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ON A CONSTRUCTION OF INDECOMPOSABLE MODULES AND APPLICATIONS

By

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1. Introduction

One of the main purposes of this paper is to introduce a new method to get a family $\{M_n\}_{n=1,2,\ldots}$ of indecomposable modules over a commutative Noetherian local ring R with the maximal ideal $\mathfrak{m}$, which will be done in Theorem (2.1) when R possesses a finitely generated R-module C of $\mathrm{depth}_R C \geqq 1$ such that $C\otimes_R \hat{R}$ (R is the completion of R with respect to the $\mathfrak{m}$-adic topology.) is indecomposable and the initial part of a minimal free resolution of C satisfies certain condition. Each M_n is a finitely generated R-module of $\dim_R M_n = \dim_R C$ and $\mathrm{depth}_R M_n = 0$ and if C is Cohen-Macaulay, then M_n is Buchsbaum (see [9] for the definition of Buchsbaum module.). Furthermore $M_n/H_{\mathfrak{m}}^0(M_n)$ ($H_{\mathfrak{m}}^0(M_n) = \bigcup_{i\geq 1}[(0):\mathfrak{m}^i]_M$) is isomorphic to the direct sum of n-copies of C. Hence in this case there are "big" indecomposable R-modules without limit.

Another aim of us is to apply Theorem (2.1) to the Buchsbaum-representation theory in the one dimensional case. We say that a Noetherian local ring R has finite Buchsbaum-representation type if there are only finitely many isomorphism classes of indecomposable Buchsbaum R-modules M which are maximal, i.e. $\dim_R M = \dim R$. In [4] S. Goto determined the structure of one-dimensional complete Noetherian local rings R of finite Buchsbaum-representation type under the hypothesis that the residue class field of R is infinite, which will be removed in section 3 of this paper. Our family constructed by Theorem (2.1) has the suffix set of non-negative integers and this enables us to develope the same arguments in [4], not assuming the infiniteness of the residue class field.

Throught this paper R is a Noetherian local ring with the maximal ideal $\mathfrak{m}$. We denote by $\hat{R}$ the completion of R with respect to the $\mathfrak{m}$-adic topology and $H_{\mathfrak{m}}^i(\cdot)$ is the i-th local cohomology functor of R relative to $\mathfrak{m}$. For each finitely generated R-module M let $\mu_R(M)$ be the number of elements in a minimal system of generaters for M and let M^n denote the direct sum of n-copies of

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M. We regard each element of M^n as column vector with entries in M.

2. Construction of indecomposable modules.

Let C be a finitely generated R-module and let

$$\sigma : 0 \longrightarrow L \longrightarrow F \xrightarrow{\varepsilon} C \longrightarrow 0$$

be the initial part of a minimal free resolution of C. We define a homomorphism

$$\rho : \mathrm{End}_R(C) \longrightarrow \mathrm{End}_R(L/\mathfrak{m}L)$$

of algebras by

$$\rho(\phi)(\bar{z})=\overline{\psi(z)}$$

for any $\phi \in \mathrm{End}_R(C)$ and $z \in L$, where $\bar{\cdot}$ denotes the reduction mod $\mathfrak{m}L$ and ψ is an R-endomorphism over F with $\varepsilon\psi=\phi\varepsilon$. The well definedness of ρ is verified as follows. If ψ' is another R-endomorphism over F with $\varepsilon\psi'=\phi\varepsilon$, then $\psi'=\psi+\delta$ for some $\delta \in \mathrm{End}_R(F)$ with $\delta(F) \subset L$. Notice that $\delta(L) \subset \mathfrak{m}L$ because $L \subset \mathfrak{m}F$. Then we have $\overline{\psi'(z)}=\overline{\psi(z)}$ for any $z \in L$. We put $A_\sigma = \mathrm{Im}\,\rho$ and we regard $L/\mathfrak{m}L$ as a (left) A_σ-module. If $\mathrm{End}_R(C)$ is generated by $\phi_1, \phi_2, \cdots, \phi_r$ as R-module, then $\rho(\phi_1), \rho(\phi_2), \cdots, \rho(\phi_r)$ generate A_σ over $R/\mathfrak{m}$. Especially A_σ is equal to $R/\mathfrak{m}$ if $\mathrm{End}_R(C)$ is a cyclic R-module.

Our main theorem is stated as follows with the above notations.

