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ON THE PRESENTATIONS OF THE FUNDAMENTAL GROUPS OF 3-MANIFOLDS

By

Moto-o TAKAHASHI

In this paper we shall treat the closed 3-manifolds obtained by Dehn surgeries along certain links and find presentations of their fundamental groups.

§1. The 3-chain link.

First we consider the 3-chain link K_1 illustrated in the Figure 1.

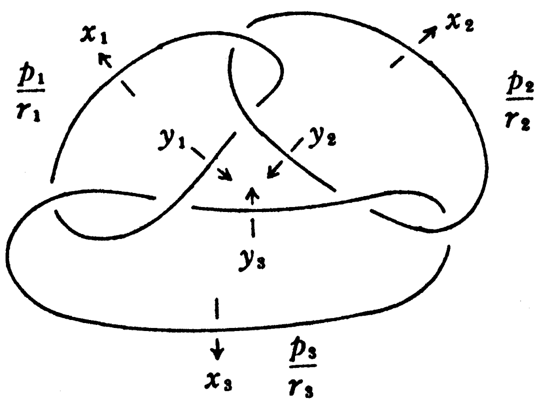

Figure 1

We do Dehn surgery along each component of K_1. Let p_1/r_1, p_2/r_2, p_3/r_3 be the surgery coefficients along three components L_1, L_2, L_3 of K_1, respectively, where p_i and r_i are co-prime integers ($i=1, 2, 3$). We denote the resulting 3-manifold by $M_1(p_1, r_1; p_2, r_2; p_3, r_3)$.

We shall find presentations of the fundamental group $\pi_1(M_1(p_1, r_1; p_2, r_2; p_3, r_3))$ of $M_1(p_1, r_1; p_2, r_2; p_3, r_3)$, by the following way.

First we shall find a presentation of the link group G of K_1.

The Wirtinger presentation of G is:

$$\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid y_2x_1=x_1x_2,\ y_3x_2=x_2x_3,\ y_1x_3=x_3x_1,$$
$$x_1y_2=y_2y_1,\ x_2y_3=y_3y_2,\ x_3y_1=y_1y_3\rangle . \qquad (1)$$

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The meridian m_i and the longitude l_i of each component L_i are:

$$m_1=x_1,\quad l_1=y_2x_3,\quad ([m_1, l_1]=1)$$

$$m_2=x_2,\quad l_2=y_3x_1,\quad ([m_2, l_2]=1)$$

$$m_3=x_3,\quad l_3=y_1x_2.\quad ([m_3, l_3]=1)$$

A presentation of $\pi_1(M_1(p_1, r_1; p_2, r_2; p_3, r_3))$ is obtained from (1) by adding the relators $m_i^{p_i}l_i^{r_i}=1$ $(i=1, 2, 3)$. But we improve this presentation.

Since $(p_i, r_i)=1$, there are integers s_i and q_i such that $r_is_i-p_iq_i=1$. Let $a_i=m_i^{s_i}l_i^{q_i}$. Then

$$m_i=a_i^{r_i}, \qquad l_i=a_i^{-p_i}.$$

So,

$$x_1=a_1^{r_1}, \qquad x_2=a_2^{r_2}, \qquad x_3=a_3^{r_3}$$

and

$$y_1=l_3x_2^{-1}=a_3^{-p_3}a_2^{-r_2},$$

$$y_2=l_1x_3^{-1}=a_1^{-p_1}a_3^{-r_3},$$

$$y_3=l_2x_1^{-1}=a_2^{-p_2}a_1^{-r_1}.$$

Substituting these in the relators of (1), we get the following three relators:

$$a_1^{p_1+r_1}a_2^{r_2}a_1^{-r_1}a_3^{r_3}=1,$$

$$a_2^{p_2+r_2}a_3^{r_3}a_2^{-r_2}a_1^{r_1}=1,$$

$$a_3^{p_3+r_3}a_1^{r_1}a_3^{-r_3}a_2^{r_2}=1.$$

Therefore we obtain the presentation:

$$\pi_1(M_1(p_1, r_1; p_2, r_2; p_3, r_3))\cong\langle a_1, a_2, a_3 \mid a_1^{p_1+r_1}a_2^{r_2}a_1^{-r_1}a_3^{r_3}=1,$$
$$a_2^{p_2+r_2}a_3^{r_3}a_2^{-r_2}a_1^{r_1}=1, \quad a_3^{p_3+r_3}a_1^{r_1}a_3^{-r_3}a_2^{r_2}=1\rangle. \tag{2}$$

REMARK. This presentation is induced by the following RR-system (c.f. [1]) illustrated in the Figure 2 and hence corresponds to a Heegaard splitting of genus 3. Actually we can easily construct a Heegaard splitting of $M_1(p_1, r_1; p_2, r_2; p_3, r_3)$ with this RR-system.

