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HYPERSURFACES IN THE QUATERNIONS II

By

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1. Introduction.

Let $\mathbf{H}=\mathrm{span}_{\mathbf{R}}\{1, i, j, k\}$ be the quaternions. We shall fix the basis $\{1, i, j, k\}$ throughout this paper. Then, we may regard $\mathbf{H}$ as a 4-dimensional Euclidean space $\mathbf{R}^4$ in the natural way. An oriented hypersurface M^3 in $\mathbf{H}$ admits a global orthonormal frame field as follows. Let (M^3, f) be an oriented hypersurface of $\mathbf{H}$ and ξ a unit normal vector field on M^3. Then $\{\xi i, \xi j, \xi k\}$ is a global orthonormal frame field on $f(M^3)$. We shall call this orthonormal frame field an associated one of $f(M^3)$ and the dual frame field of $\{\xi i, \xi j, \xi k\}$ as associated dual frame field, respectively. We may remark that the associated frame field on M^3 (intrinsically) of an oriented hypersurface (M^3, af) in $\mathbf{H}$ coincides with the associated one of the hypersurface (M^3, f) for any $a\in Sp(1)$. We note that the associated frame field of (M^3, bf) are different from the associated one of (M^3, f) for $b\in SO(4)$ and $b\notin Sp(1)$. This paper is a continuation of the previous one ([3]). Let x be the unit normal vector field of a unit 3-sphere S^3 in $\mathbf{H}$, then the vector fields $\{xi, xj, xk\}$ are killing vector fields on S^3 (see [3], [5]), and each integral curve of xi (or xj or xk) is a geodesic in S^3 and a circle in $\mathbf{H}$. We shall prove the followings:

THEOREM A. *Let* (M^3, f) *be an oriented hypersurface in the quaternions* $\mathbf{H}$ *and* ξ *a global normal vector field of* M^3 *in* $\mathbf{H}$. *If each* 1-*form of the associated dual frame field on* M^3 *is a contact form on* M^3 *and each integral curve of the associated orthonormal frame field is a circle in* $\mathbf{H}$, *then*

(1) M^3 *is locally isometric to a* 3-*dimensional round sphere in* $\mathbf{H}$ *and the immersion* f *is totally umbilic,*

or

(2) M^3 *is locally isometric to* $S^1\times\mathbf{R}^2$ (S^1 *is a circle*) *and the immertion* f *is a locally product one.*

TEEOREM B. *Let* (M^3, f) *be an oriented complete hypersurface in the quaternions* $\mathbf{H}$ *and* ξ *a global unit normal vector field of* M^3 *in* $\mathbf{H}$. *If each integral*

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curve of any one of the vector fields belonging to the associated frame field, say ξi, *is a geodesic in* M^3 (*the first curvature of the integral curve in* **H** *is non-zero*) *and* $\rho(\xi j, \xi k)\geq 0$ (*or* ≤ 0 *where* ρ *is the Ricci curvature of* M^3), *then*

(1) M^3 *is isometric to a 3-dimensional round sphere in* **H** *and the immersion* f *is totally umbilic,*

or

(2) M^3 *is isometric to* $M^1\times\mathbf{R}^2$ (M^1 *is a 1-dimensional Riemannian manifold*) *and the immertion* f *is a* (*non-totally geodesic*) *product one.*

REMARK. We shall give an example of the second case of Theorem A after the proof of Theorem A. Here, we remark that K. Nomizu and K. Yano ([15]) proved that the submanifold M^n in $\mathbf{R}^{n+p}$ is umbilical if and only if every circle in M^n is a circle in $\mathbf{R}^{n+p}$.

In this paper, all the manifolds are assumed to be connected and class C^∞ unless otherwise stated. The author would like to express his heartly thanks to Professor K. Sekigawa and Mr. T. Koda for their constant encouragement and many valuable suggestions and to the refree for his many valuable comments.

2. Preliminaries.

First, we shall recall some elementary properties of the quaternions $\mathbf{H}=\text{span}_{\mathbf{R}}\{1, i, j, k\}$ with $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$ and $ki=-ik=j$. Let $\langle , \rangle$ be the canonical inner product of **H**. For any $x\in\mathbf{H}$, we denote by $\bar{x}$ the conjugate of x. We write down some elementary formulae of **H**.

$$\begin{aligned}
&\langle xw, y\rangle=\langle x, y\bar{w}\rangle, \quad \langle wx, y\rangle=\langle x, \bar{w}y\rangle \\
&\overline{xy}=\bar{y}\bar{x}, \\
&\langle x, y\rangle=(x\bar{y}+y\bar{x})/2, \quad \langle\bar{x}, \bar{y}\rangle=\langle x, y\rangle
\end{aligned} \tag{2.1}$$

for any $x, y, w\in\mathbf{H}$ (see [2]).

We recall also some elementary formulae of hypersurfaces in the Euclidean space. We denote by $\mathbf{R}^{n+1}$ an $(n+1)$-dimensional Euclidean space. Let M^n be an n-dimensional hypersurface in $\mathbf{R}^{n+1}$. We denote by ∇, D and $\nabla^\perp$ the Riemannian connection of M^n, $\mathbf{R}^{n+1}$ and the normal connection of M^n in $\mathbf{R}^{n+1}$ respectively, and σ the second fundamental form of M^n in $\mathbf{R}^{n+1}$. Then, the Gauss formula and the Weingarten formula are given respectively by

$$\sigma(X, Y)=D_XY-\nabla_XY \tag{2.2}$$

(2.3) $$D_X\xi=-A_\xi(X)$$

for any $X, Y\in\mathcal{X}(M^n)$ ($\mathcal{X}(M^n)$ denotes the Lie algebra of all differentiable vector fields on M^n), where ξ is the unit normal vector field of M^n in $\mathbf{R}^{n+1}$ and $-A_\xi(X)$ denotes the tangential part of $D_X\xi$.