THEOREM (2.1). *Let C be a finitely generated R-module such that* $\mathrm{depth}_R C \geq 1$ *and $C \otimes_R \hat{R}$ is indecomposable and let*

$$\sigma : 0 \longrightarrow L \longrightarrow F \xrightarrow{\varepsilon} C \longrightarrow 0$$

be the initial part of a minimal free resolution of C. Suppose there exist elements x and y of L such that $\bar{x}$ and $\bar{y}$ are linearly independent over A_σ. We denote, for each integer $n \geq 1$, by N_n the R-submodule of L^n generated by

$$\begin{pmatrix} x \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \\ x \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ y \\ x \end{pmatrix}$$

and $\mathfrak{m}L^n$. We put $M_n=F^n/N_n$. Then the following statements hold.

(1) *M_n is indecomposable if A_σ is commutative.*

(2) *$M_n \ncong M_m$ if $n \neq m$.*

(3) *M_n is a maximal Buchsbaum R-module if C is maximal Cohen-Macaulay.*

Before the proof of Theorem (2.1) we show the next lemma, which may be well-known, since it plays a key role.

LEMMA (2.2). *Let A be a commutative ring with an identity element and T be an A-module. Suppose there are elements x, y of T which are linearly independent over A and P, Q are $n \times n$ ($n \geqq 1$) matrices with entries in A. Then if*

$$P\begin{bmatrix} x & y & & & \\ & x & y & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & y \\ & & & & x \end{bmatrix} Q = \left[\begin{array}{c|c} \Pi_1 & 0 \\ \hline 0 & \Pi_2 \end{array}\right]$$

for some matrices Π_1 and Π_2 with entries in T, either P or Q is singular.

PROOF. Assume that both P and Q are regular. Let

$$N = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}.$$

As x and y are linearly independent over A, we have

$$PQ = \left[\begin{array}{c|c} \Phi_1 & 0 \\ \hline 0 & \Phi_2 \end{array}\right] \quad \text{and} \quad PNQ = \left[\begin{array}{c|c} \Psi_1 & 0 \\ \hline 0 & \Psi_2 \end{array}\right]$$

for some matrices Φ_i and Ψ_i with entries in A of the same size as Π_i. Since PQ is a regular matrix, Φ_i must be square and regular. Hence we get

$$PNP^{-1} = \left[\begin{array}{c|c} \Omega_1 & 0 \\ \hline 0 & \Omega_2 \end{array}\right].$$

where $\Omega_i = \Psi_i \Phi_i^{-1}$. Take a maximal ideal J of A. For any matrix X with entries in A we denote by $\bar{X}$ the matrix of which entries are the classes of the entries of X in A/J. Then $\bar{P}$ is still regular and

$$\bar{P}\bar{N}(\bar{P})^{-1} = \left[\begin{array}{c|c} \bar{\Omega}_1 & 0 \\ \hline 0 & \bar{\Omega}_2 \end{array}\right].$$

But this contradicts the uniqueness of the Jordan's normalform.

Now let us start the proof of Theorem (2.1).

(3). Applying [4, Lemma (2.3)] to the exact sequence

$$\tau: 0 \longrightarrow L^n \longrightarrow F^n \xrightarrow{\begin{bmatrix} \varepsilon & & \\ & \ddots & \\ & & \varepsilon \end{bmatrix}} C^n \longrightarrow 0$$

and N_n we get that M_n is a maximal Buchsbaum R-module if C is maximal Cohen-Maculay.

(2). The exact sequence

$$0 \longrightarrow L^n/N_n \longrightarrow F^n/N_n \longrightarrow C^n \longrightarrow 0$$

induced from τ yields $H^0_{\mathfrak{m}}(M_n) = L^n/N_n$ and so $M_n/H^0_{\mathfrak{m}}(M_n) \cong C^n$. Hence $M_n \not\cong M_m$ if $n \neq m$.