Next we eliminate the generator a_3 in the presentation (2). From the first relator of (2),

$$a_3^{r_3}=a_1^{r_1}a_2^{-r_2}a_1^{-p_1-r_1}. \tag{3}$$

Substituting it in the second relator, we obtain

$$a_2^{p_2+r_2}a_1^{r_1}a_2^{-r_2}a_1^{-p_1-r_1}a_2^{-r_2}a_1^{r_1}=1. \tag{4}$$

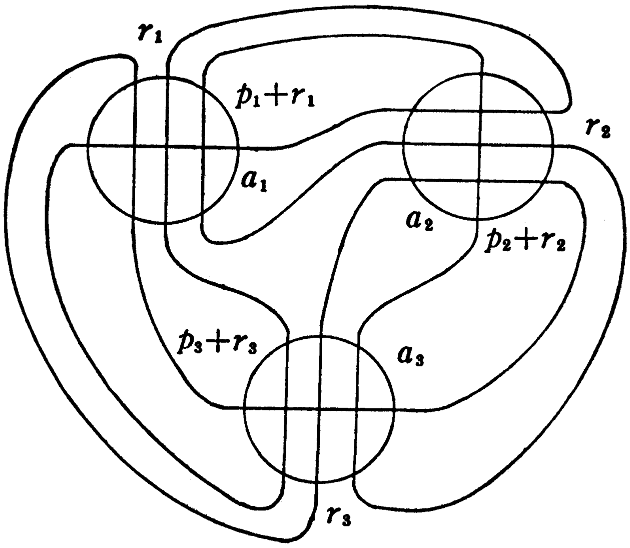


Figure 2

Moreover, from the third relator and (3),

$$a_3^{p_3+r_3}=a_2^{-r_2}a_1^{r_1}a_2^{-r_2}a_1^{-(p_1+2r_1)}\,. \tag{5}$$

But

$$[a_1^{r_1}a_2^{-r_2}a_1^{-p_1-r_1},\ a_2^{-r_2}a_1^{r_1}a_2^{-r_2}a_1^{-(p_1+2r_1)}]=1$$

is a consequence of (4). So we can eliminate a_3 by (3) and (5) (since $(p_3+r_3, r_3)=1$) and we obtain

$$(a_1^{r_1}a_2^{-r_2}a_1^{-p_1-r_1})^{p_3+r_3}=(a_2^{-r_2}a_1^{r_1}a_2^{-r_2}a_1^{-(p_1+2r_1)})^{r_3}\,.$$

In order to simplify this equality, we multiply $a_1^{-r_1}$ from the left and $a_1^{r_1}$ from the right. Then,

$$(a_2^{-r_2}a_1^{-p_1})^{p_3+r_3}=(a_1^{-r_1}a_2^{-r_2}\underline{a_1^{r_1}a_2^{-r_2}a_1^{-(p_1+r_1)}})^{r_3}\,, \tag{6}$$

By (4),

$$a_1^{r_1}a_2^{-r_2}a_1^{-(p_1+r_1)}=a_2^{-(p_2+r_2)}a_1^{-r_1}a_2^{r_2}\,.$$

Substituting this for the underlind part in (6), we obtain

$$(a_2^{-r_2}a_1^{-p_1})^{p_3+r_3}=(a_1^{-r_1}a_2^{-(p_2+2r_2)}a_1^{-r_1}a_2^{r_2})^{r_3}\,,$$

or

$$(a_1^{p_1}a_2^{r_2})^{-(p_3+r_3)}=(a_1^{-r_1}a_2^{-(p_2+2r_2)}a_1^{-r_1}a_2^{r_2})^{r_3}\,. \tag{7}$$

Now, since

$$a_1^{-r_1}a_2^{-(p_2+2r_2)}a_1^{-r_1}a_2^{r_2}=(a_1^{-r_1}a_2^{-(p_2+2r_2)}a_1^{-(p_1+r_1)})(a_1^{p_1}a_2^{r_2})$$

and

$$[a_1^{p_1}a_1^{r_2},\ a_1^{-r_1}a_2^{-(p_2+2r_2)}a_1^{-(p_1+r_1)}]=1\,,$$

by (4) it follows that

$$(a_1^{-r_1}a_2^{-(p_2+2r_2)}a_1^{-r_1}a_2^{r_2})^{r_3}=(a_1^{-r_1}a_2^{-(p_2+2r_2)}a_1^{-(p_1+r_1)})^{r_3}(a_1^{p_1}a_2^{r_2})^{r_3}.$$

So, by (7)

$$(a_1^{p_1}a_2^{r_2})^{-(p_3+2r_3)}=(a_1^{-r_1}a_2^{-(p_2+2r_2)}a_1^{-(p_1+r_1)})^{r_3}.$$

Taking the inverse we obtain

$$(a_1^{p_1}a_2^{r_2})^{p_3+2r_3}=(a_1^{p_1+r_1}a_2^{p_2+2r_2}a_1^{r_1})^{r_3}.$$

Hence

$$\begin{aligned}&\pi_1(M_1(p_1, r_1;\, p_2, r_2;\, p_3, r_3))\cong\langle a_1, a_2 \mid\\ &a_2^{p_2+r_2}a_1^{r_1}a_2^{-r_2}a_1^{-(p_1+r_1)}a_2^{-r_2}a_1^{r_1}=1,\\ &(a_1^{p_1}a_2^{r_2})^{p_3+2r_3}=(a_1^{p_1+r_1}a_2^{p_2+2r_2}a_1^{r_1})^{r_3}\rangle.\end{aligned}$$

This presentation corresponds to a Heegaard diagram of genus two.

