]\Big)^{1/2}$$
$$\leqq\text{const.}\,|z-n|^{\beta-2}.$$

Here we have set $z^\delta(\sigma_{z-n})=z_n$. Similar estimation being valid for the first term of (4-27), we finally obtain

$$\sup_{z>b}\Big(E_z\Big[\int_0^{\sigma_z}|z^\delta(t)|^\beta dt\Big]+\sum_{n=1}^{N(z)-1}E_z\Big[E_{z_n}\Big[\int_0^{\sigma_{z-n-1}}|z^\delta(t)|^\beta dt\Big]\Big]\Big)$$
$$\leqq\text{const.}\sup_{z>b}\sum_{n=0}^{N(z)-1}|z-n|^{\beta-2}<\infty.$$

The proof of Theorem 2 is now complete.

§ 5. Proof of Theorem 3.

For notational simplicity, we shall omit λ and ω from our notation in the most part of this section. Hence for example, we write $U(t)$ instead of $U_\lambda(t;\omega)$. Moreover, we consider the Ljapounov behavior at $+\infty$ only.

By Theorem 2, we have $\|U(t)\|>1$ for P-a.a. ω and for $t>0$ large enough. Hence the matrix $U(t)^*U(t)$ has two different simple eigenvalues $\|U(t)\|^2$ and $\|U(t)\|^{-2}$. Let $P(t)$ be the projection belonging to the eigenvalue $\|U(t)\|^{-2}$. Then there is an orthogonal matrix $K(t)$ such that

$$U(t)=K(t)\{\|U(t)\|(1-P(t))+\|U(t)\|^{-1}P(t)\}. \tag{5-1}$$

Now suppose we have shown that for P-a.a. ω, the following two statements hold:

(i) $\lim_{t\to\infty}P(t)\equiv P(\infty)$ exists;

(ii) $\lim_{t\to\infty}\frac{1}{t}\log\{\|U(t)\|\|P(\infty)-P(t)\|\}=-\infty$.

Then $V_\lambda^+(\omega)\equiv P(\infty)(\boldsymbol{R}^2)$ satisfies the conditions of Theorem 3. Indeed, for $v=P(\infty)w$ with $\|w\|=1$, we have from (5-1),

$$\begin{aligned}\|U(t)v\|&=\|\{\|U(t)\|(1-P(t))P(\infty)+\|U(t)\|^{-1}P(t)P(\infty)\}w\|\\&\leqq\|U(t)\|\|(1-P(t))P(\infty)\|+\|U(t)\|^{-1}\\&\leqq\|U(t)\|\|P(\infty)-P(t)\|+\|U(t)\|^{-1}.\end{aligned} \tag{5-2}$$

Since Theorem 2 implies in particular that $\lim_{t\to\infty}\frac{1}{t}\log\|U(t)\|=+\infty$, (ii) and (5-2) show

$$\lim_{t\to\infty}\frac{1}{t}\log\|U(t)v\|=-\infty.$$

On the other hand, if $v\notin V_\lambda^+(\omega)$, then

$$\lim_{t\to\infty}\|(1-P(t))v\|=\|(1-P(\infty))v\|>0.$$

Hence

$$\lim_{t\to\infty}\frac{1}{t}\log\|U(t)v\|\geqq\lim_{t\to\infty}\frac{1}{t}\log\{\|U(t)\|\|(1-P(t))v\|\}=+\infty.$$

Below, we will prove (i) and (ii) by explicitly analysing the asymptotic behavior of $P(t)$. To this end, let us introduce the polar coordinates $r_j(t)=r_j(t;\lambda,\omega)>0$, $\theta_j(t)=\theta_j(t;\lambda,\omega)$, $j=1, 2$, by

$$\varphi^+(t)+i\varphi(t)=r_1(t)\exp[i\theta_1(t)],$$

$$\psi^+(t)+i\psi(t)=r_2(t)\exp[i\theta_2(t)].$$

This determines $\theta_j(t)$ only with modulo 2π. But if we start with $\theta_1(0)=\pi/2$,

$\theta_2(0)=0$, and if we define $\theta_j(t)=2n\pi$ when we have $\theta_j(t)=0$ (mod. 2π) for the n-th time, we obtain a well-defined real valued function.

Some elementary calculations give

$$P(t)=\frac{1}{1+\beta(t)^2}\begin{bmatrix}1 & \beta(t)\\ \beta(t) & \beta(t)^2\end{bmatrix}, \tag{5-3}$$

where

$$\beta(t)=\frac{1}{2\cos(\theta_1-\theta_2)}\left[\left(\frac{r_2}{r_1}-\frac{r_1}{r_2}\right)-\left(\left(\frac{r_2}{r_1}+\frac{r_1}{r_2}\right)^2-\frac{4}{r_1^2r_2^2}\right)^{1/2}\right].$$

Since $4r_1(t)^{-2}r_2(t)^{-2}\to 0$, a.s. by Theorem 2, (i) follows from the lemma below.