(1). We shall prove that M_n is indecomposable in the following. Assume $M_n = X_1 \oplus X_2$ with non-zero R-submodules X_i. Then $\overline{X}_1 \oplus \overline{X}_2 \cong C^n$, where $\overline{X}_i = X_i/H^0_{\mathfrak{m}}(X_i)$. Since the category of finitely generated $\hat{R}$-modules is a Krull-Schmidt category and since $C \otimes_R \hat{R}$ is indecomposable, so $\overline{X}_i \otimes_R \hat{R} \cong C^{s_i} \otimes_R \hat{R}$ for some integers s_i with $s_1 + s_2 = n$. So we have $\overline{X}_i \cong C^{s_i}$ by [8, Lemma 5.8]. Because $H^0_{\mathfrak{m}}(X_i) \subset \mathfrak{m}X_i$ by $H^0_{\mathfrak{m}}(M_n) \subset \mathfrak{m}M_n$, we get a commutative diagrams

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & L^{s_i} & \longrightarrow & F^{s_i} & \longrightarrow & C^{s_i} & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \longrightarrow & N'_i & \longrightarrow & F^{s_i} & \longrightarrow & X_i & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & H^0_{\mathfrak{m}}(X_i) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns for $i=1, 2$. Then $F^{s_i}/N'_i \cong X_i$ and $\mathfrak{m}L^{s_i} \subset N'_i \subset L^{s_i}$. Let $t_i = \mu_R(N'_i)$ and let N'_i be generated by

$$\begin{pmatrix} z^{(i)}_{1,1} \\ \vdots \\ z^{(i)}_{s_i,1} \end{pmatrix}, \begin{pmatrix} z^{(i)}_{1,2} \\ \vdots \\ z^{(i)}_{s_i,2} \end{pmatrix}, \cdots, \begin{pmatrix} z^{(i)}_{1,t_i} \\ \vdots \\ z^{(i)}_{s_i,t_i} \end{pmatrix} \quad (z^{(i)}_{\nu,\mu} \in L).$$

Let N' be an R-submodule of $L^n = L^{s_1} \oplus L^{s_2}$ which is generated by

$$\begin{pmatrix} z_{1,1}^{(1)} \\ \vdots \\ z_{s_1,1}^{(1)} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} z_{1,t_1}^{(1)} \\ \vdots \\ z_{s_1,t_1}^{(1)} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline z_{1,1}^{(2)} \\ \vdots \\ z_{s_2,1}^{(2)} \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline z_{1,t_2}^{(2)} \\ \vdots \\ z_{s_2,t_2}^{(2)} \end{pmatrix}.$$

Then $F^n/N' \cong X_1 \oplus X_2$ and so $F^n/N' \cong F^n/N_n$. Hence applying [4, Lemma (2.3)] to N_n, N' and τ we have $\psi(N_n)=N'$ for some $\psi \in \mathrm{Aut}_R(F^n)$ with $\psi(L^n) \subset L^n$. Let $\xi \in \mathrm{End}_R(L^n/\mathfrak{m}L^n)$ be the endomorphism induced from ψ. We identify $\mathrm{End}_R(L^n/\mathfrak{m}L^n)$ with the matrix algebra $M_n(\Gamma)$, where $\Gamma = \mathrm{End}_R(L/\mathfrak{m}L)$. Put $\xi = [\xi_{ij}]_{1 \leq i,j \leq n}$. Since there is an automorphism $\phi \in \mathrm{Aut}_R(C^n)$ which makes the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^n & \longrightarrow & F^n & \longrightarrow & C^n & \longrightarrow & 0 \\ & & & & \psi \downarrow & & \phi \downarrow & & \\ 0 & \longrightarrow & L^n & \longrightarrow & F^n & \longrightarrow & C^n & \longrightarrow & 0 \end{array}$$

commutative, we have $\xi_{ij} \in A_\sigma$ for any $1 \leqq i$, $j \leqq n$ and $[\xi_{ij}]_{1 \leq i,j \leq n}$ is a regular matrix of $M_n(A_\sigma)$. Furthermore because $\xi(N_n/\mathfrak{m}L^n) = N'/\mathfrak{m}L^n$, we have

$$[\xi_{ij}] \begin{bmatrix} \bar{x} & \bar{y} & & \\ & \bar{x} & \bar{y} & \\ & & \ddots & \ddots \\ & & & \cdot\ \bar{y} \\ & & & \bar{x} \end{bmatrix} Q = \left[\begin{array}{ccc|ccc} \overline{z_{1,1}^{(1)}} & \cdots & \overline{z_{1,t_1}^{(1)}} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \overline{z_{s_1,1}^{(1)}} & \cdots & \overline{z_{s_1,t_1}^{(1)}} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & \overline{z_{1,1}^{(2)}} & \cdots & \overline{z_{1,t_2}^{(2)}} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \overline{z_{s_2,1}^{(2)}} & \cdots & \overline{z_{s_2,t_2}^{(2)}} \end{array}\right]$$

for some $n \times n$ regular matrix Q with entries in $R/\mathfrak{m}$ (Hence $Q \in M_n(A_\sigma)$). But this is a contradiction by Lemma (2.2) and the proof is completed.