§ 2. Some other links.

We do the same thing as did in § 1 for some other links. We describe only the results.

2.1. Consider the link K_2 illustrated in the Figure 3.

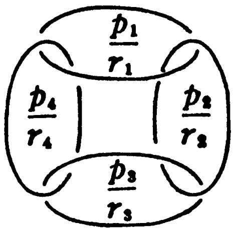


Figure 3

Let $M_2(p_1, r_1;\, p_2, r_2;\, p_3, r_3;\, p_4, r_4)$ be the 3-manifold obtained by Dehn surgery along each component of K_2 with surgery coefficients p_1/r_1, p_2/r_2, p_3/r_3, p_4/r_4. Then

$$\begin{aligned}&\pi_1(M_2(p_1, r_1;\, p_2, r_2;\, p_3, r_3;\, p_4, r_4))\cong\langle a_1, a_2, a_3, a_4 \mid\\ &a_4^{r_4}a_1^{p_1}a_2^{-r_2}=1,\ a_1^{-r_1}a_2^{p_2}a_3^{r_3}=1,\\ &a_2^{r_2}a_3^{p_3}a_4^{-r_4}=1,\ a_3^{-r_3}a_4^{p_4}a_1^{r_1}=1\rangle\\ &\cong\langle a_1, a_2 \mid (a_2^{-p_2})^{r_4}=(a_2^{r_2}a_1^{-p_1})^{p_4},\\ &(a_1^{-p_1})^{r_3}=(a_2^{-p_2}a_1^{r_1})^{p_3},\ [a_1^{p_1}, a_2^{p_2}]=1\rangle.\end{aligned}$$

The corresponding RR-system is illustrated in the Figure 4.

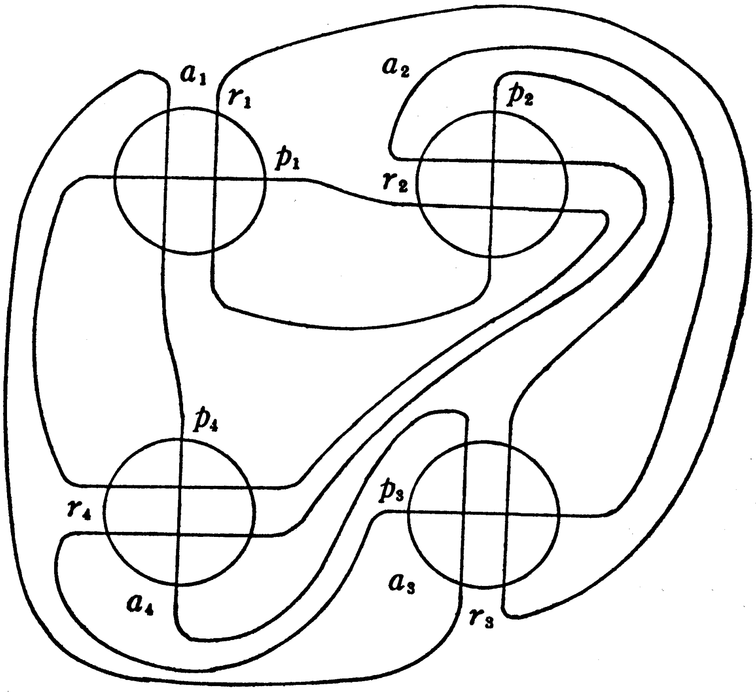


Figure 4

2.2. Consider the link K_3 illustrated in the Figure 5.

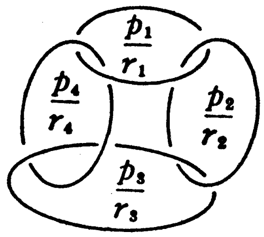


Figure 5

Let $M_3(p_1, r_1; p_2, r_2; p_3, r_3; p_4, r_4)$ be the 3-manifold obtained by Dehn surgery along each component of K_3 with surgery coefficients p_1/r_1, p_2/r_2, p_3/r_3, p_4/r_4. Then,