The tangential part $A_\xi(X)$ is related to the second fundamental form σ as follows:

(2.4) $$\langle\sigma(X, Y), \xi\rangle=\langle A_\xi(X), Y\rangle \quad \text{for any } X, Y\in\mathcal{X}(M^n).$$

Then, the Gauss, Codazzi equations are given respectively by

(2.5) $$\langle R(X, Y)Z, W\rangle=\langle\sigma(X, W), \sigma(Y, Z)\rangle-\langle\sigma(X, Z), \sigma(Y, W)\rangle$$

(2.6) $$(\nabla_X\sigma)(Y, Y)=(\nabla_Y\sigma)(X, Z)$$

for any $X, Y, Z, W\in\mathcal{X}(M^n)$, where R is the Riemannian curvature tensor of M^n defined by $R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[Y,X]}$ and $(\nabla_X\sigma)(Y, Z)=\nabla_X^\perp(\sigma(Y, Z))-\sigma(\nabla_XY, Z)-\sigma(Y, \nabla_XZ)$. We shall give some elementary formulae of an oriented hypersurface in $\mathbf{H}$ for the sake of later uses. Let (M^3, f) be an oriented hypersurface in $\mathbf{H}$. We denote by ξ the unit normal vector field of M^3 in $\mathbf{H}$. Then, we may easily see that $\{\xi i, \xi j, \xi k\}$ is a global orthonormal frame field on M^3.

By (2.1) and (2.3), we get

(2.7) $$\begin{aligned}\nabla_X(\xi i)&=\sigma(X, \xi j)k-\sigma(X, \xi k)j\,,\\ \nabla_X(\xi j)&=\sigma(X, \xi k)i-\sigma(X, \xi i)k\,,\\ \nabla_X(\xi k)&=\sigma(X, \xi i)j-\sigma(X, \xi j)i\,.\end{aligned}$$

We put the shape operator A_ξ by

(2.8) $$A_\xi=\begin{pmatrix}\alpha & \lambda & \mu\\ \lambda & \beta & \nu\\ \mu & \nu & \gamma\end{pmatrix}$$

where $\alpha=\langle\sigma(\xi i, \xi i), \xi\rangle$. $\beta=\langle\sigma(\xi j, \xi j), \xi\rangle$, $\gamma=\langle\sigma(\xi k, \xi k), \xi\rangle$, $\lambda=\langle\sigma(\xi i, \xi j), \xi\rangle$, $\mu=\langle\sigma(\xi i, \xi k), \xi\rangle$ and $\nu=\langle\sigma(\xi j, \xi k), \xi\rangle$. Then they are differentiable functions on M^3. By (2.5) and (2.8), the Ricci tensor ρ is given by

(2.9) $$\begin{pmatrix}\rho(\xi i, \xi i) & \rho(\xi i, \xi j) & \rho(\xi i, \xi k)\\ \rho(\xi j, \xi i) & \rho(\xi j, \xi j) & \rho(\xi j, \xi k)\\ \rho(\xi k, \xi i) & \rho(\xi k, \xi j) & \rho(\xi k, \xi k)\end{pmatrix} =\begin{pmatrix}\alpha(\beta+\gamma)-\lambda^2-\mu^2 & \gamma\lambda-\mu\nu & \beta\mu-\nu\lambda\\ \gamma\lambda-\mu\nu & \beta(\gamma+\alpha)-\nu^2-\lambda^2 & \alpha\nu-\lambda\mu\\ \beta\mu-\nu\lambda & \alpha\nu-\lambda\mu & \gamma(\alpha+\beta)-\mu^2-\nu^2\end{pmatrix}.$$

3. Codazzi equation, connection and differential forms with respect to the associated orthonormal frame field.

For the sake of later uses, we shall give some formulae. From the definition of $\nabla\sigma$, (2.6) and (2.7), we have

$$
\begin{aligned}
&\langle(\nabla_{\xi i}\sigma)(\xi j, \xi j), \xi\rangle=\xi i(\beta)+2\rho(\xi j, \xi k)=\xi j(\lambda)-\rho(\xi j, \xi k)\,,\\
&\langle(\nabla_{\xi i}\sigma)(\xi k, \xi k), \xi\rangle=\xi i(\gamma)-2\rho(\xi j, \xi k)=\xi k(\mu)+\rho(\xi j, \xi k)\,,\\
&\langle(\nabla_{\xi j}\sigma)(\xi k, \xi k), \xi\rangle=\xi j(\gamma)+2\rho(\xi k, \xi i)=\xi k(\nu)-\rho(\xi k, \xi i)\,,\\
(3.1)\quad &\langle(\nabla_{\xi j}\sigma)(\xi i, \xi i), \xi\rangle=\xi j(\alpha)-2\rho(\xi k, \xi i)=\xi i(\lambda)+\rho(\xi k, \xi i)\,,\\
&\langle(\nabla_{\xi k}\sigma)(\xi i, \xi i), \xi\rangle=\xi k(\alpha)+2\rho(\xi i, \xi j)=\xi i(\mu)-\rho(\xi i, \xi j)\,,\\
&\langle(\nabla_{\xi k}\sigma)(\xi j, \xi j), \xi\rangle=\xi k(\beta)-2\rho(\xi i, \xi j)=\xi j(\nu)+\rho(\xi i, \xi j)\,,\\
&\langle(\nabla_{\xi i}\sigma)(\xi j, \xi k), \xi\rangle=\xi i(\nu)+\alpha(\gamma-\beta)+\lambda^2-\mu^2\,,\\
&=\xi j(\mu)+\beta(\alpha-\gamma)+\nu^2-\lambda^2=\xi k(\lambda)+\gamma(\beta-\alpha)+\mu^2-\nu^2\,.
\end{aligned}
$$

Next, we shall give another expression of the connection of M^3 with respect to the associated orthonormal frame field $\{\xi i, \xi j, \xi k\}$. We define the map ι by

$$
\begin{aligned}
\iota\colon \mathfrak{X}(M^3) &\longrightarrow \{\mathbf{R}^3\text{-valued } C^\infty\text{-functions on } M^3\}\\
X &\longmapsto {}^t(\langle X, \xi i\rangle, \langle X, \xi j\rangle, \langle X, \xi k\rangle)\,.
\end{aligned}
$$