We note the following corollary which is a special case of Theorem (2.1).

COROLLARY (2.3). *Let C and*

$$\sigma : 0 \longrightarrow L \longrightarrow F \longrightarrow C \longrightarrow 0$$

be as in Theorem (2.1) *and let $A_\sigma = R/\mathfrak{m}$. Then if $\mu_R(L) \geqq 2$, there exists a family $\{M_n\}_{n=1,2\dots}$ of finitely generated indecomposable R-modules such that $M_n \not\cong M_m$ for $n \neq m$ and M_n is maximal Buchsbaum if C is maximal Cohen-Macaulay.*

The typical example such that A_σ is not equal to $R/\mathfrak{m}$ is the next

EXAMPLE (2.4). *Let k be any field, then the semi-group ring $R = k[\![t^3, t^4, t^5]\!]$*

has a family $\{M_n\}_{n=1,2,\dots}$ *of indecomposable maximal Buchsbaum R-modules such that* $M_n \not\cong M_m$ *if* $n \neq m$.

PROOF. Put $S=k[\![t]\!]=R+Rt+Rt^2$ and let

$$\sigma : 0 \longrightarrow L \longrightarrow R^3 \overset{\varepsilon}{\longrightarrow} S \longrightarrow 0$$

be the initial part of a minimal free resolution with

$$\varepsilon(e_1)=1, \quad \varepsilon(e_2)=t \quad \text{and} \quad \varepsilon(e_3)=t^2,$$

where e_1, e_2, e_3 are the canonical basis of R^3. Since S is an indecomposable maximal Cohen-Macaulay R-module, by Theorem (2.1) it is sufficient to show that A_σ is commutative and there exist elements x and y of L such that $\bar{x}$ and $\bar{y}$ are linearly independent over A_σ, where $\bar{\ }$ denotes the reduction mod $\mathfrak{m}L$ ($\mathfrak{m}=t^3S$). As $\mathrm{End}_R(S)$ is a commutative R-algebra which is generated by $\mathbf{1}_S$, $t\mathbf{1}_S$ and $t^2\mathbf{1}_S$ as R-module, so A_σ is commutative and $\rho(\mathbf{1}_S)=\mathbf{1}_{L/\mathfrak{m}L}$, $\rho(t\mathbf{1}_S)$ and $\rho(t^2\mathbf{1}_S)$ generate A_σ over k. We put $\xi_i=\rho(t^i\mathbf{1}_S)$ for $i=1, 2$. Let α_1 and α_2 be the R-endomorphisms over R^3 defined by the matrices

$$\begin{bmatrix} 0 & 0 & t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & t^3 & 0 \\ 0 & 0 & t^3 \\ 1 & 0 & 0 \end{bmatrix}$$

respectively. Then $\varepsilon\alpha_i=(t^i\mathbf{1}_S)\varepsilon$ for $i=1, 2$. Hence ξ_i is induced from α_i. Put

$$x=\begin{bmatrix} t^4 \\ -t^3 \\ 0 \end{bmatrix}, \quad x_1=\begin{bmatrix} 0 \\ t^4 \\ -t^3 \end{bmatrix}, \quad x_2=\begin{bmatrix} -t^6 \\ 0 \\ t^4 \end{bmatrix}, \quad y=\begin{bmatrix} t^5 \\ -t^4 \\ 0 \end{bmatrix}, \quad y_2=\begin{bmatrix} 0 \\ t^5 \\ -t^4 \end{bmatrix}, \quad y_3=\begin{bmatrix} -t^7 \\ 0 \\ t^5 \end{bmatrix}.$$

Then we have $\xi_i\bar{x}=\bar{x}_i$ and $\xi_i\bar{y}=\bar{y}_i$. Assume

$$(a_0\mathbf{1}_{L/\mathfrak{m}L}+a_1\xi_1+a_2\xi_2)\bar{x}+(b_0\mathbf{1}_{L/\mathfrak{m}L}+b_1\xi_1+b_2\xi_2)\bar{y}=0$$

with $a_i \in k$ and $b_j \in k$. Then we get

$$a_0\bar{x}+a_1\bar{x}_1+a_2\bar{x}_2+b_0\bar{y}+b_1\bar{y}_1+b_2\bar{y}_2=0\,.$$

Since $\bar{x}$, $\bar{x}_1$, $\bar{x}_2$, $\bar{y}$, $\bar{y}_1$, $\bar{y}_2$ are linearly independent over k, so

$$a_0=a_1=a_2=b_0=b_1=b_2=0\,.$$

Hence $\bar{x}$ and $\bar{y}$ are linearly independent over A_σ.