LEMMA 7. *Under the condition of Theorem* 3, $r_1(t)/r_2(t)$, $r_2(t)/r_1(t)$, *and* $\theta_1(t)-\theta_2(t)$ *have limits as* $t\to+\infty$, *for* P-*a.a.* ω.

PROOF. We shall show that for a.a. ω, the limit

$$\lim_{t\to\infty}[\log(\varphi^+(t)+i\varphi(t))-\log(\phi^+(t)+i\phi(t))]$$

exists. But if $u(t)=\varphi(t)$ or $\phi(t)$, then by Ito's formula,

$$\begin{aligned}
&\log\{u^+(t)+iu(t)\}-\log\{u^+(0)+iu(0)\}\\
&=\int_0^t\frac{iu^+(s)}{u^+(s)+iu(s)}ds+(c_\delta-\lambda)\int_0^t\frac{u(s)}{u^+(s)+iu(s)}ds\\
&\quad+v\int_0^t\frac{u(s)}{u^+(s)+iu(s)}dB(s)-\frac{v^2}{2}\int_0^t\frac{u(s)^2}{(u^+(s)+iu(s))^2}ds\\
&\quad+\int_0^{t+}\int_{|x|>\delta}[\log\{(u^+(s-)+xu(s))+iu(s)\}-\log(u^+(s-)+iu(s))]N(dsdx)\\
&\quad+\int_0^{t+}\int_{|x|\le\delta}[\log\{(u^+(s-)+xu(s))+iu(s)\}-\log(u^+(s-)+iu(s))]\tilde{N}(dsdx)\\
&\quad+\int_0^{t+}ds\int_{|x|\le\delta}\left[\log\left(\frac{(u^+(s)+xu(s))+iu(s)}{u^+(s)+iu(s)}\right)-\frac{xu(s)}{u^+(s)+iu(s)}\right]\nu(dx)\\
&\equiv A_1^u(t)+A_2^u(t)+M_1^u(t)+A_3^u(t)+S^u(t)+M_2^u(t)+A_4^u(t).
\end{aligned}$$

Noting $\varphi(t)\phi^+(t)-\varphi^+(t)\phi(t)=1$, one obtains first

$$A_1^\varphi(t)-A_1^\phi(t)=\int_0^t\frac{1}{(\varphi^+(s)+i\varphi(s))(\phi^+(s)+i\phi(s))}ds$$

and

$$A_2^\varphi(t)-A_2^\phi(t)=(c_\delta-\lambda)\int_0^t\frac{1}{(\varphi^+(s)+i\varphi(s))(\phi^+(s)+i\phi(s))}ds.$$

But $|(\varphi^+(s)+i\varphi(s))(\phi^+(s)+i\phi(s))|^{-1}=r_1(s)^{-1}r_2(s)^{-1}$ decays faster than exponentially as $s\to\infty$ by Theorem 2. Hence $A_j^\varphi(t)-A_j^\phi(t)$, $j=1, 2$, have limits as $t\to\infty$ almost surely. The same reasoning is valid for

$$A_3^{\varphi}(t)-A_3^{\phi}(t)=-\frac{v^2}{2}\int_0^t\frac{\varphi(s)\phi^+(s)+\phi(s)\varphi^+(s)+2i\varphi(s)\phi(s)}{(\varphi^+(s)+i\varphi(s))^2(\phi^+(s)+i\phi(s))^2}\,ds.$$

Now $\mathrm{Re}(M_1^{\varphi}(t)-M_1^{\phi}(t))$ and $\mathrm{Im}(M_1^{\varphi}(t)-M_1^{\phi}(t))$ are square integrable martingales, and their quadratic variational process satisfy

$$\lim_{t\to\infty}\{\langle\mathrm{Re}(M_1^{\varphi}-M_1^{\phi})\rangle(t)+\langle\mathrm{Im}(M_1^{\varphi}-M_1^{\phi})\rangle(t)\}$$

$$=\lim_{t\to\infty}v^2\int_0^t|(\varphi^+(s)+i\varphi(s))(\phi^+(s)+i\phi(s))|^{-2}ds<\infty.$$

Hence by martingale convergence theorem,

$$\lim_{t\to\infty}(M_1^{\varphi}(t)-M_1^{\phi}(t))$$

exists almost surely.

