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MINIMAL SURFACES WITH CONSTANT CURVATURE AND KÄHLER ANGLE IN COMPLEX SPACE FORMS

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Introduction.

Minimal surfaces with constant Gaussian curvature in real space forms have been classified completely (cf. [Ca-2], [Ke-1], [Br] and their references). Next natural interest is to investigate minimal surfaces with constant Gaussian curvature in complex space forms, more generally in symmetric spaces. Prof. Kenmotsu posed the following problem: Classify minimal surfaces with constant Gaussian curvature in complex space forms.

Recently, minimal 2-spheres with constant Gaussian curvature in complex projective spaces were classified independently by [B-Oh] and [B-J-R-W]. [C-Z] studied pseudo-holomorphic curves of constant curvature in complex Grassmann manifolds. For an immersion φ of a Riemann surface M into a Kähler manifold N, the *Kähler angle* θ of φ is defined to be the angle between $Jd\varphi(\partial/\partial x)$ and $d\varphi(\partial/\partial y)$, where $z=x+\sqrt{-1}\,y$ is a local complex coordinate on M and J denotes the complex structure of N. Chern and Wolfson [Ch-W] pointed out the importance of the Kähler angle in the theory of minimal surfaces in Kähler manifolds. In [B-J-R-W] and [E-G-T] they investigated minimal 2-spheres in complex projective spaces and minimal surfaces in 2-dimensional complex space forms respectively in terms of the notion of Kähler angle.

In this paper we classify minimal surfaces with constant Gaussian curvature and constant Kähler angle in complex space forms.

THEOREM A. *Let M be a minimal surface with constant Gaussian curvature K immersed fully in a complex projective space $\boldsymbol{C}P^n$ of constant holomorphic sectional curvature $c>0$. Assume that the Kähler angle θ of M is constant. Then the following:*

(1) *If $K>0$, then there exists some k with $0\leqq k\leqq n$ such that $K=c/\{2k(n-k)+n\}$, $\cos\theta=K(n-2k)/c$ and M is an open submanifold of $\psi_{n,k}(S^2)$.*

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(2) *If* $K=0$, *then* $\cos\theta=0$, *i.e.*, M *is totally real.*

(3) $K<0$ *is impossible.*

THEOREM B. *Let* M *be a minimal surface with constant Gaussian curvature* K *immersed in a complex hyperbolic space* $\boldsymbol{C}H^n$ *of constant holomorphic sectional curvature* $c<0$. *If the Kähler angle* θ *of* M *is constant, then* M *is totally geodesic, i.e.,* M *is an open submanifold of* $\boldsymbol{C}H^1$ *in* $\boldsymbol{C}H^n$ ($K=c$, $\cos\theta=1$) *or* $\boldsymbol{R}H^2$ *in* $\boldsymbol{C}H^n$ ($K=c/4$, $\cos\theta=0$).

Refer to [B-Oh] and [B-J-R-W] about the minimal immersions $\psi_{n,k}$ in (1) of Theorem A. On (2) of Theorem A totally real flat minimal surfaces in complex projective spaces were classified essentially by Kenmotsu [Ke-2]. It seems not to be known if there is a minimal surface with constant Gaussian curvature and nonconstant Kähler angle in complex space forms of nonzero constant holomorphic sectional curvature.

Eells and Wood [Ee-W] introduced the notion of universal lift for a smooth map to a complex projective space in order to investigate harmonic maps from surfaces to complex projective spaces. On the other hand Bryant [Br] defined certain fundamental operators on the space of vector-valued forms on a Riemann surface, and classified minimal surfaces with constant Gaussian curvature in real space forms by utilizing those operators. In this paper we extend Bryant's operators to the operators acting on the space of vector bundle valued forms on a Riemann surface, and apply the extended fundamental operators to the universal life for minimal immersions of surfaces. By the argument analogous to that of Bryant, we show Theorem A. By the same method we also show Theorem B.

1. Fundamental operators on the space of vector bundle valued smooth functions.

Let M be a connected Riemann surface and g_M be a Riemannian metric compatible with the holomorphic structure of M. We do not assume that M is compact or that g_M is complete. Let $T^*_{(1,0)}M$ (resp. $T^*_{(0,1)}M$) be the complex line bundle of (1, 0)-forms (resp. (0, 1)-forms). Let $\{u, \bar{u}\}$ be a unitary basis of T_xM^C with $u\in T^{(1,0)}_xM$ and $\bar{u}\in T^{(0,1)}_xM$, and $\{\omega, \bar{\omega}\}$ be its dual basis. Denote by ∇^M the Riemannian connection of (M, g_M). The curvature form R^M and the Gaussian curvature K of M are defined by $R^M(V, W)=[\nabla^M_V, \nabla^M_W]-\nabla^M_{[V,W]}$ and $K=g_M(R^M(u, \bar{u})u, \bar{u})$. Put $\tau=T^*_{(1,0)}M$ and $\tau^{-1}=T^*_{(0,1)}M$. For $m\geqq 0$ we let τ^m (resp. τ^{-m}) be the m-th power tensor product of τ (resp. τ^{-1}). Using the identi-

fication $\omega^m=(\bar{\omega})^{-m}$ for all m, we have a canonical pairing $\tau^m\times\tau^k\to\tau^{m+k}$ for all m and k. Set $\mathcal{T}=\bigoplus_{m=-\infty}^{\infty}C^\infty(\tau^m)$ as a $\boldsymbol{Z}$-graded vector space.

Let E be a complex vector bundle over M with an indefinite Hermitian fibre metric $\langle\,,\rangle^E$ and a connection ∇^E compatible with $\langle\,,\rangle^E$. Le $C^\infty(E)$ denote the vector space of all smooth sections of E defined on M. Consider the tensor product bundle $E\otimes\tau^m(m\in\boldsymbol{Z})$. For each m we equip the bundle $E\otimes\tau^m$ with the tensor product connection $D=\nabla^E\otimes\nabla^M$. Set $\mathcal{E}=\bigoplus_{m=-\infty}^{\infty}C^\infty(E\otimes\tau^m)$ as a $\boldsymbol{Z}$-graded vector space. We have a pairing $\langle\,,\rangle:\mathcal{E}\times\mathcal{E}\to\mathcal{T}$ gotten by extending the indefinite Hermitian fibre metrix $\langle\,,\rangle^E$ of E in the obvious fashion. We define operators $D'_m: C^\infty(E\otimes\tau^m)\to C^\infty(E\otimes\tau^{m+1})$ and $D''_m: C^\infty(E\otimes\tau^m)\to C^\infty(E\otimes\tau^{m-1})$ by $D'_m\sigma=(D_u\sigma)\otimes\omega$, $D''_m\sigma=(D_{\bar{u}}\sigma)\otimes\bar{\omega}$ for $\sigma\in C^\infty(E\otimes\tau^m)$. We define the fundamental operators X, Y on $\mathcal{E}$ by $X=\bigoplus_{m=-\infty}^{\infty}D'_m$, $Y=\bigoplus_{m=-\infty}^{\infty}D''_m$. Set $\Delta=XY+YX$, the Laplace-Beltrami operator on each graded piece.