$$\begin{aligned}
&\pi_1(M_3(p_1, r_1; p_2, r_2; p_3, r_3; p_4, r_4)) \cong \langle a_1, a_2, a_3, a_4 \mid \\
&a_2^{-r_2} a_1^{-p_1} a_4^{r_4}=1,\ a_2^{p_2+r_2} a_1^{r_1} a_4^{-(p_4+r_4)} a_1^{p_1+r_1}=1, \\
&a_3^{-r_3} a_1^{r_1} a_4^{-p_4}=1,\ a_3^{p_3+r_3} a_4^{p_4+r_4} a_1^{-(p_1+r_1)} a_4^{r_4}=1\rangle \\
&\cong \langle a_1, a_4 \mid \\
&(a_1^{-p_1} a_4^{r_4})^{p_2+r_2}(a_1^{r_1} a_4^{-(p_4+r_4)} a_1^{p_1+r_1})^{r_2}=1, \\
&(a_1^{r_1} a_4^{-p_4})^{p_3+r_3}(a_4^{p_4+r_4} a_1^{-(p_1+r_1)} a_4^{r_4})^{r_3}=1, \\
&[a_1^{-p_1} a_4^{r_4},\ a_1^{r_1} a_4^{-(p_4+r_4)} a_1^{p_1+r_1}]=1\rangle,
\end{aligned}$$

2.3. Consider the link K_4 illustrated in the Figure 6.

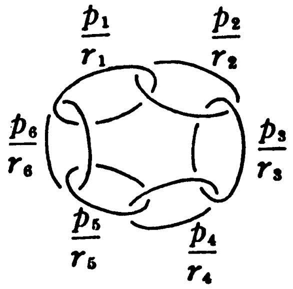


Figure 6

Let $M_4(p_1, r_1; p_2, r_2; p_3, r_3; p_4, r_4; p_5, r_5; p_6, r_6)$ be the 3-manifold obtained by Dehn surgery along each component of K_4 with surgery coefficients p_1/r_1, p_2/r_2, p_3/r_3, p_4/r_4, p_5/r_5, p_6/r_6. Then,

$$\pi_1(M_4(p_1, r_1; p_2, r_2; p_3, r_3; p_4, r_4; p_5, r_5; p_6, r_6))$$

$$\cong\langle a_1, a_2, a_3, a_4, a_5, a_6 \mid a_1^{-r_1}a_2^{p_2}a_3^{r_3}=1,$$

$$a_2^{r_2}a_3^{p_3}a_4^{-r_4}=1,\ a_3^{-r_3}a_4^{p_4}a_5^{r_5}=1,\ a_4^{r_4}a_5^{p_5}a_6^{-r_6}=1,$$

$$a_5^{-r_5}a_6^{p_6}a_1^{r_1}=1,\ a_6^{r_6}a_1^{p_1}a_2^{-r_2}=1\rangle.$$

Note that

$$a_2^{p_2}a_4^{p_4}a_6^{p_6}=1 \quad \text{and} \quad a_1^{p_1}a_3^{p_3}a_5^{p_5}=1$$

are consequences of the relators of this presentation. This presentation is expressed by the following 4-regular planar graph with labels (Figure 7).

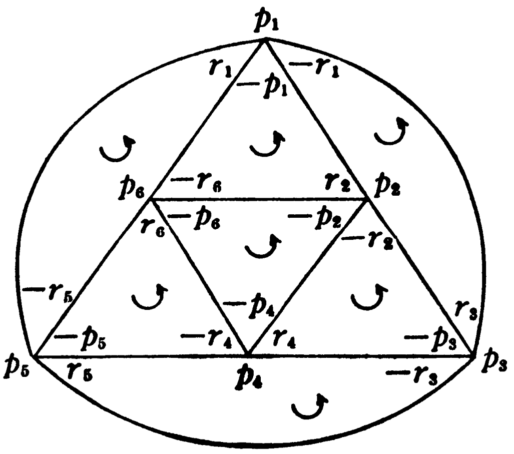


Figure 7

2.4. Consider the link L_{2n} illustrated in the Figure 8.

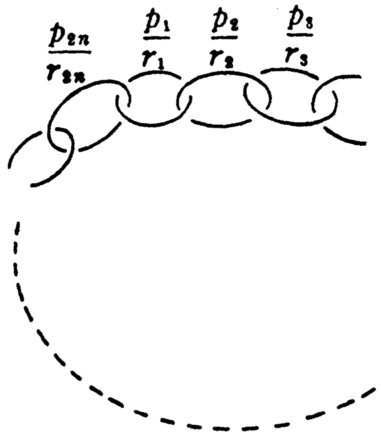


Figure 8

Let $M'_{2n}(p_1, r_1; p_2, r_2; \cdots; p_{2n}, r_{2n})$ be the 3-ma·
gery along each component of L_{2n} with surger
p_{2n}/r_{2n}. Then,

$$\pi_1(M'_{2n}(p_1, r_1; p_2, r_2; \cdots; p_{2n}, r_{2n}))$$

$$\cong\langle a_1, a_2, \cdots, a_{2n} \mid a_{2i}^{r_{2i}} a_{2i+1}^{p_{2i+1}} a_{2i+2}^{-r_{2i+2}} =$$

$$a_{2i-1}^{-r_{2i-1}} a_{2i}^{p_{2i}} a_{2i+1}^{r_{2i+1}} = 1, \quad (i=1, 2, \cdots, n) \text{ (n}$$

For example, if $n=5$ then the presentation is exp
4-regular graph with labels (Figure 9).