Next consider $M_2^{\varphi}(t)-M_2^{\phi}(t)$. We have

$$\langle\mathrm{Re}(M_2^{\varphi}-M_2^{\phi})\rangle(t)+\langle\mathrm{Im}(M_2^{\varphi}-M_2^{\phi})\rangle(t)$$

$$=\int_0^t ds\int_{|x|\le\delta}\left|\log\left(\frac{(\varphi^+(s)+x\varphi(s))+i\varphi(s)}{\varphi^+(s)+i\varphi(s)}\,\frac{\phi^+(s)+i\phi(s)}{(\phi^+(s)+x\phi(s))+i\phi(s)}\right)\right|^2\nu(dx).$$

It is easily seen that the quantity inside $\log\{\ \}$ remains on the same branch such that $\log 1=0$. Therefore

$$\log\left(\frac{(\varphi^++x\varphi)+i\varphi}{\varphi^++i\varphi}\,\frac{\phi^++i\phi}{(\phi^++x\phi)+i\phi}\right)=\log(1+\eta(s,\ x))$$

with

$$\eta(s,\ x)=\frac{x}{\{(\phi^+(s)+x\phi(s))+i\phi(s)\}(\varphi^+(s)+i\varphi(s))}$$

is small if $|\eta(s,\ x)|$ is small, namely it is $O(|\eta(s,\ x)|)$. On the other hand,

$$|\eta(s,\ x)|=|x|\left|\frac{\phi^+(s)+i\phi(s)}{(\phi^+(s)+x\phi(s))+i\phi(s)}\right|\frac{1}{r_1(s)r_2(s)}\leqq C|x|\frac{1}{r_1(s)r_2(s)},$$

where C is a constant not depending on x, s, and ω. Hence for P-a.a. ω,

$$\int_{|x|\le\delta}|\log(1+\eta(s,\ x,\ \omega))|^2\nu(dx)\leqq\mathrm{const.}\left(\int_{|x|\le\delta}x^2\nu(dx)\right)\frac{1}{r_1(s)^2r_2(s)^2}$$

for s sufficiently large, and we get

$$\langle\mathrm{Re}(M_2^{\varphi}-M_2^{\phi})\rangle(\infty)+\langle\mathrm{Im}(M_2^{\varphi}-M_2^{\phi})\rangle(\infty)<\infty.$$

Again by martingale convergence theorem, $M_2^{\varphi}(t)-M_2^{\phi}(t)$ has a limit as $t\to\infty$ almost surely.

As for $A_4^{\varphi}(t)-A_4^{\phi}(t)$, we have

$$A_4^{\varphi}(t)-A_4^{\phi}(t)$$
$$=\int_0^t ds\int_{|x|\le\delta}\Big[\log\Big(\frac{(\varphi^+(s)+x\varphi(s))+i\varphi(s)}{\varphi^+(s)+i\varphi(s)}\frac{\phi^+(s)+i\phi(s)}{(\phi^+(s)+x\phi(s))+i\phi(s)}\Big)$$
$$-\frac{x\varphi(s)}{\varphi^+(s)+i\varphi(s)}+\frac{x\phi(s)}{\phi^+(s)+i\phi(s)}\Big]\nu(dx)$$
$$=\int_0^t ds\int_{|x|\le\delta}\Big[\log\Big(1+\frac{(\phi^+(s)+x\phi(s))+i\phi(s)}{\phi^+(s)+i\phi(s)}\frac{x}{(\phi^+(s)+i\phi(s))(\varphi^+(s)+i\varphi(s))}\Big)$$
$$-\frac{x}{(\phi^+(s)+i\phi(s))(\varphi^+(s)+i\varphi(s))}\Big]\nu(dx).$$

As before, for P-a. a. ω and for s large enough, the integrand above is estimated from above by

$$\Big(\frac{(\phi^+(s)+x\phi(s))+i\phi(s)}{\phi^+(s)+i\phi(s)}-1\Big)\frac{x}{(\phi^+(s)+i\phi(s))(\varphi^+(s)+i\varphi(s))}+O\Big(\frac{x^2}{r_1(s)^2r^2(s)^2}\Big)$$
$$=\frac{\phi(s)}{\phi^+(s)+i\phi(s)}\frac{x^2}{(\phi^+(s)+i\phi(s))(\varphi^+(s)+i\varphi(s))}+O\Big(\frac{x^2}{r_1(s)^2r_2(s)^2}\Big).$$

Hence

$$\int_{|x|\le\delta}[\cdots]\nu(dx)\le\text{const.}\Big(\int_{|x|\le\delta}x^2\nu(dx)\Big)\frac{1}{r_1(s)r_2(s)},$$

for s large enough, which implies the existence of $\lim_{t\to\infty}(A_4^{\varphi}(t)-A_4^{\phi}(t))$.