PROPOSITION 1.1. *Assume that the curvature form R^E of the bundle E satisfies the condition*

$$\Delta R^E=R^E(u,\bar{u})=\lambda\cdot I \tag{1.1}$$

for some real valued function λ on M. Then for any $\sigma\in C^\infty(E\otimes\tau^m)$,

$$[X,Y]\sigma=(\lambda-mK)\sigma\,. \tag{1.2}$$

Moreover if λ and K are constant, then we get

$$\begin{aligned}&[H,X]=K\cdot X\,,\quad [H,Y]=-K\cdot Y\,,\\ &[X,Y]=-H\,,\end{aligned} \tag{1.3}$$

where the operator H on $\mathcal{E}$ is defined by $H=-\bigoplus_{m=-\infty}^{\infty}(\lambda-mK)I_m$ and $I_m: C^\infty(E\otimes\tau^m)\to C^\infty(E\otimes\tau^m)$ is the identity.

PROOF. Let $\sigma\in C^\infty(E\otimes\tau^m)$ and write $\sigma=s\otimes(\omega)^m$ locally, where s is a local smooth section of E. Since $R^M(u,\bar{u})\omega=-K\cdot\omega$, $R^M(u,\bar{u})\bar{\omega}=K\cdot\bar{\omega}$, by (1.1) we have

$$\begin{aligned}[X,Y]\sigma&=(R^E(u,\bar{u})s)\otimes(\omega)^m+(-mK)s(\omega)^m\\ &=(\lambda-mK)\sigma\,.\end{aligned}$$

If λ and K are constant, we have the first formula of (1.3),

$$[H, X]\sigma = H(X\sigma) - X(H\sigma)$$
$$= (-\lambda + (m+1)K)X\sigma - (-\lambda + mK)X\sigma$$
$$= K \cdot X\sigma .$$

The second formula of (1.3) is similar. q.e.d.

REMARK. (1) In case $\lambda=0$ these are just the formulas used by [Br].

(2) Generally a holomorphic connection satisfying the condition (1.1) for some constant λ over a Kähler manifold is called an Einstein-holomorphic connection.

2. Harmonic map equation to a complex projective space.

Let $\boldsymbol{C}^{n+1}$ denote the complex $(n+1)$-space equipped with the standard Hermitian inner product $\langle v, w\rangle = \sum_{i=0}^{n} v^i \overline{w^i}$ for $v=(v^0, \cdots, v^n)$, $w=(w^0, \cdots, w^n) \in \boldsymbol{C}^{n+1}$. Let $\boldsymbol{CP}^n$ be an n-dimensional complex projective space and $\pi : \boldsymbol{C}^{n+1}\setminus\{0\} \to \boldsymbol{CP}^n$ be its canonical projection. $\boldsymbol{C}^{n+1}\setminus\{0\}$ is a principal bundle over $\boldsymbol{CP}^n$ with the structure group $\boldsymbol{C}^*$, where $\boldsymbol{C}^*$ denotes the group of non-zero complex numbers. For a positive constant c, set $S^{2n+1}(c)=\{v\in\boldsymbol{C}^{n+1}; \langle v, v\rangle = 1/c\}$. The Hopf fibration $\pi : S^{2n+1}(c/4) \to \boldsymbol{CP}^n$ is obtained by restricting the canonical projection $\pi : \boldsymbol{C}^{n+1}\setminus\{0\} \to \boldsymbol{CP}^n$. The Fubini-Study metric on $\boldsymbol{CP}^n$ with constant holomorphic sectional curvature $c(>0)$ is characterized by the fact that the Hopf fibration $\pi : S^{2n+1}(c/4) \to \boldsymbol{CP}^n$ is a Riemannian submersion. We endow $\boldsymbol{CP}^n$ with the Fubini-Study metric g of constant holomorphic sectional curvature c. Let L be the universal bundle over $\boldsymbol{CP}^n$; the fibre L_x over any $x\in\boldsymbol{CP}^n$ can be identified with the complex 1-dimensional subspace of $\boldsymbol{C}^{n+1}$ determined by x. Thus L is identified as a holomorphic subbundle of the trivial bundle $\underline{\boldsymbol{C}}^{n+1}=\boldsymbol{CP}^n\times\boldsymbol{C}^{n+1}$ over $\boldsymbol{CP}^n$. Let $L^\perp$ be the subbundle of $\underline{\boldsymbol{C}}^{n+1}$ whose fibre at x is the orthogonal complement of L_x in $\boldsymbol{C}^{n+1}$. $L^\perp \cong \underline{\boldsymbol{C}}^{n+1}/L$ can be given a holomorphic structure. We endow the bundles L and $L^\perp$ with the Hermitian connected structure induced from the Hermitian inner product $\langle , \rangle$ of $\boldsymbol{C}^{n+1}$. We give $L^*\otimes L^\perp$ the tensor product Hermitian connected structure, where L^* denotes the dual bundle of L. Then there exists a natural bundle isomorphism $h : T^{(1,0)}\boldsymbol{CP}^n \to L^*\otimes L^\perp$ preserving connects and satisfying $\langle h(Z), h(W)\rangle = (c/2)g(Z, \overline{W})$ for $Z, W \in T_x^{(1,0)}\boldsymbol{CP}^n$ (cf. [Ee-W, p. 224]).

Let $\varphi : M \to \boldsymbol{CP}^n$ be a smooth map from a Riemann surface to a complex projective space. We say φ is *full* if its image lies in no proper complex projective subspace of $\boldsymbol{CP}^n$. Denote by $d\varphi^{(1,0)}(\xi)$ the (1, 0)-component of $d\varphi(\xi)$ for

$\xi \in T_x M^c$.