Then, for any $Y\in\mathfrak{X}(M^3)$, we get

$$Y=\iota(Y)\cdot \boldsymbol{f} \tag{3.2}$$

where "$\cdot$" is a formal inner product of $\mathbf{R}^3$ and $\boldsymbol{f}={}^t(\xi i, \xi j, \xi k)$. Then the induced connection ∇ is given by

$$\nabla_X Y=\{X(\iota(Y))+\iota(Y)\times\Theta(X)\}\cdot\boldsymbol{f} \tag{3.3}$$

where $\Theta(X):={}^t(\langle\sigma(X, \xi i), \xi\rangle, \langle\sigma(X, \xi j), \xi\rangle, \langle\sigma(X, \xi k), \xi\rangle)$ and $\times$ is the canonical exterior product of $\mathbf{R}^3$. In fact, by (2.7) and (3.2), we get

$$
\begin{aligned}
(3.4)\quad \nabla_X Y&=\nabla_X(\iota(Y)\cdot\boldsymbol{f})=X(\iota(Y))\cdot\boldsymbol{f}+\iota(Y)\cdot\nabla_X\boldsymbol{f}\\
&=X(\iota(Y))\cdot\boldsymbol{f}+\iota(Y)\cdot(\Theta(X)\times\boldsymbol{f})\,.
\end{aligned}
$$

On one hand, the property of the exterior product of product of $\mathbf{R}^3$, we get

$$\iota(Y)\cdot(\Theta(X)\times\boldsymbol{f})=(\iota(Y)\times\Theta(X))\cdot\boldsymbol{f}\,. \tag{3.5}$$

From (3.4) and (3.5), we have (3.3).

Lastly, we give the expression of the dual 1-forms and connection forms with respect to the associated orthonormal frame field $\{\xi i, \xi j, \xi k\}$. We put

$$f_1 := \xi i, \qquad f_2 := \xi j, \qquad f_3 := \xi k.$$

Let $\{\omega_i\}_{i=1,2,3}$ be the dual 1-forms on M^3 of $\{f_i\}_{i=1,2,3}$ and $\omega_{ij}(X) := \langle \nabla_X f_i, f_j \rangle$ the connection forms on M^3. From (2.7), we get

$$\omega_{12} = -\sum_{j=1}^{3} h_{3j}\omega_j, \qquad \omega_{23} = -\sum_{j=1}^{3} h_{1j}\omega_j, \qquad \omega_{31} = -\sum_{j=1}^{3} h_{2j}\omega_j. \tag{3.6}$$

where $h_{ij} := \langle \sigma(f_i, f_j), \xi \rangle$.

On the other hand, Cartan's structure equations are given by

$$d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j \tag{3.7}$$

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} + (1/2) \sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l \tag{3.8}$$

where $R_{ijkl} := \langle R(f_i, f_j)f_k, f_l \rangle$.

LEMMA 3.1. *The* 1-*form* ω_1 *is a contact form if and only if* $h_{22}+h_{33}=\beta+\gamma \neq 0$ *everywhere on* M^3.

PROOF. By (3.6) and (3.7), we get

$$d\omega_1 = -h_{13}\omega_1 \wedge \omega_2 - h_{12}\omega_3 \wedge \omega_1 + (h_{22}+h_{33})\omega_2 \wedge \omega_3. \tag{3.9}$$

On one hand, the 1-form ω_1 is a contact form if and only if $\omega_1 \wedge d\omega_1 \neq 0$ everywhere on M^3. By (3.9), we get

$$0 \neq \omega_1 \wedge d\omega_1 = (h_{22}+h_{33})\omega_1 \wedge \omega_2 \wedge \omega_3.$$

4. Circles of the hypersurfaces in H.

In this section, we suppose that (M^3, f) is an oriented hypersurface in **H** and ξ is the unit normal vector field of M^3 in **H**.

PROPOSITION 4.1.

(1) *Any integral curve of* ξi *is a circle* (*an extrinsic circle*) *in* **H** *if and only if*

$$\langle (\nabla_{\xi k}\sigma)(\xi i, \xi i), \xi \rangle + \rho(\xi i, \xi j) = 0,$$

$$\langle (\nabla_{\xi j}\sigma)(\xi i, \xi i), \xi \rangle - \rho(\xi i, \xi k) = 0,$$

$$\langle (\nabla_{\xi i}\sigma)(\xi i, \xi i), \xi \rangle = 0.$$

(2) *Any integral curve of* ξi *is a circle* (*an intrinsic circle*) *in* M^3 *if and*

only if

$$\langle(\nabla_{\xi k}\sigma)(\xi i, \xi i), \xi\rangle+\rho(\xi i, \xi j)=\langle\sigma(\xi i, \xi i), \sigma(\xi i, \xi j)\rangle=0,$$

$$\langle(\nabla_{\xi j}\sigma)(\xi i, \xi i), \xi\langle-\rho(\xi i, \xi k)+\langle\sigma(\xi i, \xi i), \sigma(\xi i, \xi k)\rangle=0,$$

(3) *Any integral curve of* ξi *is a geodesic in* M^3 *if and only if*

$$\langle\sigma(\xi i, \xi j), \xi\rangle=\langle\sigma(\xi i, \xi k), \xi\rangle=0.$$

PROOF. In [6], a curve $c(t)$ in $\mathbf{R}^n$ is a circle in $\mathbf{R}^n$ if and only if

$$D_{\dot c}(D_{\dot c}\dot c)+\|D_{\dot c}\dot c\|^2\dot c=0.$$

Therefore, the integral curve of ξi is a circle in $\mathbf{H}$ if and only if

$$D_{\xi i}(D_{\xi i}(\xi i))+\|D_{\xi i}(\xi i)\|^2\xi i=0. \tag{4.1}$$

By (2.2) and (2.7), we get

$$\begin{aligned}D_{\xi i}(D_{\xi i}(\xi i))&=D_{\xi i}(\nabla_{\xi i}(\xi i)+\sigma(\xi i, \xi i))\\&=\nabla_{\xi i}(\nabla_{\xi i}(\xi i))+\sigma(\xi i, \nabla_{\xi i}(\xi i))-A_{\sigma(\xi i, \xi i)}(\xi i)+\nabla^{\perp}_{\xi i}(\sigma(\xi i, \xi i)).\end{aligned} \tag{4.2}$$

On one hand, by (2.7) and (2.8), we get

$$\sigma(\xi i, \nabla_{\xi i}(\xi i))=0, \qquad \sigma(\xi i, \xi i)=\alpha\xi. \tag{4.3}$$