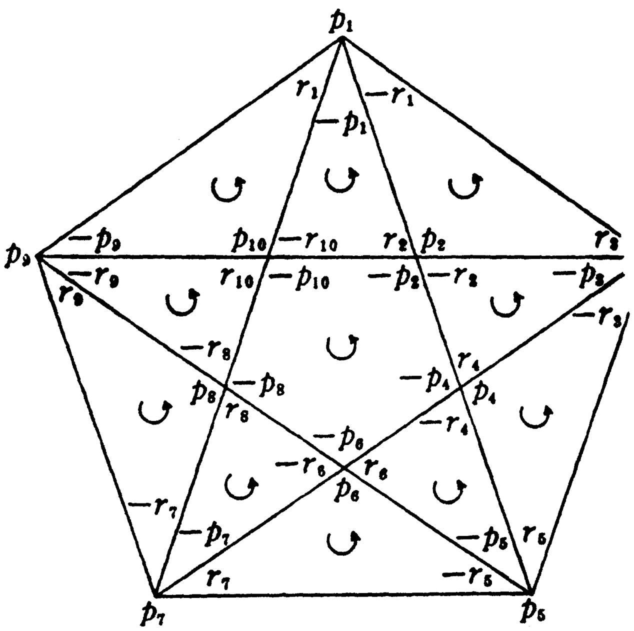


Figure 9

the results in § 2.
:ted planar graph with the following label for
ıere p, r are co-prime integers. We call such

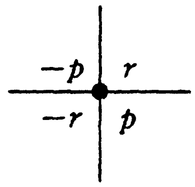


Figure 10

rable. We color the faces by two colors (say, red
4-color problem. We assume that G is drawn on
)[3].

copies of G. We assume that G_1, G_2, G_3, G_4 are

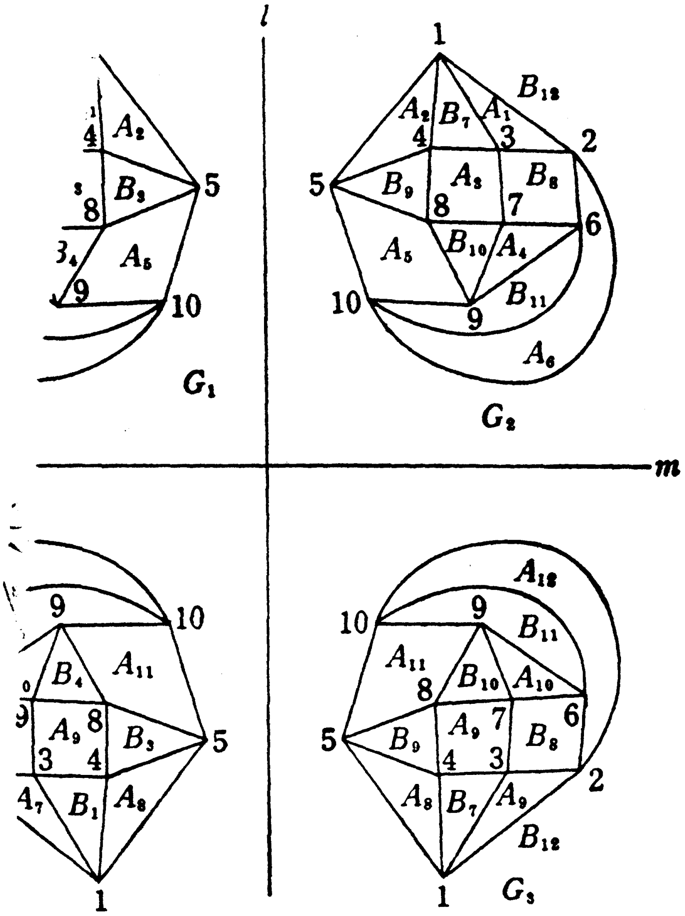


Figure 11

drawn on the boundaries of the 3-disks D_1^3, D_2^3, D_3^3, G_4^3, respectively, and the faces of G_1, G_2, G_3, G_4 are colored in the same way as G. Moreover we assume that G_2, G_4 are mirror images of G_1, G_3, The Figure 11 is an example. (This figure is symmetric with respect to the lines l and m.)

We glue the corresponding points of ∂D_1^3, ∂D_2^3, ∂D_3^3, ∂D_4^3, in the following way. The corresponding points in the red faces of G_1 and G_2 are glued together; the corresponding points in the red faces of G_3 and G_4 are glued together; the corresponding points in the blue faces of G_1 and G_4 are glued together; the corresponding points in the blue faces of G_2 and G_3 are glued together.

red faces	blue faces
$G_1 \longleftrightarrow G_2$	$G_1 \longleftrightarrow G_4$
$G_3 \longleftrightarrow G_4$	$G_2 \longleftrightarrow G_3$

Then the corresponding vertices of G_1, G_2, G_3, G_4 are glued together. We remove the interiors of regular neighborhoods of these vertices. Then we obtain a 3-manifold, whose boundary consists of the same number of tori as the number of vertices of G. We denote this manifold by $M'(G)$.