Consider the exact sequence of vector bundles over CP^n:

$$0 \longrightarrow L \xrightarrow{i} \underline{C}^{n+1} \xrightarrow{j} L^{\perp} \longrightarrow 0$$

where i is the natural inclusion and j is given by the orthogonal projection along L. Tensoring with L^* and pulling back via a map $\varphi: M \to CP^n$ gives the exact sequence over M

$$0 \longrightarrow \varphi^{-1}(L^* \otimes L) \xrightarrow{i} \varphi^{-1}(L^* \otimes \underline{C}^{n+1}) \xrightarrow{j} \varphi^{-1}(L^* \otimes L^{\perp}) \longrightarrow 0 .$$

Note that the bundle $\varphi^{-1}(L^* \otimes L)$ has the "identity" section, which we denote simply by 1. We call the section $\Phi = i(1) \in C^{\infty}(\varphi^{-1}(L^* \otimes \underline{C}^{n+1}))$ the *universal lift* of φ (cf. [Ee-W]). We give the bundles $\varphi^{-1}L$, $\varphi^{-1}L^*$, $\varphi^{-1}L^{\perp}$, $\varphi^{-1}(L \otimes \underline{C}^{n+1})$ and $\varphi^{-1}(L_*, L^{\perp})$ the pull-back Hermitian connected structures. Pulling back $h: T^{(1,0)}CP^n \to L^* \otimes L^{\perp}$ by φ, we get a connection-preserving bundle isomorphism

$$h: \varphi^{-1}(T^{(1,0)}CP^n) \longrightarrow \varphi^{-1}(L^* \otimes L^{\perp}) \tag{2.1}$$

satisfying

$$\langle h((d\varphi)^{(1,0)}(\xi)),\ h((d\varphi)^{(1,0)}(\eta)) \rangle = (c/2) g((d\varphi)^{(1,0)}(\xi),\ \overline{((d\varphi)^{(1,0)}(\eta))})$$

for any $\xi, \eta \in T_x M^c$.

Set $E = \varphi^{-1}(L^* \otimes \underline{C}^{n+1}) = (\varphi^{-1}L^*) \otimes C^{n+1}$ and denote by D the covariant differentiation in the bundle E. We apply results of Section 1 to the bundle E and use the formulation and notation in Section 1.

Now we give a description of the curvature form for the bundle E. Let ω be the fundamental 2-form of (CP^n, g) defined by $\omega(Z, W) = g(Z, JW)$ for $Z, W \in T_x CP^n$, where J denotes the canonical complex structure CP^n. For any $V \in C^{\infty}(E)$ and $\rho \in C^{\infty}(\varphi^{-1}L)$,

$$\begin{aligned}(R^E(u, \bar{u})V)(\rho) &= R^{C^{n+1}}(u, \bar{u})(V(\rho)) - V(R^{\varphi^{-1}L}(u, \bar{u})\rho) \\ &= -V(\varphi^{-1}R^L(u, \bar{u})(\rho)) .\end{aligned}$$

Since it is known that the curvature form of the universal bundle L is given by $R^L = -(c/2)\sqrt{-1}\omega$, we get

$$R^E(u, \bar{u}) = (c/2)\sqrt{-1}(\varphi^*\omega)(u, \bar{u}) .$$

Hence we can write

$$AR^E = (c/2) \cdot \mu I , \tag{2.2}$$

where μ is a smooth function on M defined by

$$\mu = \sqrt{-1}(\varphi^*\omega)(u, \bar{u}) .$$

μ is called the *Kähler function* of a map φ (cf. [E-G-T, p. 573]). When φ is an isometric immersion, the function μ is related to the Kähler angle θ of φ by $\mu=\cos\theta$. Then $\mu=1$ (resp. $\mu=-1$) if and only if φ is holomorphic (resp. anti-holomorphic), and $\mu=0$ if and only if φ is totally real.

It is easily shown that if a smooth map $\varphi: M\to \boldsymbol{C}P^n$ satisfies $\mu\equiv 0$, there are a covering space $\nu: \tilde{M}\to M$ and a horizontal smooth map $\tilde{\varphi}: \tilde{M}\to S^{2n+1}(c/4)$ relative to the Hopf fibration $\pi: S^{2n+1}(c/4)\to \boldsymbol{C}P^n$ such that $\pi\circ\tilde{\varphi}=\varphi\circ\nu$. Moreover φ is harmonic if and only if $\tilde{\varphi}$ is harmonic. Therefore every minimal surface in $\boldsymbol{C}P^n$ with $\mu\equiv 0$ can be locally and isometrically lifted to a minimal surface in $S^{2n+1}(c/4)$.

PROPOSITION 2.1. (i) *Φ always satisfies*

$$\langle \Phi, \Phi\rangle=1\,. \tag{2.3}$$

(ii) *For any $\xi\in C^\infty(TM^C)$, $D_\xi\Phi\in C^\infty(E)$ has image in $\varphi^{-1}L^\perp$. In particular Φ always satisfies*

$$\langle X\Phi, \Phi\rangle=0\,,\qquad \langle Y\Phi, \Phi\rangle=0\,. \tag{2.4}$$

Thus we may regard $D_\xi\Phi$ as a section of $\varphi^{-1}(L^*\otimes L^\perp)$.

(iii) *Under the isomorphism* (2.1),

$$h((d\varphi)^{(1,0)}(\xi))=D_\xi\Phi \tag{2.5}$$

for any $\xi\in T_xM^C$.

(iv) *A smooth map $\varphi: M\to \boldsymbol{C}P^n$ is harmonic if and only if*

$$\Delta\Phi+|D\Phi|^2\Phi=0\,. \tag{2.6}$$

This proposition is essentially due to Lemma 4.3 and Propositions 4.5, 4.6 in [Ee-W]. In [Ee-W] they introduced the notion of complex isotropy of a map. A smooth map $\varphi: M\to \boldsymbol{C}P^n$ is called *complex isotropic* if

$$\langle X^p\Phi, Y^q\Phi\rangle\equiv 0 \tag{2.7}$$

for all $p, q\geqq 0$ with $p+q\geqq 1$.

Suppose that $\varphi: M\to \boldsymbol{C}P^n$ is a minimal surface with constant Gaussian curvature K. If φ is complex isotropic, then we have $K>0$. Because, according to [Ee-W], φ has a horizontal holomorphic lift of $\tilde{\varphi}$ relative to a twistor fibration $\mathcal{H}_{r,s}\to \boldsymbol{C}P^n$. Here $\mathcal{H}_{r,s}$ is endowed with the structure of a homogeneous Kähler submanifold in a complex projective space. Hence $\tilde{\varphi}$ induces a holomorphic isometric immersion of M into a complex projective space. Thus by

virtue of a result of Calabi [Ca-1], K must be positive.