By (2.7), (4.2) and (4.3), we get

$$\begin{aligned}&D_{\xi i}(D_{\xi i}(\xi i))\\&=\nabla_{\xi i}\{\langle\sigma(\xi i, \xi j), \xi\rangle\xi k-\langle\sigma(\xi i, \xi k), \xi\rangle\xi j\}-\alpha A_{\xi}(\xi i)+(\nabla_{\xi i}\sigma)(\xi i, \xi i)\\&=(\xi i\langle\sigma(\xi i, \xi j), \xi\rangle)\xi k+(\langle\sigma(\xi i, \xi j), \xi\rangle)\nabla_{\xi i}(\xi k)-(\xi i\langle\sigma(\xi i, \xi k), \xi\rangle)\xi j\\&\quad-(\langle\sigma(\xi i, \xi k), \xi\rangle)\nabla_{\xi i}(\xi j)-\alpha\{\alpha\xi i+\lambda\xi j+\mu\xi k\}+(\nabla_{\xi i}\sigma)(\xi i, \xi i)\\&=\{\langle(\nabla_{\xi i}\sigma)(\xi i, \xi j)+\sigma(\nabla_{\xi i}(\xi i), \xi j)+\sigma(\xi i, \nabla_{\xi i}(\xi j)), \xi\rangle\}\xi k\\&\quad+\langle\sigma(\xi i, \xi j), \xi\rangle\{\langle\sigma(\xi i, \xi i), \xi\rangle\xi j-\langle\sigma(\xi i, \xi j), \xi\rangle\xi i\}\\&\quad-\{\langle(\nabla_{\xi i}\sigma)(\xi i, \xi k)+\sigma(\nabla_{\xi i}(\xi i), \xi k)+\sigma(\xi i, \nabla_{\xi i}(\xi k)), \xi\rangle\}\xi j\\&\quad-\langle\sigma(\xi i, \xi k), \xi\rangle\{\langle\sigma(\xi i, \xi k), \xi\rangle\xi i-\langle\sigma(\xi i, \xi i), \xi\rangle\xi k\}\\&\quad-\alpha^2\xi i-(\alpha\lambda)\xi j-(\alpha\mu)\xi k+(\nabla_{\xi i}\sigma)(\xi i, \xi i)\\&=-\{\alpha^2+\lambda^2+\mu^2\}\xi i-\{\langle(\nabla_{\xi i}\sigma)(\xi i, \xi k)+\sigma(\nabla_{\xi i}(\xi i), \xi k)+\sigma(\xi i, \nabla_{\xi i}(\xi k)), \xi\rangle\}\xi j\\&\quad+\{\langle\nabla_{\xi i}\sigma)(\xi i, \xi j)+\sigma(\nabla_{\xi i}(\xi i), \xi j)+\sigma(\xi i, \nabla_{\xi i}(\xi j)), \xi\rangle\}\xi k+(\nabla_{\xi i}\sigma)(\xi i, \xi i).\end{aligned} \tag{4.4}$$

On the other hand, by (2.7), we get

$$
\begin{aligned}
&\langle\sigma(\xi i, \nabla_{\xi i}(\xi k)), \xi\rangle=\langle\sigma(\xi i, \nabla_{\xi i}(\xi j)), \xi\rangle=0\,, \\
&\langle\sigma(\nabla_{\xi i}(\xi i), \xi k), \xi\rangle=\rho(\xi i, \xi j)\,, \\
&\langle\sigma(\nabla_{\xi i}(\xi i), \xi j), \xi\rangle=-\rho(\xi i, \xi k)\,, \\
&\|D_{\xi i}(\xi i)\|^2=\alpha^2+\lambda^2+\mu^2\,.
\end{aligned} \tag{4.5}
$$

By (4.1), (4.4) and (4.5), we have (1). Similarly, we have (2). By (2.7), we have (3) immediately. □

THEOREM 4.2. *Let* (M^3, *f*) *be an oriented hypersurface in* **H** *and* ξ *the unit normal vector field of* M^3 *in* **H**. *If both integral curves of the two vector fields* $\{\xi i, \xi j\}$ (*or* $\{\xi j, \xi k\}$ *or* $\{\xi k, \xi i\}$) *are intrinsic circles and extrinsic circles simultaneously, then*

(1) M^3 *is locally isometric to a* 3*-dimensional round sphere in* **H** *and the immersion f is totally umbilic,*

or

(2) M^3 *is locally isometric to* $M^1\times \boldsymbol{R}^2$ (M^1 *is a* 1*-dimensional Riemannian manifold*) *and the immertion f is a locally product one.*

PROOF. We suppose that the integral curves of $\{\xi i, \xi j\}$ are intrinsic circles and extrinsic circles simultaneously. By the assumtion and (1), (2) of Proposition 4.1, we get

$$\alpha\lambda=\alpha\mu=\beta\lambda=\beta\mu=0 \quad \text{on } M^3\,. \tag{4.6}$$

First, we assume that $\alpha\beta$ is not identically zero. Let U be a connected component of the set $\{p\in M^3 | (\alpha\beta)(p)\neq 0\}$. Then, by (4.6), we get

$$\lambda=\mu=\nu=0 \quad \text{on } U\,. \tag{4.7}$$

Hence $\{\xi i, \xi j, \xi k\}$ are eigenvector fields of A_ξ on U. By $(3.7)_7$, we have

$$\alpha(\gamma-\beta)=\beta(\alpha-\gamma)=\gamma(\beta-\alpha) \quad \text{on } U\,. \tag{4.8}$$

By (4.8), we have $\alpha=\beta=\gamma$ on U. Hence each point of U is an umbilical point, so U is an open and closed non-empty subset in M^3. Since M^3 is connected, it is a round sphere.

Next, we assume that $\alpha\beta$ is identically zero on M^3. In this case, the proof is divided the following three cases

Case (1) α is not identically zero (or β is not identically zero).

Case (2) α is identically zero on M^3 and there exist a point $p\in M^3$ such that $\beta(p)\neq 0$.

Case (3) α and β are identically zero.