In the neighborhood of a vertex the situation is as shown in the Figure 12.

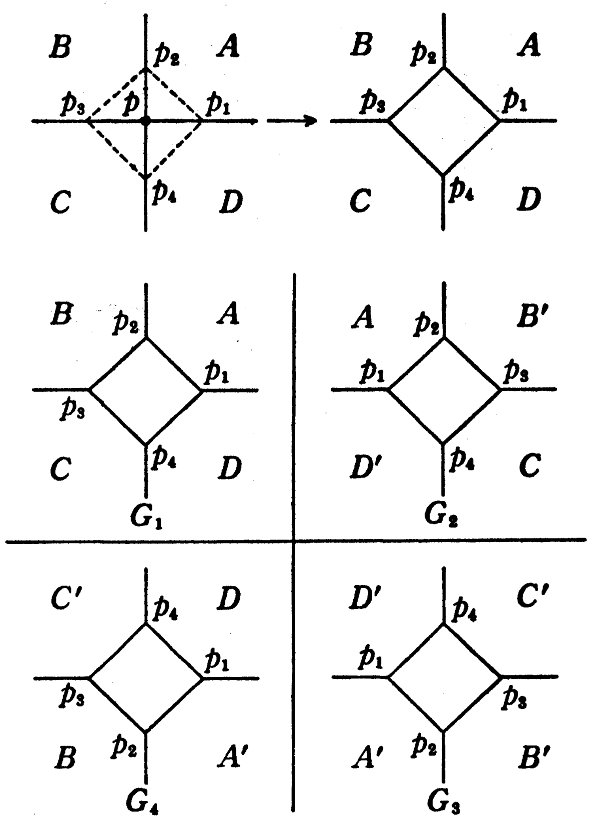
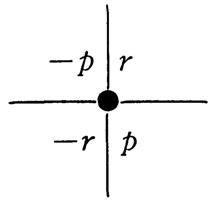
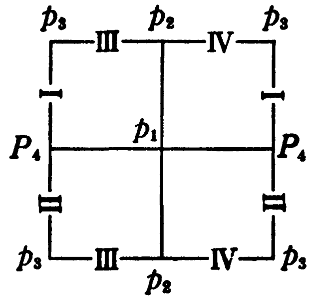


Figure 12

Here, if the label at a vertex is



then we do Dehn surgery (Dehn filling) on the corresponding boundary torus along the loop of slope p/r. Examples are shown in the Figure 13.



boundary torus

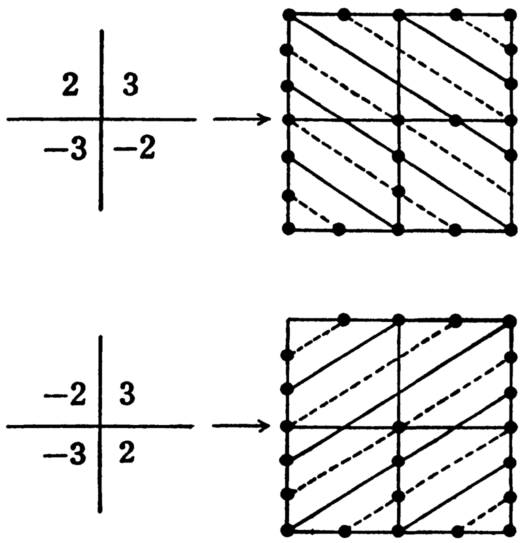


Figure 13

Then we obtain a closed orientable 3-manifold, which we denote by $M(G)$. We say that the graph G represents $M(G)$. For this the following theorem holds.

THEOREM 1. *Let M be a closed orientable connected 3-mainfold. In order for M to be representable by an M-graph it is necessary and sufficient that M is* *homeomorphic to the 2-fold branched covering space of S^3 branched along a link.*

PROOF. [Necessity] Suppose that M is represented by an M-graph G. We change G to a link L in the following way.

For every vertex of G with label as shown in the Figure 14 (we can assume $p\geqq 0$, $r\geqq 0$) we insert the rational tangle shown in the Figure 15. (The Figure 16