PROPOSITION 2.2. (i) *φ is conformal if and only if Φ satisfies*

$$\langle X\Phi, Y\Phi\rangle=0\,. \tag{2.8}$$

(ii) *φ is an isometric immersion if and only if Φ satisfies* (2.8) *and*

$$\langle X\Phi, X\Phi\rangle+\langle Y\Phi, Y\Phi\rangle=c/2\,. \tag{2.9}$$

(iii) *The Kähler function μ of a map φ is given by*

$$\langle X\Phi, X\Phi\rangle-\langle Y\Phi, Y\Phi\rangle=(c/2)\mu\,. \tag{2.10}$$

(iv) *φ is a minimal isometric immersion if and only if Φ satisfies* (2.8) *and*

$$\Delta\Phi+(c/2)\Phi=0\,. \tag{2.11}$$

PROOF. By (2.1) we have

$$\langle D_\xi\Phi, D_\eta\Phi\rangle=(c/4)((\varphi_*g)(\xi, \eta)+\sqrt{-1}(\varphi^*\omega)(\xi, \eta) \tag{2.12}$$

for $\xi, \eta\in T_xM$. Let $\{e_1, e_2\}$ be an orthonormal basis of T_xM so that $u=(1/\sqrt{2})(e_1-\sqrt{-1}e_2)$, $\bar{u}=(1/\sqrt{2})(e_1+\sqrt{-1}e_2)$. Using (2.12), we compute

$$\begin{aligned}\langle X\Phi, Y\Phi\rangle=(c/2)\{&(1/4)((\varphi^*g)(e_1, e_1)-(\varphi^*g))e_2, e_2))\\ &-(\sqrt{-1}/2)(\varphi^*g)(e_1, e_2)\},\end{aligned} \tag{2.13}$$

$$\begin{aligned}\langle X\Phi, X\Phi\rangle=(c/2)\{&(1/4)((\varphi^*g)(e_1, e_1)+(\varphi^*g)(e_2, e_2))\\ &-(1/2)(\varphi^*\omega)(e_1, e_2)\},\end{aligned} \tag{2.14}$$

$$\begin{aligned}\langle Y\Phi, Y\Phi\rangle=(c/2)\{&(1/4)((\varphi^*g)(e_1, e_1)+(\varphi^*g)(e_2, e_2))\\ &+(1/2)(\varphi^*\omega)(e_1, e_2)\}.\end{aligned} \tag{2.15}$$

(2.13) implies (i). From (2.14) and (2.15) we get (ii) and (iii). If φ is a minimal isometric immersion, by (iv) of Proposition 2.1 and (ii) we get (2.8) and (2.11). Conversely suppose (2.8) and (2.11). By (2.3), (2.4) we compute $\langle X\Phi, X\Phi\rangle+\langle Y\Phi, Y\Phi\rangle=-\langle YX\Phi, \Phi\rangle-\langle XY\Phi, \Phi\rangle=-\langle\Delta\Phi, \Phi\rangle=(c/2)$. Hence φ is a minimal isometric immersion. So we get (iv). q.e.d.

3. Minimal surfaces with constant curvature and Kähler angle in a complex projective space.

Let M be a Riemann surface with a Hermitian metric g_M and K denote its Gaussian curvature. Let $\varphi: M\to \boldsymbol{C}P^n$ be a smooth map and $\mu=\sqrt{-1}(\varphi^*\omega)(u, \bar{u})$ be the Kähler function of φ, where u and $\bar{u}$ denote a unit (1, 0)-vector on M

and its conjugate. In this section *we assume that K and μ are constant on M.* Consider the bundle $E=\varphi^{-1}(L^*\otimes\underline{C}^{n+1})$ and the universal lift $\Phi\in C^\infty(E)$ of φ.

PROPOSITION 3.1. *Suppose that a section Ψ of $C^\infty(E)$ satisfies $\Delta\Psi+(c/2)\Psi=0$. Then, for each $m\geqq 0$,*

$$YX^{m+1}\Psi=(1/2)[m(m+1)K-(c/2)\{1+(2m+1)\mu\}]X^m\Psi\,, \tag{3.1}$$

$$XY^{m+1}\Psi=(1/2)[m(m+1)K-(c/2)\{1-(2m+1)\mu\}]Y^m\Psi\,. \tag{3.2}$$

PROOF. We show (3.1) and (3.2) by the induction on k. Since $\Psi\in\mathcal{E}$ has degree 0, $H\Psi=-(c/2)\mu\cdot\Psi$.

$$\Delta\Psi=(XY+YX)\Psi=-(c/2)\Psi\,,$$

$$(-H)\Psi=(XY-YX)\Psi=(c/2)\mu\cdot\Psi\,.$$

It follows that

$$YX\Psi=-(c/2)(1+\mu)/2\cdot\Psi\,,\qquad XY\Psi=-(c/2)(1-\mu)/2\cdot\Psi\,.$$

This verifies our claim when $m=0$. Now suppose that

$$YX^m\Psi=(1/2)[m(m-1)\cdot K-(c/2)\{1+(2m-1)\mu\}]X^{m-1}\Psi\,,$$

and

$$XY^m\Psi=(1/2)[m(m-1)\cdot K-(c/2)\{1-(2m-1)\mu\}]Y^{m-1}\Psi\,.$$

We compute

$$\begin{aligned}
YX^{m+1}\Psi&=XY(X^m\Psi)-[X,Y]X^m\Psi\\
&=X(YX^m\Psi)+H(X^m\Psi)\\
&=(1/2)[m(m-1)\cdot K-(c/2)\cdot\{1+(2m-1)\mu\}]X^m\Psi,\\
&\quad-\{(c/2)\mu-m\cdot K\}X^m\Psi\\
&=(1/2)[m(m+1)\cdot K-(c/2)\{1+(2m+1)\mu\}]X^m\Psi\,,\\
XY^{m+1}\Psi&=Y(XY^m\Psi)-H(Y^m\Psi)\\
&=(1/2)[m(m-1)K-(c/2)\{1-(2m-1)\mu\}]Y^m\Psi\\
&\quad+\{(c/2)\mu+mK\}Y^m\Psi\\
&=(1/2)[m(m+1)K-(c/2)\{1-(2m+1)\mu\}]Y^m\Psi\,.
\end{aligned}$$

So the induction is complete. q.e.d.

PROPOSITION 3.2. *Suppose that Φ satisfies $\Delta\Phi+(c/2)\Phi=0$. Then, for each $m\geqq 0$,*

$$\langle X^{m+1}\Phi,\ X^{m}\Phi\rangle=\langle Y^{m}\Phi,\ Y^{m+1}\Phi\rangle=0$$

and $\langle X^{m}\Phi,\ X^{m}\Phi\rangle=A_m$, $\langle Y^{m}\Phi,\ Y^{m}\Phi\rangle=B_m$, *where* A_m *and* B_m *are constants depending only on* m, K *and* μ *satisfying*

$$A_0=B_0=1\,,$$

$$A_{m+1}=(1/2)[(c/2)\{1+(2m+1)\mu\}-m(m+1)\cdot K]\cdot A_m$$

and

$$B_{m+1}=(1/2)[(c/2)\{1-(2m+1)\mu\}-m(m+1)\cdot K]\cdot B_m\,.$$

PROOF. We show this proposition by the induction on m. (2.3) and (2.4) verify our claim when $m=0$. Suppose that our claim is true for $m=p$. Applying Y to $\langle X^{p+1}\Phi,\ X^{p}\Phi\rangle=0$, we get

$$\langle YX^{p+1}\Phi,\ X^{p}\Phi\rangle+\langle X^{p+1}\Phi,\ X^{p}{}_{+}{}^{1}\Phi\rangle=0\,.$$