3. Curve singularities of finite Buchsbaum-representation type.

This section is devoted to verifying that we can avoid the restriction on the residue class field in [4, Theorem (1.1)]. But we will not look over the whole arguments developed in [4] since it is sufficient to prove only [4, Corollary (2.4)]. [4, Theorem (2.7)], [4, Theorem (3.1)] and [4, Proposition (6.1)] without the hypothesis that the residue class field is infinite. Throught this section we assume that R is complete and $\dim R=1$.

We begin with the following

PROPOSIEION (3.1) (cf. [4, Corollary (2.4)]). *Let R have finite Buchsbaum-representation type and I be an ideal of R such that R/I is a Cohen-Macaulay ring of* $\dim R/I=1$. *Then* $\mu_R(I)\leqq 1$.

PROOF. Applying Corollary (2.3) to the exact sequence

$$\sigma: 0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

we have $\mu_R(I)\leqq 1$.

THEOREM (3.2) (cf. [4, Theorem (2.7)]). *Let R be a Cohen-Macaulay ring with the canonical module K_R. If R has finite Buchsbaum-representation type, then $v(R)\leqq 2$, where $v(R)$ denotes the embedding dimension of R.*

PROOF. Let

$$\sigma: 0 \longrightarrow M \longrightarrow F \longrightarrow K_R \longrightarrow 0$$

be the initial part of a minimal free resolution of K_R. Since $\mathrm{End}_R(K_R)=R$, we have $A_\sigma=R/\mathfrak{m}$. Hence $\mu_R(M)\leqq 1$ by Corollary (2.3). Then the proof of [4, Theorem (2.7)] works for the rest.

THEOREM (3.3) (cf. [4, Theorem (3.2)]). *Let P be a regular local ring of* $\dim P=2$ *and let $R=P/fP$ with $f\in P$. We denote the integral closure of R in its total quotient ring by $\bar{R}$. If $\bar{R}$ is module-finite over R and $e(R)\geqq 3$, where $e(R)$ denote the multiplicity of R, then there exists a family $\{M_n\}_{n=1,2,\dots}$ of indecomposable maximal Buchsbaum R-modules such that $M_n\not\cong M_m$ if $n\neq m$.*

PROOF. Let L be the first syzygy module of $\mathfrak{m}$. Since $\mathfrak{m}$ is an indecomposable maximal Cohen-Macaulay R-module, the minimal free resolution of $\mathfrak{m}$ is periodic of period 2 and L is indecomposable by [2] and [7]. Hence we have an exact seqeence

$$\sigma: 0 \longrightarrow \mathfrak{m} \longrightarrow F \longrightarrow L \longrightarrow 0$$

with F R-free. We put $A=\{x\in\bar{R} \mid x\mathfrak{m}\subset\mathfrak{m}\}$, which we identify with $\mathrm{End}_R(\mathfrak{m})$ as algebras. Then by [4, Proposition (3.4)] there is an element $h\in A$ such that $A=R+Rh$ and $h\mathfrak{m}\subset\mathfrak{m}^2$. Hence $\mathrm{End}_R(\mathfrak{m}/\mathfrak{m}^2)=A_\sigma=R/\mathfrak{m}$. Since R is not regular, $\mu_R(\mathfrak{m})\geqq 2$ and so we can get the required family by Corollary (2.3).

By Theorem (3.2), Theorem (3.3) and [1] we have the next

THEOREM (3.4) (cf. [4, Theorem (3.1)]). *Let R be a Cohen-Macaulay ring. Then R is reduced and $e(R)\leqq 2$ if R has finite Buchsbaum-representation type.*

Finally we prove the following

PROPOSITION (3.5) (cf. [4, Proposition (6.1)]). *If R has finite Buchsbaum-representation type, then $e(R)\leqq 2$ and $v(R)\leqq 2$.*

PROOF. Our method of proof is almost the same as the proof of [4, Proposition (6.1)] and so see it for the detail.