Finally, let us consider $S^{\varphi}(t)-S^{\phi}(t)$. As was already considered in § 4, let σ_n be the n-th time at which $|\Delta Q_{\omega}^{\delta}(t)|>\delta$. Then

$$S^{\varphi}(t)-S^{\phi}(t)=\sum_{\sigma_n\le t}\log\Big[\frac{\varphi^+(\sigma_n)+i\varphi(\sigma_n)}{\phi^+(\sigma_n)+i\phi(\sigma_n)}\frac{\phi^+(\sigma_n)+i\phi(\sigma_n)}{\varphi^+(\sigma_n)+i\varphi(\sigma_n)}\Big]$$
$$=\sum_{\sigma_n\le t}\log\Big[\frac{i+\varphi^+(\sigma_n)/\varphi(\sigma_n)}{i+\phi^+(\sigma_n)/\phi(\sigma_n)}\frac{i+\phi^+(\sigma_n-)/\phi(\sigma_n)}{i+\varphi^+(\sigma_n-)/\varphi(\sigma_n)}\Big]$$
$$=\sum_{\sigma_n\le t}\log\Big[1-\frac{1}{(\phi^+(\sigma_n)+i\phi(\sigma_n))\varphi(\sigma_n)}\Big]$$
$$+\sum_{\sigma_n\le t}\log\Big[1+\frac{1}{(\varphi^+(\sigma_n-)+i\varphi(\sigma_n))\phi(\sigma_n)}\Big].$$

We consider the first summation only, the other one being treated similarly. In order to prove that the limit

$$\lim_{t\to\infty}\sum_{\sigma_n\le t}\log\Big[1-\frac{1}{(\phi^+(\sigma_n)+i\phi(\sigma_n))\varphi(\sigma_n)}\Big]$$
$$=\sum_{n=1}^{\infty}\log\Big[1-\frac{1}{(\phi^+(\sigma_n)+i\phi(\sigma_n))\varphi(\sigma_n)}\Big]$$

exists, it is sufficient to show that with probability one,

$$|(\psi^*(\sigma_n)+i\psi(\sigma_n))^{-1}\varphi(\sigma_n)^{-1}|^2$$

converges to 0 faster than exponentially as $n\to\infty$. To this end, note that $\cot\theta_1(t)=z(t;0,\omega)=\varphi^+(t)/\varphi(t)$, and $\varphi(t)=r_1(t)\sin\theta_1(t)$. Then

$$\begin{aligned}&\log|\psi^+(\sigma_n)+i\psi(\sigma_n))^{-1}\varphi(\sigma_n)^{-1}|^2\\&=-\log\{r_2(\sigma_n)^{-2}r_1(\sigma_n)^{-2}(1+z(\sigma_n)^2)\}\\&=-\log r_2(\sigma_n)^2-\log r_1(\sigma_n)^2+\log(1+z(\sigma_n)^2).\end{aligned}$$

But from § 4, we have

$$-\log r_1(\sigma_n)^2+\log(1+z(\sigma_n)^2)=-\log r_1(\sigma_n-)^2-\log(1+z(\sigma_n-)^2),$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log(1+z(\sigma_n-)^2)=0\,,\quad \text{a.s..}$$

Hence applying Theorem 2,

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\log|(\psi^+(\sigma_n)+i\psi(\sigma_n))^{-1}\varphi(\sigma_n)^{-1}|^2\\&=-\lim_{n\to\infty}\frac{1}{n}\log r_1(\sigma_n-)^2-\lim_{n\to\infty}\frac{1}{n}\log r_2(\sigma_n)^2=-\infty\,.\end{aligned}$$

This completes the proof of Lemma 7.

When $f(t)$ and $g(t)$ are positive real function on $(0,\infty)$, we shall write $f(t)\asymp g(t)$ if

$$0<\varliminf_{t\to\infty}\frac{g(t)}{f(t)}\leqq\varlimsup_{t\to\infty}\frac{g(t)}{f(t)}<\infty\,.$$

Then Theorem 7 implies in particular $r_1(t)\asymp r_2(t)$. From this and (5-3), we see that (ii) follows from the lemma below.

LEMMA 8. *Let* $S_+(t)=\sum_{\sigma_n\leq t}\log(1+z(\sigma_n)^2)$, *where* $z(t)=\varphi^+(t)/\varphi(t)$ *or* $\psi^+(t)/\psi(t)$. *Then under the condition of Theorem* 3, *we have for any* $\varepsilon>0$,

$$\left|\frac{r_2(t)}{r_1(t)}-\left(\frac{r_2}{r_1}\right)(\infty)\right|=O(e^{-(1-\varepsilon)S_+(t)});$$

$$\left|\frac{r_1(t)}{r_2(t)}-\left(\frac{r_1}{r_2}\right)(\infty)\right|=O(e^{-(1-\varepsilon)S_+(t)});$$

$$|(\theta_1(t)-\theta_2(t))-(\theta_1-\theta_2)(\infty)|=O(e^{-(1-\varepsilon)S_+(t)}).$$

Iu particular, for $j=1,2$,

$$\lim_{t\to\infty}\frac{1}{t}\log\left(r_j(t)\left|\frac{r_2(t)}{r_1(t)}-\left(\frac{r_2}{r_1}\right)(\infty)\right|\right)=-\infty\,,$$

and similar relation holds when we replace r_2/r_1 *by* r_2/r_1 *or* $\theta_1-\theta_2$.