So by (3.1) and the assumption of the induction we have

$$\begin{aligned}\langle X^{p+1}\Phi,\ X^{p+1}\Phi\rangle&=-(1/2)[p(p+1)\cdot K-(c/2)\{1+(2p+1)\mu\}]\langle X^{p}\Phi,\ X^{p}\Phi\rangle\\&=A_{p+1}\,.\end{aligned}$$

Applying X to this equation, we get

$$\langle X^{p+2}\Phi,\ X^{p+1}\Phi\rangle+\langle X^{p+1}\Phi,\ YX^{p+1}\Phi\rangle=XA_{p+1}=0\,.$$

By (3.1) we have

$$\begin{aligned}\langle X^{p+2}\Phi,\ X^{p+1}\Phi\rangle&=-(1/2)[p(p+1)\cdot K-(c/2)\{1+(2p+1)\mu\}]\langle X^{p+1}\Phi,\ X^{p}\Phi\rangle\\&=0\,.\end{aligned}$$

Similarly by (3.2) and the assumption of the induction we have $\langle Y^{p+1}\Phi,\ Y^{p+1}\Phi\rangle=B_{p+1}$ is constant and $\langle Y^{p+2}\Phi,\ Y^{p+1}\Phi\rangle=0$. So the induction is complete. q.e.d.

Put

$$a_m=(1/2)[(c/2)\{1+(2m+1)\mu\}-m(m+1)\cdot K]\,,$$

and

$$b_m=(1/2)[(c/2)\{1-(2m+1)\mu\}-m(m+1)\cdot K]\,.$$

Then from Propositions 3.1 and 3.2 we get

(3.3) $$A_{p+1}=a_pA_p\,,\qquad B_{p+1}=b_pB_p\,,$$

(3.4) $$Y^{q}X^{p}\Phi=(-1)^{q}a_{p-1}\cdots a_{p-q}X^{p-q}\Phi$$

(3.5) $$X^{q}Y^{p}\Phi=(-1)^{q}b_{p-1}\cdots b_{p-q}Y^{p-q}\Phi$$

for $p\geqq q\geqq 0$.

LEMMA 3.3. *Suppose that* Φ *satisfies* $\Delta\Phi+(c/2)\Phi=0$. *If* $A_{m+1}=0$ *for some* $m\geqq 0$, *then* Φ *is complex isotropic and satisfies*

$$\langle X^p\Phi, X^q\Phi\rangle=0 \tag{3.6}$$

for any p *and* q *with* $p, q\geqq m+1$ *or* $m\geqq p>q\geqq 0$. *Similarly, if* $B_{m+1}=0$ *for some* $m\geqq 0$, *then* Φ *is complex isotropic and satisfies*

$$\langle Y^p\Phi, Y^q\Phi\rangle=0 \tag{3.7}$$

for any p *and* q *with* $p, q\geqq m+1$ *or* $m\geqq p>q\geqq 0$.

PROOF. Assume that $A_{m+1}=0$ for some $m\geqq 0$ and let m be the smallest integer satisfying $A_{m+1}=0$. From (3.3) we have $A_p=0$ for all $p\geqq m+1$. Applying X to $\langle X^m\Phi, X^{m-1}\Phi\rangle=0$, we get

$$\langle X^{m+1}\Phi, X^{m-1}\Phi\rangle+\langle X^m\Phi, YX^{m-1}\Phi\rangle=0.$$

Since $X^{m+1}\Phi=0$, by (3.4) we have

$$a_{m-2}\cdot\langle X^m\Phi, X^{m-2}\Phi\rangle=0.$$

Since $A_m\neq 0$, from (3.3) we see $a_{m-2}\neq 0$. Hence $\langle X^m\Phi, X^{m-2}\Phi\rangle=0$. Similarly, applying X to this equation, we have $\langle X^m\Phi, X^{m-3}\Phi\rangle=0$. Inductively we get $\langle X^m\Phi, X^q\Phi\rangle=0$ for each q with $0\leqq q\leqq m-1$. Applying Y to $\langle X^m\Phi, X^q\Phi\rangle=0$ for each q with $0\leqq q\leqq m-2$,

$$\langle YX^m\Phi, X^q\Phi\rangle+\langle X^m\Phi, X^{q+1}\Phi\rangle=0.$$

By (3.4) we get $a_{m-1}\cdot\langle X^{m-1}\Phi, X^q\Phi\rangle=0$. Since $a_{m-1}\neq 0$, we have $\langle X^{m-1}\Phi, X^q\Phi\rangle=0$ for each q with $0\leqq q\leqq m-2$. Inductively, we obtain $\langle X^p\Phi, X^q\Phi\rangle=0$ for any p, q with $m\geqq p>q\geqq 0$. So we get (3.6). In particular $\langle X^p\Phi, \Phi\rangle=0$ for all $p\geqq 1$. We show the complex isotropy of Φ by the induction on $p+q$. (2.4) shows our claim when $p+q=1$. We suppose that $\langle X^p\Phi, Y^q\Phi\rangle=0$ for any p and q with $k>p+q\geqq 1$. Using this assumption repeatedly, we compute, for each p, q with $p+q=k$,

$$\begin{aligned}\langle X^p\Phi, Y^q\Phi\rangle&=X\langle X^p\Phi, Y^{q-1}\Phi\rangle-\langle X^{p+1}\Phi, Y^{q-1}\Phi\rangle\\&=-\langle X^{p+1}\Phi, Y^{q-1}\Phi\rangle\\&=-X\langle X^{p+1}\Phi, Y^{q-2}\Phi\rangle+\langle X^{p+2}\Phi, Y^{q-2}\Phi\rangle\\&\quad-\langle X^{p+2}\Phi, Y^{q-2}\Phi\rangle\\&=(-1)^q\langle X^{p+q}\Phi, \Phi\rangle=0.\end{aligned}$$

Therefore we get the complex isotropy of φ. When $B_{m+1}=0$ for some $m\geqq 0$,

similarly we can show (3.7) and the complex isotropy of φ. q.e.d.

We shall study a map $\varphi: M\to \boldsymbol{C}P^n$ satisfying $\Delta\Phi+(c/2)\Phi=0$ in each case: $K>0$, $K=0$, $K<0$.