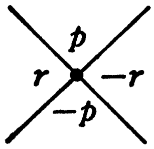


Figure 14

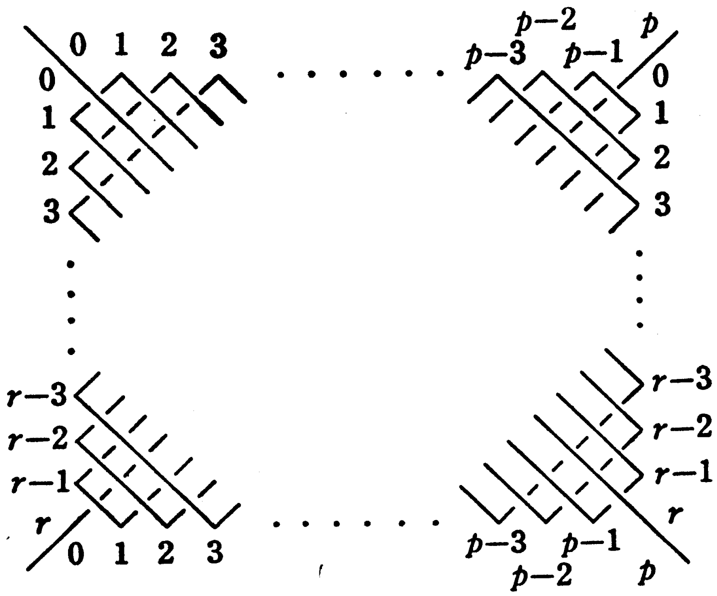


Figure 15

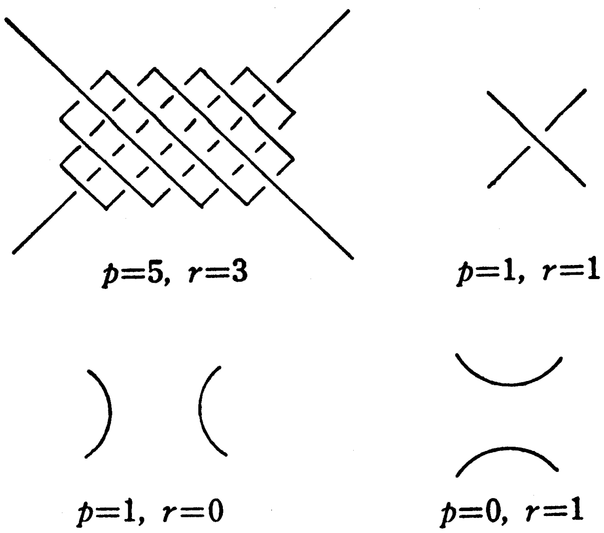


Figure 16

shows examples.)

Now it is not hard to see that M is homeomorphic to the 2-fold branched covering space of S^3 branched along the link L now constructed.

[sufficiency] Suppose that M is the 2-fold branched covering space of S^3 branched along a link L. Consider a regular projection P of L on a plane. We change P to an M-graph G by changing each crossing point to a vertex with label as shown in the Figure 17.

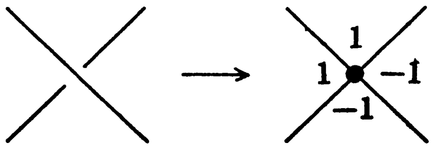


Figure 17

Then as above M is represented by this M-graph. q. e. d.

Next we shall find a presentation of the fundamental group of $M'(G)$.

Let $\mathcal{F}$ be the set of all faces of G and let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be the set of all red (resp. blue) faces of G. Let $\mathcal{V}$ be the set of all vertices of G. For each vertex V we correspond generators b_V, c_V and write the following at the vertex V.

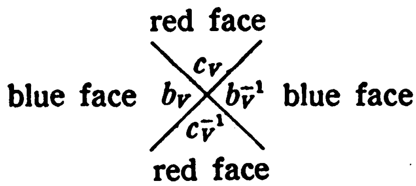
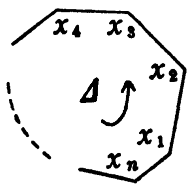


Figure 18

For each face Δ of G, we correspond the relator $r_\Delta=1$ in the following way as illustrated in the Figure 19.



$r_\Delta \equiv x_1 x_2 x_3 x_4 \cdots x_n$

Figure 19

THEOREM 2.

$$\pi_1(M'(G)) \cong \langle \{b_V, c_V : V \varepsilon \mathcal{F}\} \mid \{[b_V, c_V]=1 : V \varepsilon \mathcal{V}\}, \{r_\Delta=1 : \Delta \varepsilon \mathcal{F}\} \rangle .$$

Each relator $r_\Delta=1$ $(\Delta \varepsilon \mathcal{F}_i)$ *is a consequence of* $\{r_{\Delta'}=1 : \Delta' \varepsilon \mathcal{F}_i - \{\Delta\}\}$, *for* $i=$

1, 2. *So two of the relators of the above presentation are redundant.*

PROOF. Let

$$J_i = D_i^3 - \{\text{vertices}\}, \quad (i=1, 2, 3, 4).$$

Let X be the space obtained from J_1 and J_2 by glueinng the corresponding points of the red faces of them.