Case (1): Let U be a connected component of $\{p\in M^3 \mid \alpha(p)\neq 0\}$. Then we get

$$\beta=\lambda=\mu=0 \quad \text{on } U. \tag{4.9}$$

By the assumption (the integral curve of ξj is an intrinsic and extrinsic circle simultaneously), we get

$$\begin{aligned} &\langle(\nabla_{\xi j}\sigma)(\xi j, \xi j), \xi\rangle+\rho(\xi j, \xi k)=0, \\ &\langle(\nabla_{\xi k}\sigma)(\xi j, \xi j), \xi\rangle-\rho(\xi j, \xi i)=0, \\ &\langle(\nabla_{\xi j}\sigma)(\xi j, \xi j), \xi\rangle=0. \end{aligned} \tag{4.10}$$

By $(3.1)_1$, (4.9) and (4.10), we get

$$\rho(\xi j, \xi k)=0 \quad \text{on } U. \tag{4.11}$$

By (4.9) and (4.11), we get

$$\rho(\xi i, \xi j)=\rho(\xi j, \xi k)=\rho(\xi k, \xi i)=0 \quad \text{on } U.$$

Hence Theorem B in [3], we have each point of U is an umbilical point or locally flat. The former case, by (4.9), we get $\alpha=\beta=\gamma=\lambda=\mu=\nu=0$. This is a contradiction. Hence the former case does not occur. The latter case, by (2.9), we get

$$\gamma=\nu=0 \quad \text{on } U. \tag{4.12}$$

From (1) of Proposition 4.1, (3.1) and (4.12), we get

$$X(\alpha)=0 \quad \text{for any } X\in T_pM^3 \quad (p\in U). \tag{4.13}$$

Hence α is nonzero constant on U. Therefore U is an open and closed subset in M^3, since M^3 is connected, we have $U=M^3$. By (4.13) and Theorem B in [3], M^3 is locally isometric to $S^1\times\mathbf{R}^2$ (where S^1 is a circle in $\mathbf{R}^2$).

Case (2): By the same argument of Case (1), we get the conclusion.

Case (3): Taking account of the assumption, (3.1), (1) of Proposition 4.1 and (4.10), we get

$$\rho(\xi i, \xi j)=\rho(\xi j, \xi k)=\rho(\xi k, \xi i)=0 \quad \text{on } M^3.$$

Hence Theorem B in [3], we have the desired conclusion.

5. The Proof of Theorem A.

LEMMA 5.1. *Each integral curve of the associated orthonormal frame field* $\{\xi i, \xi j, \xi k\}$ *of* M^3 *is a circle in* $\mathbf{H}$ *if and only if*

(1) $\xi j(\alpha)-3\rho(\xi i, \xi k)=0$,

(2) $\xi k(\alpha)+3\rho(\xi i, \xi j)=0$,

(3) $\xi k(\beta)-3\rho(\xi i, \xi j)=0$,

(4) $\xi i(\beta)+3\rho(\xi j, \xi k)=0$,

(5) $\xi i(\gamma)-3\rho(\xi j, \xi k)=0$,

(6) $\xi j(\gamma)+3\rho(\xi k, \xi i)=0$,

(7) $\xi i(\alpha)=\xi i(\lambda)=\xi i(\mu)=0$,

(8) $\xi j(\beta)=\xi j(\nu)=\xi j(\lambda)=0$,

(9) $\xi k(\gamma)=\xi k(\mu)=\xi k(\nu)=0$.

PROOF. From (1) of Proposition 4.1, (3.1), by the direct calculation, we can easily get the desired equations above. □

From Lemma 5.1, we may note that *the mean curvature of* M^3 *is constant* if M^3 satisfies the assumption of Theorem A.

LEMMA 5.2. *If each integral curve of the associated orthonormal frame field* $\{\xi i, \xi j, \xi k\}$ *of* M^3 *is a circle in* $\mathbf{H}$ *and if* $\{\omega_i\}_{i=1,2,3}$ *are contact forms, then* λ, μ *and* ν *are constant functions on* M^3.

PROOF. By (2.7), we get

$$(5.1)\qquad \begin{aligned} [\xi i, \xi j]&=\mu\xi i+\nu\xi j-(\alpha+\beta)\xi k,\\ [\xi j, \xi k]&=\lambda\xi j+\mu\xi k-(\beta+\gamma)ki,\\ [\xi k, \xi i]&=\nu\xi k+\lambda\xi i-(\gamma+\alpha)\xi j. \end{aligned}$$

By the definition of the Lie bracket, and (7), (8) of Lemma 5.1, we get

$$(5.2)\qquad [\xi i, \xi j](\lambda)=\xi i(\xi j(\lambda))-\xi j(\xi i(\lambda))=0.$$

By (5.1), (5.2) and (7), (8) of Lemma 5.1, we get

$$0=[\xi i, \xi j](\lambda)=-(\alpha+\beta)\xi k(\lambda).$$

Since ω_1 is a contact form and Lemma 3.1, we have

$$\xi k(\lambda)=0.$$

From this and (7), (8) of Lemma 5.1, λ is a constant function on M^3. Similarly, we see that μ and ν are constant functions on M^3. □

LEMMA 5.3. *Let* (M^3, f) *be an oriented hypersurface in* **H**. *If each integral curve of the associated orthonormal frame field* $\{\xi i, \xi j, \xi k\}$ *is a circle in* **H**. *Then we have*

(1) $(\alpha+\beta)\rho(\xi i, \xi j)+\nu\rho(\xi k, \xi i)+3\mu\rho(\xi j, \xi k)=0$,

(2) $(\gamma+\alpha)\rho(\xi k, \xi i)+\nu\rho(\xi i, \xi j)+3\lambda\rho(\xi j, \xi k)=0$,

(3) $(\beta+\gamma)\rho(\xi j, \xi k)+\mu\rho(\xi i, \xi j)+3\lambda\rho(\xi k, \xi i)=0$,

(4) $(\alpha+\beta)\rho(\xi i, \xi j)+\mu\rho(\xi j, \xi k)+3\nu\rho(\xi k, \xi i)=0$,

(5) $(\gamma+\alpha)\rho(\xi k, \xi i)+\lambda\rho(\xi j, \xi k)+3\nu\rho(\xi i, \xi j)=0$,

(6) $(\beta+\gamma)\rho(\xi j, \xi k)+\lambda\rho(\xi k, \xi i)+3\mu\rho(\xi i, \xi j)=0$,

(7) $\mu\rho(\xi i, \xi j)-\lambda\rho(\xi k, \xi i)=0$,

(8) $\lambda\rho(\xi j, \xi k)-\nu\rho(\xi i, \xi j)=0$,

(9) $\nu\rho(\xi k, \xi i)-\mu\rho(\xi k, \xi j)=0$.