PROPOSITION 3.4. *Suppose that Φ satisfies $\Delta\Phi+(c/2)\Phi=0$ and φ is full. If $K>0$, then $\varphi: M\to \boldsymbol{C}P^n$ is a minimal isometric immersion and there exists some l with $0\leqq l\leqq n$ such that $k=c/\{2l(n-l)+n\}$, $\mu=K\cdot(n-2l)/c$ and $\varphi(M)$ is an open submanifold of $\Psi_{n,l}: S^2\to \boldsymbol{C}P^n$.*

PROOF. Since $K>0$, $a_m, b_m\to-\infty$ as $m\to\infty$. Since $A_m\geqq 0$, $B_m\geqq 0$ for all m, by (3.3) there are $k, l\geqq 2$ such that $a_k=b_l=0$, $a_{k-1}\neq 0$ and $b_{l-1}\neq 0$. From $a_k=b_l=0$ we have

$$\begin{cases} (c/2)\{1+(2k+1)\mu\}-k(k+1)\cdot K=0\,, \\ (c/2)\{1-(2l+1)\mu\}-l(l+1)K=0\,. \end{cases}$$

By a simple computation we get $\mu=K(k-l)/c$ and $K=c/(2kl+k+l)$. We have $A_p=B_q=0$ for any $p\geqq k+1$, $q\geqq l+1$, and $A_p=\langle X^p\Phi, X^p\Phi\rangle>0$, $B_p=\langle Y^p\Phi, Y^q\Phi\rangle>0$ for any $0\leqq p\leqq k$, $0\leqq q\leqq l$. Set $Z_0=\Phi$, $Z_p=(1/\sqrt{A_p})\cdot X^p\Phi$ for each $1\leqq p\leqq k$, and $Z_{-q}=(-1)^q(1/\sqrt{B_q})\cdot Y^q\Phi$ for each $1\leqq q\leqq l$. Then by Lemma 3.3 we have $\langle Z_p, Z_q\rangle=\delta_{p,q}$ for $-l\leqq p, q\leqq k$. If we regard each Z_p as a vector bundle E-valued function on the bundle $SO(M)$ of orthonormal frames compatible with the orientation of M, then $\{Z_p(\rho), Z_0(\rho), Z_{-q}(\rho);\ 1\leqq p\leqq k,\ 1\leqq q\leqq l\}$ is unitary in $\boldsymbol{C}^{n+1}$ for any unit element $\rho\in\varphi^{-1}L$ at every point of $SO(M)$. Hence $\{Z_p, Z_0, Z_{-q};\ 1\leqq p\leqq k,\ 1\leqq q\leqq l\}$ is projective unitary in $\boldsymbol{C}^{n+1}$ at every point of $SO(M)$. By (3.3), (3.4), (3.3) we compute

$$\text{(3.8)}\qquad \begin{cases} DZ_p=XZ_p+YZ_p \\ \qquad\quad =\sqrt{a_p}\cdot Z_{p+1}-\sqrt{a_{p-1}}\cdot Z_{p-1} \qquad \text{for } 1\leqq p\leqq k\,, \\ DZ_0=\sqrt{a_0}\cdot Z_1-\sqrt{b_0}\cdot Z_{-1}\,, \qquad \text{and} \\ DZ_{-q}=\sqrt{b_{q-1}}\cdot Z_{-(q-1)}-\sqrt{b_q}\cdot Z_{-(q+1)} \qquad \text{for } 1\leqq q\leqq l\,. \end{cases}$$

From these equations and the fullness of φ we see $k+l=n$. So we get $\mu=K(n-2l)/c$, $K=c/\{2l(n-l)+n\}$. Moreover we have

$$\text{(3.9)}\qquad a_p=(c/2)\cdot(n-l-p)(l+p+1)/\{2l(n-l)+n\}$$

for $0\leqq p\leqq k-1$ and

$$\text{(3.10)}\qquad b_p=(c/2)\cdot(l-q)(n-l+q+1)/\{2l(n-l)+n\}$$

for $0\leqq q\leqq l-1$. $\{Z_p, Z_0, Z_{-q};\ 1\leqq p\leqq k,\ 1\leqq q\leqq l\}$ can be regarded as a map from

$SO(M)$ to a projective unitary group $PU(n+1)$. Using (3.8), (3.9), (3.10) and results of [B-Oh, § 2], by virtue of the congruence theorem for smooth maps to a homogeneous space (cf. [Gr] or [Je]) we conclude that φ is locally congruent with $\psi_{n,l}$. q.e.d.

REMARK. From the complex isotropy of φ, we also can get the conclusion of this proposition by results of [Ee-W], [Ca-1], [B-Oh] and [B-J-R-W].

PROPOSITION 3.5. *Suppose that* Φ *satisfies* $\Delta\Phi+(c/2)\Phi=0$. *If* $K=0$, *then* $\mu=0$.

PROOF. In this case $a_m=(c/4)\{1+(2m+1)\mu\}$ and $b_m=(c/4)\{1-(2m+1)\mu\}$. If $\mu\neq 0$, then $a_m\to-\infty$ or $b_m\to-\infty$ as $m\to\infty$. By (3.3) we get $A_m=0$ or $B_m=0$ for some $m\geqq 1$. By virtue of Lemma 3.3 φ is complex isotropic. From (iv) of Proposition 2.2 φ is a complex isotropic, minimal isometric immersion. But since $K=0$, it's impossible. Therefore we have $\mu=0$. q.e.d.

By the argument similar to that of [Br, Theorem 2.3] we show the following.

PROPOSITION 3.6. *Suppose that* Φ *satisfies* $\Delta\Phi+(c/2)\Phi=0$. *Then* $K<0$ *is impossible.*

PROOF. Suppose $K<0$. If $A_m=0$ or $B_m=0$ for some $m\geqq 1$, then by Lemma 3.3 and (iv) of Proposition 2.2 φ becomes a complex isotropic minimal isometric immersion. But since $K<0$, it's impossible. Therefore $A_m>0$ and $B_m>0$ for all $m\geqq 0$. From (3.3) $a_m>0$, $b_m>0$ for all $m\geqq 0$. We fix an integer m with $m\geqq 2$. For any integer p with $p\geqq m$, applying X^{m-1} to the equation $\langle X^{p+1}\Phi, X^p\Phi\rangle=0$, by (3.3), (3.4) we compute

$$X^{m-1}\langle X^{p+1}\Phi, X^p\Phi\rangle=\sum_{r=0}^{m-1}\binom{m-1}{r}\langle X^{r+p+1}\Phi, Y^{m-1-r}X^p\Phi\rangle$$

$$=\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^{m-1}a_p\cdots a_{p-(m-1-r)}\langle X^{r+p+1}\Phi, X^{p-(m-1-r)}\Phi\rangle$$

$$=\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^{m-1-r}(A_p/A_{p-m+1+r})\langle X^{p+1+r}\Phi, X^{p-m+1+r}\Phi\rangle$$

$$=0.$$

Hence we have

$$\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^{m-1-r}(1/A_{p+r})\langle X^{p+r+m}\Phi, X^{p+r}\Phi\rangle=0$$

for each $p\geqq 1$. This equation says that the sequence $\{(1/A_p)\langle X^{p+m}\Phi, X^p\Phi\rangle;$ $p\in Z,\ p\geqq 1\}$ satisfies a difference equation of order $m-1$: $\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^{m-1-r}x_{p+r}$ $=0$. By a well-known result about difference equations, there exists a polynomial $R_m(s)$, of degree at most $m-2$ in s with coefficients in $C^\infty(\tau^m)$ so that

$$(1/A_p)\langle X^{p+m}\Phi, X^p\Phi\rangle=R_m(p) \tag{3.6}$$

for all $p\geqq 1$. For $p\geqq 0$, define $Z_p=(1/\sqrt{A_p})X^p\Phi$. Then we have $\langle Z_p, Z_p\rangle=1$, $\langle Z_{p+1}, Z_p\rangle=0$. When $m\geqq 2$, for all $p\geqq 1$

$$\begin{aligned}\langle Z_{p+m}, Z_p\rangle&=(1/\sqrt{A_{p+m}}\sqrt{A_p})\langle X^{p+m}\Phi, X^p\Phi\rangle\\&=\sqrt{A_p/A_{p+m}}\cdot R_m(p).\end{aligned}$$