PROOF. Let $r(t)=r_1(t)$ or $r_2(t)$. The results of § 4 tell us that we can write

$$r(t)=\exp\left[X(t)+\frac{1}{2}S_+(t)\right], \tag{5-4}$$

where $X(t)=O(t)$ and $(1/t)S_+(t)\to\infty$ a.s. as $t\to\infty$. Therefore the second part of the lemma is a direct consequence of the first one. This, in turn, reduces to proving

$$\left|\log\left(\frac{\varphi^+(t)+i\varphi(t)}{\phi^+(t)+i\phi(t)}\right)-\log\left(\frac{\varphi^++i\varphi}{\phi^++i\phi}\right)\right|=O(e^{-(1-\varepsilon)S_+(t)}).$$

Indeed,

$$\begin{aligned}
&|r_1(t)/r_2(t)-(r_1/r_2)(\infty)|\\
&=|\exp[\log(r_1(t)/r_2(t))]-\exp[\log((r_1/r_2)(\infty))]|\\
&=O\Big(|\log(r_1(t)/r_2(t))-\log((r_1/r_2)(\infty))|\Big)\\
&=O\left(\left|\log\left(\frac{\varphi^+(t)+i\varphi(t)}{\phi^+(t)+i\phi(t)}\right)-\log\left(\left(\frac{\varphi^++i\varphi}{\phi^++i\phi}\right)\right)\right|\right)\\
&=O(e^{-(1-\varepsilon)S_+(t)})
\end{aligned}$$

gives the first estimate of the lemma. The second one is obtained in the same way. Finally

$$\begin{aligned}
&|(\theta_1(t)-\theta_2(t))-(\theta_1-\theta_2)(\infty)|\\
&=\left|\mathrm{Im}\left(\log\left(\frac{\varphi^+(t)+i\varphi(t)}{\phi^+(t)+i\phi(t)}\right)-\log\left(\left(\frac{\varphi^++i\varphi}{\phi^++i\phi}\right)(\infty)\right)\right)\right|\\
&=O(e^{-(1-\varepsilon)S_+(t)})
\end{aligned}$$

gives the third estimate.

Now let $\tilde{A}_j=A_j^\varphi-A_j^\phi$, $j=1, 2, 3, 4$, $\tilde{M}_j(t)=M_j^\varphi(t)-M_j^\phi(t)$, $j=1, 2$, and $\tilde{S}(t)=S^\varphi(t)-S^\phi(t)$. We will estimate the rate of convergence to zero of these processes.

First from Lemma 7 and its proof, there is a constant C_ω for P-a. a. ω such that

$$|\tilde{A}_1(t)-\tilde{A}_1(\infty)|\leqq C_\omega\int_t^\infty r_1(s)^{-2}ds\,.$$

Since $S_+(t)$ is non-decreasing, we have

$$\int_t^\infty r_1(s)^{-2}ds\leqq e^{-(1-\varepsilon)S_+(t)}\int_0^\infty e^{-2X(t)-\varepsilon S_+(t)}ds\,,$$

for any $\varepsilon>0$. The last integral being convergent almost surely, we get

$$|\tilde{A}_1(t)-\tilde{A}_1(\infty)|=O(e^{-(1-\varepsilon)S_+(t)})\,.$$

In the same manner, we can show

$$|\tilde{A}_j(t)-\tilde{A}_j(\infty)|=O(e^{-(1-\varepsilon)S_+(t)})\,. \quad j=2,\,3,\,4\,.$$

Next consider $\tilde{M}_1(t)$. Since $\{\mathrm{Re}\,(\tilde{M}_1(t))\}$ and $\{\mathrm{Im}\,(\tilde{M}_1(t))\}$ are continuous square integrable martingales, we can construct two Brownian motions $\{\tilde{B}_j(t)\}$, $j=1, 2$, such that

$$\mathrm{Re}\,\tilde{M}_1(t)=\tilde{B}_1(\langle\mathrm{Re}\,\tilde{M}_1\rangle(t)) \quad \text{and} \quad \mathrm{Im}\,\tilde{M}_1(t)=\tilde{B}_1(\langle\mathrm{Im}\,\tilde{M}_1\rangle(t))\,.$$