$\pi_1(X)$ is a free group. Now we take a base point O in the interior of J_2 and define a loop b_V for each $V \varepsilon \mathcal{V}$ as follows.

b_V starts from O, proceeds in J_2 and reaches a point A_1 of a red face Δ_1 with vertex V, and then proceeds in J_1 and reaches a point A_2 of another red face Δ_2 with vertex V and again proceeds in J_2 and returns to O.

$$b_V : O \xrightarrow{J_2} A_1 \xrightarrow{J_1} A_2 \xrightarrow{J_2} O$$

It is easy to see that

$$\pi_1(X) \cong \langle \{b_V : V \varepsilon \mathcal{V}\} \mid \{r_\Delta = 1 : \Delta \varepsilon \mathcal{F}_2\} \rangle ,$$

and that each $r_\Delta = 1$ is a consequence of $\{r_{\Delta'} = 1 : \Delta' \varepsilon \mathcal{F}_2 - \{\Delta\}\}$.

Next let Y be the space obtained by glueing the corresponding points of blue faces of J_2 and J_3. We define the loop c_V $(V \varepsilon \mathcal{V})$ as follows. c_V starts from O, proceeds in J_2 and reaches a point B_1 of a blue face Δ_3 with vertex V, and then proceeds in J_3 and reaches a point B_2 of another blue face Δ_4 with vertex V and again proceeds in J_2 and returns to O.

$$c_V : O \xrightarrow{J_2} B_1 \xrightarrow{J_3} B_2 \xrightarrow{J_2} O .$$

As before, we have that

$$\pi_1(Y) \cong \langle \{c_V : V \varepsilon \mathcal{V}] \mid \{r_\Delta = 1 : \Delta \varepsilon \mathcal{F}_1\} \rangle ,$$

and that each $r_\Delta = 1$ is a consequence of $\{r_{\Delta'} = 1 : \Delta' \varepsilon \mathcal{F}_1 - \{\Delta\}\}$.

Next let Z be the space obtained from $J_1 \cup J_2 \cup J_3$ by glueing the corresponding points of red faces of J_1 and J_2 and by glueing the corresponding points of blue faces of J_2 and J_3.

Then,

$$Z = X \cup Y , \qquad X \cap Y = J_2 .$$

By using van Kampen theorem we obtain

$$\begin{aligned} \pi_1(Z) &\cong \pi_1(X) * \pi_1(Y) \\ &\cong \langle \{b_V, c_V : V \varepsilon \mathcal{V}\} \mid \{r_\Delta = 1 : \Delta \varepsilon \mathcal{F}\} \rangle . \end{aligned}$$

Finally let U be the space obtained from $Z\cup J_4$ by glueing the corresponding points of blue faces of J_1 and J_4 and by glueing the corresponding points of red faces of J_3 and J_4. Then,

$$U\cap J_4=\partial D_4^3-\{\text{vertices}\}.$$

By using van Kampen theorem again, we obtain

$$\pi_1(U)\cong\langle\{b_V, c_V: V\varepsilon\mathcal{V}\} \mid \{[b_V, c_V]=1: V\varepsilon\mathcal{V}\}, \{r_\Delta=1: \Delta\varepsilon\mathcal{F}\}\rangle.$$

Now it is obvious that $\pi_1(U)\cong\pi_1(M'(G))$. Hence we have the theorem.

Next let G be an M-graph, and let V be a vertex with label as shown in the Figure 20.

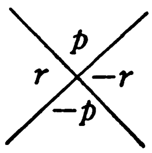


Figure 20

To this vertex we correspond a generator a_V and write the following at the vertex V.

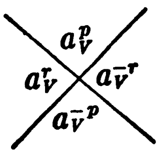


Figure 21

For each face Δ of G we correspond a relator $s_\Delta=1$ in the following way.

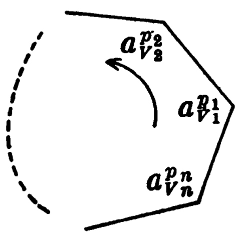


Figure 22

$$s_\Delta=a_{V_1}^{p_1}a_{V_2}^{p_2}\cdots a_{V_n}^{p_n}.$$

Then we have the following theorem.

THEOREM 3. $\pi_1(M(G))\cong\langle\{a_V: V\varepsilon\mathcal{V}\}\mid\{s_\Delta=1: \Delta\varepsilon\mathcal{F}\}\rangle$. *Each relator* $s_\Delta=1$ $(\Delta\varepsilon\mathcal{F}_i)$ *is a consequence of other* $s_{\Delta'}=1$ $(\Delta'\varepsilon\mathcal{F}_i)$ *for* $i=1, 2$. *So two of the relators of the above presentation is redundant.*

PROOF. A presentation of $\pi_1(M(G))$ is obtained from that of $\pi_1(M'(G))$ in Theorem 2 by adding the relator $b_V{}^{p_V}=c_V{}^{r_V}$ for each $V\varepsilon\mathcal{V}$. Since $[b_V, c_V]=1$, $b_V=a_V{}^{r_V}$, $c_V=a_V{}^{p_V}$ for some $a_V\varepsilon\pi_1(M(G))$, as in § 1. So the theorem is obvious from Theorem 2.

Reference

[1] Osborne, R. P. and Stevens, R. S., Group presentations correspoding to spines of 3-manifolds, II, Trans. Amer. Math. Soc. **234** (1977), 213–243.