Since $K<0$, we have

$$\sqrt{A_p/A_{p+m}}<C_m/p^m$$

for some positive constant C_m which depends on K and μ. Because from (3.3) we compute

$$\begin{aligned}\sqrt{A_p/A_{p+m}}&=(a_{p+m-1})^{-2}\cdot(a_{p+m-2})^{-2}\cdot\cdots\cdot(a_p)^{-2}\\&=[(c/4)\{1+(2p+2m-1)\mu\}-(p+m-1)(p+m)K/2]^{-2}\\&\cdots[(c/4)\{1+(2p+1)\mu\}-p(p+1)K/2]^{-2}<C_m/p^m.\end{aligned}$$

Since $R_m(p)$ is of degree at most $m-2$, when $m\geqq 1$

$$\lim_{p\to\infty}\langle Z_{p+m}, Z_p\rangle(u^m)=0$$

for each unit vector $u\in T^{(1,0)}M$. Let $u\in T^{(1,0)}M$ be a fixed unit (1, 0)-vector at $x\in M$ and $\rho\in L_x$ be a fixed unit element. We define the vectors W_p in $\boldsymbol{C}^{n+1}$ by $((Z_p)_x(u^p))(\rho)=W_p$. Then $\langle Z_{p+m}, Z_p\rangle(u^m)=\langle W_{p+m}, W_p\rangle$.

Let $r>n$ be any integer and let $\varepsilon>0$ be small. By the above argument, there exist an integer p so large that $|\langle W_{p+k}, W_{p+l}\rangle|<\varepsilon$ for all $k\neq l$, $0\leqq k$, $l\leqq r$, while $\langle W_{p+k}, W_{p+k}\rangle=\langle Z_{pk}, Z_{p+k}\rangle=1$ for all k. Taking ε sufficiently small, this implies that the $r+1$ vectors $\{W_p, \cdots, W_{p+r}\}$ are linearly independent in $\boldsymbol{C}^{n+1}$. Since $r>n$, this is impossible. q. e. d.

Combining Propositions 3.4, 3.5 and 3.6, by (iv) of Proposition 2.2 we obtain Theorem A. We remark about the case $K=0$. Let $\varphi: M\to\boldsymbol{C}P^n$ be a totally real flat minimal surface. By the total realness of φ, φ can be locally lifted to a flat minimal surface $\tilde{\varphi}: M\to S^{2n+1}(c/4)$. By Theorem 3.1 of [Br], $\tilde{\varphi}$ extends to a minimal immersion of $\boldsymbol{C}$. So φ also extends to a totally real minimal

immersion of C into CP^n. Such minimal immersions are completely classified by [Ke-2].

4. The case when the ambient space is a complex hyperbolic space.

In C^{n+1} we consider an indefinite Hermitian inner product $\langle , \rangle_{1,n}$ defined by

$$\langle z, w\rangle_{1,n}=-z^0\bar{w}^0+\sum_{i=1}^{n} z^i\bar{w}^i .$$

Fixing any negative constant c, we let $H^{2n+1}(c)=\{z\in C^{n+1}; \langle z, z\rangle_{1,n}=1/c\}$. The group $S^1=\{e^{\sqrt{-1}\theta}\}$ acts freely on $H^{2n+1}(c/4)$ by $z\to e^{\sqrt{-1}\theta}z$. An n-dimensional complex hyperbolic space CH^n is the base manifold of the principal S^1-bundle $H^{2n+1}(c/4)$ with the projectionn $\pi: H^{2n+1}(c/4)\to CH^n$. For each $z\in H^{2n+1}(c/4)$, we define a subspace $\mathcal{H}_z$ of $T_zH^{2n+1}(c/4)$ by $\mathcal{H}_z=\{w\in C^{n+1}; \langle z, w\rangle_{1,n}=0\}$. The restriction of $\langle , \rangle_{1,n}$ to each $\mathcal{H}_z$ is positive definite. Then we can define a Riemannian metric g on CH^n so that $d\pi: (\mathcal{H}_z, (\ ,\ ))\to (T_{\pi(z)}CH^n, g_{\pi(z)})$ is a linear isometry for each $z\in H^{2n+1}(c/4)$, where $(\ ,\ )=\mathrm{Re}\langle , \rangle_{1,n}$. g gives the standard Kähler structure on CH^n of constant holomorphic sectional curvature c. We define a holomorphic line subbundle L_1 of the trivial bundle $\underline{C}^{n+1}$ over CH^n by $(L_1)_x=C\cdot z$ for $x=\pi(z)\in CH^n$ and $z\in H^{2n+1}(c/4)$. The restriction of $\langle , \rangle_{1,n}$ to L_1 defines an indefinite (negative definite) Hermitian fibre metric $\langle , \rangle$ of L_1. Then L_1 has the Hermitian connection with respect to the holomorphic structure and the indefinite Hermitian fibre metric. Let $L_1^{\perp}$ be the complex vector subbundle of C^{n+1} defined by $(L_1^{\perp})_x=\{w\in C^{n+1}; \langle w, z\rangle_{1,n}=0$ for all $z\in(L)_x\}$. We have an orthogonal direct sum $\underline{C}^{n+1}=L_1\oplus L_1^{\perp}$ with respect to $\langle , \rangle_{1,n}$. We endow the bundle $L_1^{\perp}$ with the Hermitian fibre metric $\langle , \rangle$ by restrictinng $\langle , \rangle_{1,n}$ to $L_1^{\perp}$. $L_1^{\perp}$ has the holomorphic structure through the bundle isomorphism $L_1^{\perp}\cong \underline{C}^{n+1}/L_1$. With respect to them $L_1^{\perp}$ has the Hermitian connection. Now we consider the tensor product bundle $L_1^*\otimes L_1^{\perp}$ with the Hermitian connected structure induced from those of L_1 and $L_1^{\perp}$. Then there exists a connection-preserving biholomorphic isomorphism $h: T^{(1,0)}CH^n\to L_1^*\otimes L_1^{\perp}$ such that $\langle h(Z), h(W)\rangle = -(c/2)g(Z, \overline{W})$ for $Z, W\in T_x^{(1,0)}CH^n$.