Let $I=H_\mathfrak{m}^0(R)$. Then R/I is a Cohen-Macaulay ring of finite Buchsbaum-representation type. Hence we get by Theorem (3.4) that R/I is reduced and $e(R/I)=e(R)\leqq 2$. As $\mu_R(I)\leqq 1$ by Proposition (3.1) and as $v(R/I)\leqq 2$ by Theorem (3.2), we have $v(R)\leqq 3$. We show $v(R)\neq 3$ in the following. Assume $v(R)=3$. Then $v(R/I)=2$ and $\mu_R(I)=1$, hence $e(R/I)=2$. We put $I=zR$. By [4, Claim 1 in the proof of Proposition (6.1)] R/I is an integral domain. We denote the normalization of R/I by S. Let $\bar{\mathfrak{m}}$ be the maximal ideal of R/I and let $\bar{\ }$ denote the reduction mod I. Then there are elements x and y of $\mathfrak{m}$ such that $\bar{\mathfrak{m}}=(\bar{x}, \bar{y})$ and $S=R+Rt$, where $t=\bar{x}/(\bar{y})^n$ for suitable $n\geqq 1$. Furthermore we get an exact sequence

$$R^4 \xrightarrow{\begin{bmatrix} z & 0 & x & bx+ay^n \\ 0 & z & y^n & x \end{bmatrix}} R^2 \xrightarrow{\ \varepsilon\ } S \to 0$$

for some $a\in R$ and $b\in R$ with $t^2=\bar{a}+\bar{b}t$, where

$$\varepsilon\begin{bmatrix} 1 \\ 0 \end{bmatrix}=1 \quad \text{and} \quad \varepsilon\begin{bmatrix} 0 \\ 1 \end{bmatrix}=-t.$$

We note if $a\in\mathfrak{m}$, then $b\in\mathfrak{m}$. Let $L=\mathrm{Ker}\,\varepsilon$. Then L is generated by

$$v_1=\begin{bmatrix} z \\ 0 \end{bmatrix},\quad v_2=\begin{bmatrix} 0 \\ z \end{bmatrix},\quad v_3=\begin{bmatrix} x \\ y^n \end{bmatrix},\quad v_4=\begin{bmatrix} bx+ay^n \\ x \end{bmatrix}.$$

We apply Theorem (2.1) to the exact sequence

$$\sigma : 0 \longrightarrow L \longrightarrow R^2 \overset{\varepsilon}{\longrightarrow} S \longrightarrow 0 .$$

Since $\mathrm{End}_R(S)$ is a commutative R-algebra which is generated by $\mathbf{1}_S$ and $t\mathbf{1}_S$ as R-module, so A_σ is commutative and $\rho(\mathbf{1}_S)=\mathbf{1}_{L/\mathfrak{m}L}$ and $\rho(t\mathbf{1}_S)$ generate A_σ over $R/\mathfrak{m}$. We put $\xi=\rho(t\mathbf{1}_S)$. Because the following diagram

$$\begin{array}{ccc} R^2 & \xrightarrow{\ \varepsilon\ } & S \\ {\scriptstyle \begin{bmatrix} 0 & a \\ 1 & -b \end{bmatrix}} \downarrow & & \downarrow {\scriptstyle t\mathbf{1}_S} \\ R^2 & \xrightarrow{\ \varepsilon\ } & S \end{array}$$

is commutative, we have

$$\xi\bar{v}_1=\bar{v}_2, \quad \xi\bar{v}_2=(a \bmod \mathfrak{m})\bar{v}_1-(b \bmod \mathfrak{m})\bar{v}_2 ,$$

$$\xi\bar{v}_3=\bar{v}_4-(b \bmod \mathfrak{m})\bar{v}_3, \quad \xi\bar{v}_4=(a \bmod \mathfrak{m})\bar{v}_3 ,$$

where $\bar{v}_i=v_i \bmod \mathfrak{m}L$ for $1\leqq i\leqq 4$. Hence if $a\in\mathfrak{m}$, then $\bar{v}_1$ and $\bar{v}_4$ are linearely independent over A_σ and if $a\in\mathfrak{m}$, then $\bar{v}_1$ and $\bar{v}_3$ are linearly independent over A_σ. But this is a contradiction by Theorem (2.1).

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