But from the local Hölder continuity of the Brownian motion, we have

$$\sup_{t,s\in[0,T]}|\tilde{B}_1(t)-B_1(s)|/|t-s|^{1/2-\delta}<\infty\,, \quad \text{a.s.},$$

for any $\delta>0$ and $T>0$. Hence on the set $\{\langle\mathrm{Re}\,\tilde{M}_1\rangle(\infty)<T\}$, we have

$$\begin{aligned}|\mathrm{Re}\,\tilde{M}_1(t)-\mathrm{Re}\,\tilde{M}_1(\infty)| &\leqq O\Big(|\langle\mathrm{Re}\,\tilde{M}_1\rangle(t)-\langle\mathrm{Re}\,\tilde{M}_1\rangle(\infty)|^{1/2-\delta}\Big)\\ &\leqq O\Big(\Big(\int_t^\infty r_1(s)^{-4}ds\Big)^{1/2-\delta}\Big)\\ &=O(e^{-(1-\varepsilon)(1/2-\delta)S_+(t)})\,,\end{aligned}$$

for any $\varepsilon>0$. Since the same result holds for $\{\mathrm{Im}\,\tilde{M}_1(t)\}$, and since $\bigcup_{T>0}\{\langle\mathrm{Re}\,\tilde{M}_1\rangle(\infty)<T\}=\bigcup_{T>0}\{\langle\mathrm{Im}\,\tilde{M}_1\rangle(\infty)<T\}=\Omega$ up to null sets, we finally obtain, for any $\varepsilon>0$,

$$|\tilde{M}_1(t)-\tilde{M}_1(\infty)|=O(e^{-(1-\varepsilon)S_+(t)})\,, \quad \text{a.s.}$$

As for $\{\tilde{M}_2(t)\}$, we could not find out how to use its martingale property, because, being discontinuous, there is no good representation theorem like that for $\{\tilde{M}_1(t)\}$. But if we assume $\int_{|x|<1}|x|\nu(dx)<\infty$, then $\{\tilde{M}_2(t)\}$ is locally of bounded variation, and we can treat this in the same manner as $\tilde{A}_j(t)$'s. Indeed, under the assumption,

$$|Q^\delta|(t)\equiv\int_{|x|\leq\delta}|x|N((0,\,t]\times dx)$$

is well defined, and it is an increasing process with ergodic increments. Hence

$$\lim_{t\to\infty}\frac{1}{t}|Q^\delta|(t)=E[|Q^\delta|(1)]\,, \quad \text{a.s.}$$

From this we obtain, using the notation in the proof of Lemma 7,

$$|\tilde{M}_2(\infty)-\tilde{M}_2(t)|\leqq\int_{t+}^\infty\int_{|x|\leq\delta}|\log(1+\eta(s,\,x))|N(dsdx)$$

$$\leqq C_\omega \int_{t+}^{\infty}\int_{|x|\leqq\delta} |x| r_1(s)^{-2} N(dsdx)$$

$$= C_\omega \int_{t+}^{\infty} r_1(s)^{-2} d|Q^\delta|(s)$$

$$\leqq C_\omega e^{-(1-\varepsilon)S_+(t)} \int_0^\infty e^{-2X(s)-\varepsilon S_+(s)} d|Q^\delta|(s)$$

$$= O(e^{-(1-\varepsilon)S_+(t)}), \qquad \text{for } P\text{-a.a. } \omega.$$

Finally let us consider $\tilde{S}(t)$. Again using the notation in the proof of Lemma 7, we obtain without difficulty the following estimates: for t large enough,

$$\Big|\sum_{\sigma_n>t} \log\Big[1-\frac{1}{(\phi^+(\sigma_n)+i\phi(\sigma_n))\varphi(\sigma_n)}\Big]\Big|$$

$$\leqq \text{const.} \sum_{\sigma_n>t} r_2(\sigma_n)^{-1} r_1(\sigma_n)^{-1}(1+z_1(\sigma_n)^2)^{1/2}$$

$$\leqq \text{const.} \sum_{\sigma_n>t} r_1(\sigma_n)^{-1} r_1(\sigma_n -)^{-1}(1+z_1(\sigma_n -)^2)^{-1/2}$$

$$\leqq \text{const.}\ e^{-(1-\varepsilon)S_n(t)} \sum_{n=0}^{\infty} e^{-X(\sigma_n)-X(\sigma_n)-\varepsilon S_n(\sigma_{n-1})}(1+z_1(\sigma_n -)^2)^{1/2}$$

$$= O(e^{-(1-\varepsilon)S_+(t)}),$$

for any $\varepsilon>0$. Here we have set $z_1(t)=\varphi^+(t)/\varphi(t)$. Similarly we can show

$$\Big|\sum_{\sigma_n>t} \log\Big[1+\frac{1}{(\varphi^+(\sigma_n)+i\varphi(\sigma_n))\phi(\sigma_n)}\Big]\Big| = O(e^{-(1-\varepsilon)S_+(t)}).$$

and consequently

$$|S(\infty)-S(t)| = O(e^{-(1-\varepsilon)S_+(t)}) \cdot$$

This completes the proof of Lemma 8, and hence of Theorem 3.

§ 6. Proof of Theorem 4.