Let $\varphi: M\to CH^n$ be a smooth map from a Riemann surface. We consider the exact sequence of the complex bundles equipped with pull-back indefinite Hermitian connected structure:

$$0\longrightarrow \varphi^{-1}(L_1^*\otimes L_1)\xrightarrow{i} \varphi^{-1}(L_1^*\otimes\underline{C}^{n+1})\xrightarrow{j} \varphi^{-1}(L_1^*\otimes L_1^{\perp})\longrightarrow 0,$$

where i is the inclusion map and j is the orthogonal projection along L_1 relative to $\langle , \rangle_{1,n}$. Set $E=\varphi^{-1}(L_1^*\otimes\underline{C}^{n+1})$. We call the section $i(1)\in C^{\infty}(E)$ the

universal lift Φ of φ, where 1 denotes the identity section. Let $\langle , \rangle$ and D denote the indefinite Hermitian fibre metric and the covariant differentiation in the bundle E. Then we have the following:

(1) $\langle \Phi, \Phi \rangle = -1$.

(2) For any $\xi \in C^\infty(TM^c)$, $D_\xi \Phi$ has image in $\varphi^{-1}L_1^+$.

Moreover $h((d\varphi)^{(1,0)}) = D\Phi$.

(3) φ is harmonic if and only if in any chart

$$D''D'\Phi - \langle D'\Phi, D'\Phi \rangle \Phi = 0$$

or

$$D'D''\Phi - \langle D''\Phi, D''\Phi \rangle \Phi = 0 .$$

More generally the similar formulation for indefinite complex space forms was given by [Er-G] in detail.

Let M be a Riemann surface with a compatible Riemannian metric and $\varphi : M \to \boldsymbol{C}H^n$ be a smooth map. The following is shown easily:

(i) $\langle \Phi, \Phi \rangle = -1$, $\langle X\Phi, \Phi \rangle = \langle Y\Phi, \Phi \rangle = 0$.

(ii) φ is conformal if and only if $\langle X\Phi, Y\Phi \rangle = 0$.

(iii) φ is an isometric immersion if and only if $\langle X\Phi, Y\Phi \rangle = 0$ and $\langle X\Phi, X\Phi \rangle + \langle Y\Phi, Y\Phi \rangle = -c/2$.

(v) Let θ be the Kähler angle of φ and put $\mu = \cos\theta$. Then $\langle X\Phi, X\Phi \rangle - \langle Y\Phi, Y\Phi \rangle = -(c/2)\mu$.

(vi) Suppose that φ is an isometric immersion. Then φ is minimal (or harmonic) if and only if $\Delta\Phi + (c/2)\Phi = 0$.

Suppose that K and μ are constant and Φ satisfies $\Delta\Phi + (c/2)\Phi = 0$. Following the calculations in Section 3, we easily establish the same formulas as in Propositions 3.1 and 3.2 for a negative constant c. So we get, for $m \geqq 0$,

$$A_{m+1} = a_m A_m , \qquad B_{m+1} = b_m B_m \tag{4.1}$$

where $A_m = \langle X^m\Phi, X^m\Phi \rangle$, $B_m = \langle Y^m\Phi, Y^m\Phi \rangle$,

$$a_m = (1/2)[(c/2)\{1 + (2m+1)\mu\} - m(m+1)\cdot K] , \tag{4.2}$$

$$b_m = (1/2)[(c/2)\{1 - (2m+1)\mu\} - m(m+1)\cdot K] . \tag{4.3}$$

Now assume that $\varphi : M \to \boldsymbol{C}H^n$ is a minimal surface with constant Gaussian curvature K and constant Kähler angle θ. By the equation of Gauss we have

$$K = (c/4)(1 + 3\mu^2) - (1/2)\|\alpha\|^2 < 0 , \tag{4.4}$$

where $\|\alpha\|$ denotes the length of the second fundamental form α of φ. By (4.4) we compute

$$a_0=(c/4)(1+\mu)\leqq 0,$$

$$b_0=(c/4)(1-\mu)\leqq 0,$$

$$a_1=(1/2)\{(c/2)3\mu(1-\mu)+\|\alpha\|^2\},$$

$$b_1=(1/2)\{(c/2)3(-\mu)(1+\mu)+\|\alpha\|^2\},$$

$$a_2=(1/2)\{(c/2)(-2+5\mu-9\mu^2)+3\|\alpha\|^2\}>0,$$

$$b_2=(1/2)\{(c/2)(-2-5\mu-9\mu^2)+3\|\alpha\|^2\}>0.$$

Therefore form (4.2), (4.3) we have $a_m>0$, $b_m>0$ for any $m\geqq 2$. We see that if $\mu\leqq 0$ (resp. $\mu\geqq 0$), then $a_1\geqq 0$ (resp. $b_1\geqq 0$).

LEMMA 4.1. *The case* $a_0<0$ *and* $a_1>0$ *is impossible. Similarly, the case* $b_0<0$ *and* $b_1>0$ *is also impossible.*

PROOF. Suppose that $a_0<0$ and $a_1>0$. Since $a_m>0$ for any $m\geqq 2$, by (4.1) we have $A_m>0$ for all $m\geqq 1$. By the argument similar to the proof of Proposition 3.6, we derive the same identies as (3.6). We define $Z_p=(1/\sqrt{A_p})X^p\Phi$ for $p\geqq 1$. Then we have $\langle Z_p, Z_p\rangle\equiv 1$ and the same estimate $\lim_{p\to\infty}\langle Z_{p+m}, Z_p\rangle=0$ pointwise on M for any $m>0$. Hence we again derive a contradiction from the finite dimensionality of $\boldsymbol{C}^{n+1}$. When $b_0<0$ and $b_1>0$, by the similar argument we can derive a contradiction. q.e.d.

PROOF OF THEOREM B. First suppose that $-1\leqq\mu\leqq 0$. Then $a_1\geqq 0$. If $\mu=-1$, we have $b_0<0$, $b_1=(1/2)\|\alpha\|^2$. By Lemma 4.1 we get $b_1=0$, i.e. M is totally geodesic and anti-holomorphic. If $-1<\mu\leqq 0$, we have $a_0<0$. By Lemma 4.1 we get $a_1=0$. Hence we get $\mu=0$ and $\|\alpha\|^2=0$. Thus M is totally real and totally geodesic.

Next suppose that $0\leqq\mu\leqq 1$. Then $b_1\leqq 0$. If $\mu=1$, we have $a_0<0$, $a_1=(1/2)\|\alpha\|^2$. By Lemma 4.1 M is holomorphic and totally geodesic. If $0\leqq\mu<1$, we have $b_0<0$ and $b_1=0$ by Lemma 4.1. Therefore M is totally real and totally geodesic. q.e.d.

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