PROOF. By (5.1) and (1), (2), (7) of Lemma 5.1, we get

$$[\xi i, \xi j](\alpha)=\mu\xi i(\alpha)+\nu\xi j(\alpha)-(\alpha+\beta)\xi k(\alpha)$$
$$=3\{\nu\rho(\xi k, \xi i)+(\alpha+\beta)\rho(\xi i, \xi j)\}. \tag{5.4}$$

By (2.9), (1), (4) of Lemma 5.1, Lemma 5.2 and the definition of Lie bracket, we get

$$[\xi i, \xi j](\alpha)=\xi i(\xi j(\alpha))-\xi j(\xi i(\alpha))=\xi i(\xi j(\alpha))$$
$$=3\xi i(\rho(\xi k, \xi i))=3\xi i(\beta\mu-\nu\lambda)=3\xi i(\beta)\mu$$
$$=-9\mu\rho(\xi j, \xi k). \tag{5.5}$$

Hence, by (5.4) and (5.5), we have (1). Similarly, by (5.1) and (1), (2), (7) of Lemma 5.1, we get

$$[\xi j, \xi k](\alpha)=\lambda\xi j(\alpha)+\mu\xi k(\alpha)-(\beta+\gamma)\xi i(\alpha)$$
$$=3\{\lambda\rho(\xi k, \xi i)-\mu\rho(\xi i, \xi j)\}. \tag{5.6}$$

By (2.9), Lemma 5.1, Lemma 5.2 and the definition of Lie bracket, we get

$$[\xi j, \xi k](\alpha)=\xi j(\xi k(\alpha))-\xi k(\xi j(\alpha))=\xi j(-3\rho(\xi i, \xi j))-\xi k(3\rho(\xi k, \xi i))$$
$$=-3\{\xi j(\gamma\lambda-\mu\nu)+\xi k(\beta\mu-\nu\lambda)\}=-3\{\xi j(\gamma)\lambda+\xi k(\beta)\mu\}$$
$$=9\{\lambda\rho(\xi k, \xi i)-\mu\rho(\xi i, \xi j)\}. \tag{5.7}$$

From (5.6) and (5.7), we have (7). Similarly, by calculating

$$[\xi k, \xi i](\alpha),\ [\xi i, \xi j](\beta),\ [\xi j, \xi k](\beta),\ [\xi k, \xi i](\beta),$$
$$[\xi i, \xi j](\gamma),\ [\xi j, \xi k](\gamma) \text{ and } [\xi k, \xi i](\gamma),$$

we have (3)$\sim$(6), (8) and (9). □

Now, we are in a crucial position to prove Theorem A. For simplicity, we put

$$A:=\rho(\xi i, \xi j), \qquad B:=\rho(\xi j, \xi k), \qquad C:=\rho(\xi k, \xi i).$$

Then the equations of Lemma 5.3 can be rewritten as follows:

$$(5.8)\qquad \begin{aligned} &(\alpha+\beta)A+\nu C+3\mu B=0,\\ &(\beta+\gamma)B+\mu A+3\lambda C=0,\\ &(\gamma+\alpha)C+\lambda B+3\nu A=0,\\ &(\alpha+\beta)A+\mu B+3\nu C=0,\\ &(\beta+\gamma)B+\lambda C+3\mu A=0,\\ &(\gamma+\alpha)C+\nu A+3\lambda B=0,\\ &\mu A-\lambda C=\lambda B-\nu A=\nu C-\mu B=0. \end{aligned}$$

By (5.8), we get

$$(5.9)\qquad \{(\alpha+\beta)\lambda+4\mu\nu\}C=\{(\beta+\gamma)\nu+4\lambda\mu\}C=\{(\gamma+\alpha)\mu+4\nu\lambda\}C=0$$

We suppose that C is not identically 0 on M^3. Let U be a connected component of the set $\{p\in M^3 \mid C(p)\neq 0\}$. By (5.9), we get

$$(5.10)\qquad (\alpha+\beta)\lambda+4\mu\nu=(\beta+\gamma)\nu+4\lambda\mu=(\gamma+\alpha)\mu+4\nu\lambda=0 \quad \text{on } U.$$

By (5.10), we get

$$(5.11)\qquad \alpha\lambda\mu\nu+2\nu^2(\lambda^2+\mu^2)-2\lambda^2\mu^2=0 \quad \text{on } U.$$

We differentiate (5.11) by the direction ξj and by Lemma 5.2, we get

$$(5.12)\qquad \xi j(\alpha)\lambda\mu\nu=0 \quad \text{on } U.$$

Suppose that there is a point p in U such that $\lambda\mu\nu(p)=0$. Since λ, μ and ν are constant functions on U, because of Lemma 5.2, we may suppose that $\lambda=0$. By (5.10) and the fact that $\{\omega_i\}_{i=1,2,3}$ are contact forms on M^3, we get $\lambda=\mu=\nu=0$ on U. From this and (6) of Lemma 5.3, we have $C=0$ on U. This is a contradiction. Hence $\lambda\mu\nu\neq 0$ on U. By (5.12), we get

$$(5.13)\qquad \xi j(\alpha)=0 \quad \text{on } U.$$

By (5.11) and the same argument as above, we get

(5.14) $$\xi k(\alpha)=0 \quad \text{on } U.$$

By (5.13) and (5.14), (1), (2) of Lemma 5.1, we get

(5.15) $$\rho(\xi i, \xi j)=\rho(\xi i, \xi k)=0 \quad \text{on } U.$$

On one hand, since $\lambda\mu\nu\neq 0$ on U, by (5.15) and (8) of Lemma 5.3, we have $C=\rho(\xi j, \xi k)=0$ on U. This is a contradiction. Hence we have

(5.16) $$C=0 \quad \text{on } M^3.$$

By (5.8), (5.16) and $\{\omega_i\}_{i=1,2,3}$ are contact forms on M^3, we get

$$A=B=C=0.$$

This means that $\{\xi i, \xi j, \xi k\}$ is a Ricci adapted frame on M^3 and hence, by Theorem B in [3] (also see [4]), M^3 is totally umbilic or locally flat. On the other hand, by Lemmas 5.1 and 5.2, α, β, γ, λ, μ and ν are constant functions on M^3. Hence M^3 is a isoparametric hypersurface in $\mathbf{R}^4$. We conclude that M^3 is locally isometric to $S^1\times\mathbf{R}^2$ or a 3-dimensional sphere. This completes the proof of Theorem A. □

Now, we shall give an example $(S^1\times\mathbf{R}^2, f)$ of the second case of Theorem A.