Let $V_\lambda^+(\omega)$ be the one-dimensional subspace of $\boldsymbol{R}^2$ which was constructed in § 5, and let $v_1=P(\infty)w\in V_\lambda^+(\omega)\backslash\{0\}$ with $\|w\|=1$, $v_2\notin V_\lambda^+(\omega)$, $\|v_2\|=1$. Our purpose here is to obtain more precise estimate of the asymptotic behaviors of $\|U(t)v_1\|$ and $\|U(t)v_2\|$ than those in § 5. This, on the other hand, reduces to estimating the growth of $S_+(t)$ which appeared in Lemma 8. In fact, we have from (5-1),

$$\|U(t)v_1\| \leqq \|U(t)\| \|P(\infty)-P(t)\| + \|U(t)\|^{-1},$$

$$\|U(t)v_1\| \geqq \|U(t)\|^{-1} \|P(t)v_1\|,$$

and

$$\|U(t)\| \|(1-P(t))v_2\| \leqq \|U(t)v_2\| \leqq \|U(t)\|.$$

But as we have already seen, it holds that

$$\lim_{t\to\infty} \|P(t)v_1\| = \|P(\infty)v_1\| > 0 .$$

$$\lim_{t\to\infty} \|(1-P(t))v_2\| = \|(1-P(\infty))v_2\| > 0 ,$$

$$\|U(t)\| \asymp r_1(t) ,$$

and

$$\text{const.}\, e^{(1/2-\varepsilon)S_+(t)} \leqq r_1(t) \leqq \text{const.}\, e^{(1/2+\varepsilon)S_+(t)} ,$$

the constants depending on $\varepsilon>0$ and ω.

Combining these, we see that for each $\varepsilon>0$, there are positive constants $C_{j,\varepsilon,\omega}$, $j=1, 2, 3, 4$, for P--a.a. ω such that

$$C_{1,\varepsilon,\omega} e^{-(1/2+\varepsilon)S_+(t)} \leqq \|U(t)v_1\| \leqq C_{2,\varepsilon,\omega} e^{-(1/2-\varepsilon)S_+(t)} ,$$

$$C_{3,\varepsilon,\omega} e^{(1/2-\varepsilon)S_+(t)} \leqq \|U(t)v_2\| \leqq C_{4,\varepsilon,\omega} e^{(1/2+\varepsilon)S_+(t)} .$$

Therefore Theorem 4 follows immediately from the lemma below.

LEMMA 9. *Under the conditions of Theorem* 4, *we have*

$$\varlimsup_{t\to\infty} \frac{1}{t^\alpha} \lambda_{(k)}\big(S_+(t)\big) = 0 , \quad a.s., \quad for\ \alpha > \beta^{-1} ,$$

$$\varliminf_{t\to\infty} \frac{1}{t^\alpha} \lambda_{(k)}\big(S_+(t)\big) = +\infty , \quad a.s., \quad for\ < \alpha < \beta^{-1} .$$

For the proof of this lemma, we need the following

LEMMA 10. *Let $X_n \geqq 0$, $n=1, 2, 3, \cdots$, be a sequence of i.i.d. random variables with the common distribution function $F(x)$. Suppose that for some $k \geqq 0$ and $\beta \geqq 0$,*

$$1 - F(\varepsilon_{(k)}(x)) = x^{-\beta} L(x) ,$$

where $L(x)$ is slowly varying at $+\infty$. In case $k=0$, we shall further assume $\beta \leqq 1$. Then

(6-1) $$\varlimsup_{n\to\infty} \frac{1}{n^\alpha} \lambda_{(k)}\Big(\sum_{j=1}^{n} X_j\Big) = 0 . \quad a.s., \quad for\ \alpha > \beta^{-1} ,$$

and

(6-2) $$\varliminf_{n\to\infty} \frac{1}{n^\alpha} \lambda_{(k)}\Big(\max_{1\leq j\leq n} X_j\Big) = +\infty , \quad a.s., \quad for\ 0 < \alpha < \beta^{-1} .$$

PROOF. First suppose $k=0$, $0 \leqq \beta \leqq 1$. Them for any $\alpha > \beta^{-1}$, we have $E[X_1^{1/\alpha}] < \infty$ with $0 < 1/\alpha < 1$. Hence (6-1) for $k=0$, namely

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{\alpha}}\sum_{j=1}^{n}X_j=0\,,\quad \text{a.s.}$$

is a consequence of a well known theorem (see Neveu [27], Proposition 4.7.1.).