Let b be a fixed unit vector of $\mathbf{H}$ and $c(t)$ the curve in $\mathbf{H}$ defined by

$$c(t)=\cos t\,(m_1bi+m_2bj+m_3bk)+(\sin t)b$$

where $(m_1)^2+(m_2)^2+(m_3)^2=1$ and $m_1m_2m_3\neq 0$ (fixed). We put $\mathbf{R}_b^2:=\mathrm{span}_{\mathbf{R}}\{b, mbi+m_2bj+m_3bk\}$. Let c and d be orthonomal vectors in $\mathbf{H}$ which are orthogonal to $\mathbf{R}_b^2$. We define the immersion f following:

$$f: S^1\times\mathbf{R}^2 \longrightarrow \mathbf{H}$$

$$(t, x, y) \longrightarrow (\cos t)(m_1bi+m_2bj+m_3bk)+(\sin t)b+xc+yd$$

By direct calculation, the second fundamental form with respect to $\{\xi i, \xi j, \xi k\}$ (where $\xi=c(t)$, is given by

$$A_\xi=-\begin{bmatrix}(m_1)^2 & m_1m_2 & m_1m_3\\ m_1m_2 & (m_2)^2 & m_2m_3\\ m_1m_3 & m_2m_3 & (m_3)^2\end{bmatrix}$$

Hence by Lemma 3.1, $\{\omega_i\}_{i=1,2,3}$ are contact forms on $S^1\times\mathbf{R}^2$.

6. Proof of Theorem B.

By the assumption, (each integral curve of ξi is a geodesic in M^3) and (3) of Proposition 4.1, the shape operator A_ξ with respect to the associated orthonormal frame field $\{\xi i, \xi j, \xi k\}$ is given by

$$A_\xi=\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & \nu \\ 0 & \nu & \gamma \end{bmatrix}. \tag{6.1}$$

In particular, we easily see that ξi is a eigenvector field of A_ξ. Hence, we put $e_1=\xi i$ and e_2, e_3 other (local) eigenvector field of A_ξ (i.e., $A_\xi(e_i)=\lambda_i e_i$, $i=1,2,3$). Then we may put

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \xi i \\ \xi j \\ \xi k \end{bmatrix} \tag{6.2}$$

where θ is a continious function M^3, The functions α, β, γ and ν are represented by the eigenvalues $\{\lambda_i\}_{i=1,2,3}$ of A_ξ and θ in such a way that

$$\begin{aligned} &\alpha=\lambda\,, \qquad \beta=\langle\sigma(\xi j, \xi j), \xi\rangle=\lambda_2\cos^2\theta+\lambda_3\sin^2\theta\,, \\ &\gamma=\langle\sigma(\xi k, \xi k), \xi\rangle=\lambda_2\sin^2\theta+\lambda_3\cos^2\theta\,, \\ &\nu=\langle\sigma(\xi j, \xi k), \xi\rangle=(\lambda_3-\lambda_2)\sin\theta\cos\theta\,. \end{aligned} \tag{6.3}$$

LEMMA 6.1. *If each integral ccrve of ξi is a geodesic in* M^3 *and θ is a differentiable function, then we have*

$$\begin{aligned} &\nabla_{e_1}e_1=0\,, \qquad \nabla_{e_2}e_1=\lambda_2 e_3\,, \qquad \nabla_{e_3}e_1=-\lambda_3 e_2\,, \\ &\nabla_{e_1}e_2=(\lambda_1+e_1(\theta))e_3\,, \quad \nabla_{e_2}e_2=-e_2(\theta)\,, \quad \nabla_{e_3}e_2=\lambda_3 e_1-e_3(\theta)e_3\,, \\ &\nabla_{e_1}e_3=(\lambda_1+e_1(\theta))e_2\,, \quad \nabla_{e_2}e_3=-\lambda_2 e_1+e_2(\theta)e_2\,, \quad \nabla_{e_3}e_3=e_3(\theta)e_2\,. \end{aligned}$$

PROOF. We put $e={}^t(e_1, e_2, e_3)$. By (6.2), we get

$$\boldsymbol{f}={}^t\varphi e \tag{6.4}$$

where

$$\varphi=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

By (3.3) and (6.4), we get

(6.5) $\nabla_X Y=[\varphi\{X(\iota(Y))+\iota(Y)\times\Theta(X)\}]\cdot e$ for any $X, Y\in\mathfrak{X}(M^3)$ ·

From the definition of ι, Θ and $\{e_i\}_{i=1,2,3}$, we get

$$\iota(e_1)=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \iota(e_2)=\begin{bmatrix}0\\\cos\theta\\-\sin\theta\end{bmatrix},\quad \iota(e_3)=\begin{bmatrix}0\\\sin\theta\\\cos\theta\end{bmatrix}, \tag{6.6}$$

$$\Theta(e_1)=\begin{bmatrix}\lambda_1\\0\\0\end{bmatrix},\quad \Theta(e_2)=\lambda_2\begin{bmatrix}0\\\cos\theta\\-\sin\theta\end{bmatrix},\quad \Theta(e_3)=\lambda_3\begin{bmatrix}0\\\sin\theta\\\cos\theta\end{bmatrix}. \tag{6.7}$$

From the definition of vector cross product of $\mathbf{R}^3$, we get

$$\iota(e_1)\times\iota(e_2)=\iota(e_3)\,,\qquad \iota(e_2)\times\iota(e_3)=\iota(e_1)\,,\qquad \iota(e_3)\times\iota(e_1)=\iota(e_2)\,. \tag{6.8}$$