Next consider (6-1) for $k\geqq 1$ and $\beta\geqq 0$. From the elementary inequality

$$\lambda_{(k)}(xy)\leqq\lambda_{(k)}(u)+\lambda_{(k)}(y)\,,\quad x,\,y\geqq 0\,,$$

we get

$$\lambda_{(k)}\Big(\sum_{j=1}^{n}X_j\Big)\leqq\lambda_{(k)}\Big(n\cdot\max_{1\leq j\leq n}X_j\Big)$$

$$\leqq\lambda_{(k)}(n)+\max_{1\leq j\leq n}\lambda_{(k)}(X_j)\,.$$

Hence (6-1) is reduced to proving

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{\alpha}}\max_{1\leq j\leq n}\lambda_{(k)}(X_j)=0\quad \text{a.s.}.$$

On the other hand, for any sequence $a_n\geqq 0$, $B_n>0$ of real numbers such that $B_n\uparrow\infty$, one has

$$\overline{\lim_{n\to\infty}}\frac{1}{B_n}\max_{1\leq j\leq n}a_j=\overline{\lim}\,\frac{1}{B_n}a_n\,.$$

Hence the problem is further reduced to showing

(6-3) $$\overline{\lim_{n\to\infty}}\frac{1}{n^{\alpha}}\lambda_{(k)}(X_n)=0\,,\quad \text{a.s.}.$$

But for any $\varepsilon>0$, we have

$$\sum_{n=1}^{\infty}P\Big(\frac{1}{n^{\alpha}}\lambda_{(k)}(X_n)>\varepsilon\Big)=\sum_{n}(1-F(\varepsilon n^{\alpha}))=\sum_{n}\varepsilon^{-\beta}n^{-\alpha\beta}L(x)<\infty\,,$$

provided $\alpha>\beta^{-1}$. Therefore by Borel-Cantelli's lemma,

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{\alpha}}\lambda_{(k)}(X_n)\leqq\varepsilon\,,\quad \text{a.s.}.$$

Letting $\varepsilon\downarrow 0$, we arrive at (6-3).

Now let us turn to the proof of (6-2). For each $K>0$, we have from the assumption,

$$\sum_{n=1}^{\infty}P\Big(\frac{1}{n^{\alpha}}\lambda_{(k)}\Big(\max_{1\leq j\leq n}X_j\Big)\leqq Kn^{\alpha}\Big)=\sum_{n}\Big(P(\lambda_{(k)}(X_1)\leqq Kn^{\alpha})\Big)^{n}$$

$$=\sum_{n}\exp\left[n\log F(Kn^{\alpha})\right]$$

$$\leqq\sum_{n}\exp\left[-n(1-F(Kn^{\alpha}))\right]$$

$$=\sum_{n}\exp\left[-K^{-\beta}n^{1-\alpha\beta}L(Kn^{\alpha})\right]<\infty\,,$$

whenever $0<\alpha<\beta^{-1}$. Hence again by Borel-Cantelli's lemma,

$$\varliminf_{n\to\infty}\frac{1}{n^{\alpha}}\lambda_{(k)}\Big(\max_{1\leq j\leq n}X_j\Big)>K, \quad \text{a.s..}$$

Letting $K\uparrow\infty$, we obtain (6-2).

Now we can proceed to the proof of Lemma 9. First note that $\log(1+|\Delta Q(\sigma_n)|)$, $n\geqq 1$, are i.i.d. with distribution function $F(x)$ given by

$$1-F(x)=P(|\Delta Q(\sigma_1)|>e^x-1)=\nu(\boldsymbol{R}_\sigma)^{-1}\int_{|y|>e^x-1}\nu(dy).$$

Hence the assumption of Theorem 4 implies that we can apply Lemma 10 to $X_n=\log(1+|\Delta Q(\sigma_n)|)$, $n\geqq 1$, with given k and β.

It is clear that

$$\lambda_{(k)}(S_+(t))=\lambda_{(k)}\Big(\sum_{\sigma_n\leq t}\log(1+(z(\sigma_n-)+\Delta Q(\sigma_n))^2)\Big)$$

$$\leqq\sum_{\sigma_n\leq t}\lambda_{(k)}\Big(\log(1+2z(\sigma_n-)^2)\Big)+\sum_{\sigma_n\leq t}\lambda_{(k)}\Big(\log(1+2(\Delta Q(\sigma_n))^2)\Big).$$

The first term on the right hand side is $O(t)$ by virture of the results of §4. Hence the first part of Lemma 9 follows from (6-1). On the other hand,

$$\lambda_{(k)}(S_+(t))\geqq\lambda_{(k)}\Big(\log\Big\{1+\Big(\max_{n:\sigma_n\leq t}|z(\sigma_n-)+\Delta Q(\sigma_n)|\Big)^2\Big\}\Big).$$

Again noting that $\max_{n:\sigma_n\leq t}\log(1+z(\sigma_n-)^2)=O(t)$, we have

$$\varliminf_{t\to\infty}\frac{1}{t^{\alpha}}\lambda_{(k)}(S_+(t))=\varliminf_{t\to\infty}\frac{1}{t^{\alpha}}\lambda_{(k)}\Big(\max_{\sigma_n\leq t}\log(1+|\Delta Q(\sigma_n)|)\Big)$$

$$=+\infty, \quad \text{a.s.}$$

from (6-2), showing the second part of Lemma 9.

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