From (6.4)~(6.8), we get

$$\nabla_{e_1}e_2=\left[\varphi\begin{bmatrix}0\\-(\lambda_1+e_1(\theta)\sin\theta\\-(\lambda_1+e_1(\theta))\cos\theta\end{bmatrix}\right]\cdot e=-(\lambda_1+e_1(\theta))e_3\,.$$

By the same calculation, we get the other equalities. □

LEMMA 6.2. *If each integral curve of ξi is a geodesic in M^3 and θ is a differentiable function M^3, then we have*

(1) $(\lambda_3-\lambda_2)(\lambda_1+e_1(\theta))+\lambda_2(\lambda_3-\lambda_1)=0\,,$

(2) $(\lambda_3-\lambda_2)(\lambda_1+e_1(\theta))+\lambda_3(\lambda_1-\lambda_2)=0\,,$

(3) $2\lambda_2\lambda_3-\lambda_1(\lambda_2+\lambda_3)=0\,,$

(4) $e_2(\lambda_3)+e_3(\theta)(\lambda_2-\lambda_3)=0\,,$

(5) $e_3(\lambda_2)+e_2(\theta)(\lambda_2-\lambda_3)=0\,,$

(6) $e_1(\lambda_2)=e_1(\lambda_3)=0\,,$

(7) $e_2(\lambda_1)=e_3(\lambda_1)=0\,.$

PROOF. By (2.6) and Lemma 6.1, we get

$$\begin{aligned}0&=(\nabla_{e_1}A_\xi)(e_2)-(\nabla_{e_2}A_\xi)(e_1)\\&=\nabla_{e_1}(A_\xi(e_2))-A_\xi(\nabla_{e_1}e_2)-\nabla_{e_3}(A_\xi(e_1))+A_\xi(\nabla_{e_3}e_1)\\&=\nabla_{e_1}(\lambda_2e_2)-A_\xi(\nabla_{e_1}e_2)-\nabla_{e_2}(\lambda_1e_1)+A_\xi(\nabla_{e_3}e_1)\\&=e_1(\lambda_2)e_2-e_2(\lambda_1)e_1+\{(\lambda_1+e_1(\theta))(\lambda_3-\lambda_2)+\lambda_2(\lambda_3-\lambda_1)\}e_3\,.\end{aligned}$$

Hence we have (1), $(6)_1$ and $(7)_1$. By the same way, we get the remaining equalities. □

LEMMA 6.3. $\lambda_2=\lambda_3$ *on* M^3.

PROOF. We suppose it is not so and derive the contradiction. Let V be a set $\{p\in M^3 \mid \lambda_2(p)\neq\lambda_3(p)\}$ $(\neq\phi)$ and U a connected component of V. Then θ is differentiable function on U. Hence we can apply Lemma 6.2. By (3) of Lemma 6.2, we get

$$\lambda_2+\lambda_3\neq 0 \quad \text{on } U. \tag{6.8}$$

In fact, if there is a point x in U at which $(\lambda_2+\lambda_3)(x)=0$, by (3) of Lemma 6.2, we get $\lambda_2=\lambda_3=0$ at x. This is a contradiction. □

Also, by (3), (6) of Lemma 6.2, we get

$$e_1(\lambda_1)(\lambda_2+\lambda_3)=0 \quad \text{on } U. \tag{6.9}$$

By (6.8) and (6.9), we have

$$e_1(\lambda_1)=0 \quad \text{on } U. \tag{6.10}$$

By (6.10) and (7) of Lemma 6.2, we see that $\lambda_1=\alpha$ $(\neq 0)$ is a constant function on U. Let $\tau(t)=\exp_p(t\xi i)$ be a geodesic emanating from $p\in U$ in the direction $\xi i\in T_p(M^3)$. Since M^3 is complete, $\tau(t)\in M^3$ for any $t\in\mathbf{R}$. More, by (6) of Lemma 6.2, we have $\tau(t)\in U$ for any $t\in\mathbf{R}$. From these facts and (1) of Proposition 4.1, $\tau(t)$ is *a closed circle in* $\mathbf{H}$ and contained in U. By $(2.9)_1$, we get

$$\xi i(\beta)=-3\rho(\xi j, \xi k) \quad (d\beta(\tau(t))/dt=-3\rho(\xi j, \xi k)). \tag{6.11}$$

Since β is a differentiable function on M^3, $\beta(\tau(t))$ is a periodic function on $\tau(t)$. By the assumption ($\rho(\xi j, \xi k)\geqq 0$ or $\leqq 0$) and (6.11), $\beta(\tau(t))$ is a decreasing (or increasing) function on the closed circle $\tau(t)$. Hence $\beta(\tau(t))$ is a constant function. Therefore, by (6.11), $\{\xi i, \xi j, \xi k\}$ is a Ricci adapted frame on U. By Theorem B in [3], U is totally umbilic or locally flat. We see immediately that U cannot be totally umbilic. So, U must be locally flat and hence the following two cases are possible:

(*) $\lambda_1=\lambda_2=0$ and $\lambda_3\neq 0$

or

(**) $\lambda_1=\lambda_3=0$ and $\lambda_2\neq 0$.

However, both two cases contradicts the assumption $(\lambda_1=\alpha\neq 0)$. □

From (6.3) and Lemma 6.3, we get

(6.12) $$\alpha=\lambda_1, \quad \beta=\gamma \quad \text{and} \quad \nu=0.$$

By (6.12), we see that $\{\xi i, \xi j, \xi k\}$ are eigenvector field of A_ξ. Hence we may put $\theta=0$ identically on M^3. From this argument, the following two cases may occur:

(i) $\beta=\gamma$ is not identically 0 on M^3,

or

(ii) $\beta=\gamma=0$ identically on M^3.

Case (i). Let V be a connected component of the open set $\{p\in M^3 | \beta(p)\neq 0\}$. Since $\theta=0$, by (1), (2) of Lemma 6.2, we get

$$\alpha=\beta=\gamma=\lambda_1=\lambda_2=\lambda_3\neq 0 \qquad \text{on } V.$$

Hence each point of V is a umbilical point, so V is a open and closed subset in M^3. Since M^3 is connected, it is a round sphere.

Case (ii). In this case, M^3 is Ricci flat and $\alpha\neq 0$, hence it is (non-totally geodesic) flat. This completes the proof of Theorem B.

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