TSUKUBA J. MATH.
Vol. 13 No. 1 (1989). 113–129

ON LORENTZ MANIFOLDS WITH ABUNDANT ISOMETRIES

By

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0. Introduction.

Let M be an n-dimensional Lorentz manifold with metric $\langle , \rangle$ of signature $(-, +, \cdots, +)$. Then there is no r-dimensional isometry group whose isotropy subgroup at every point is compact for $n(n-1)/2+1<r\leqq n(n+1)2$ (c.f., [5], Proposition). In [6], we determined n-dimensional Lorentz manifolds M which admit an $n(n-1)/2+1$-dimensional isometry group with compact isotropy subgroup at every point for $n\geqq 4$.

The first purpose of this note is to determine simply connected M admitting an $n(n-1)/2$-dimensional isometry group with compact isotropy subgroup at every point for $n\geqq 4$ (see § 2). We will prove the following Theorem A.

THEOREM A. *Let $(M, \langle , \rangle)$ be a simply connected n-dimensional Lorentz manifold admitting a connected $n(n-1)/2$-dimensional isometry group with compact isotropy subgroup at every point in M $(n\geqq 4)$. Then M is isometric to the warped product manifold $(I\times N, -dt^2+\phi(t)ds_N^2)$ where I is an open interval and N is the simply connected $(n-1)$-dimensional Riemannian manifold with metric ds_N^2 of constant curvature and $\phi(t)$ is a positive function on I.*

For isometry groups whose dimension are less than $n(n-1)/2$, we will have the following proposition in § 1.

PROPOSITION 1.1. *If $n\geqq 6$, there is no r-dimensional isometry group with compact isotropy subgroup at every point for $(n-1)(n-2)/2+3\leqq r\leqq n(n-1)/2-1$.*

In view of Proposition 1.1, it is natural to ask which Lorentz manifold of dimension n admits an $(n-1)(n-2)/2+2$-dimensional isometry group with compact isotropy subgroup. The second purpose of this note is to determine simply connected manifold M admitting an isometry group of dimension $(n-1)(n-2)/2+2$ with compact isotropy subgroup at every point (see § 3). We will prove the following Theorem B.

Received August 24, 1987. Revised April 7, 1988.

THEOREM B. *Let* $(M, \langle\ ,\ \rangle)$ *be a simply connected n-dimensional Lorentz manifold adimitting a connected* $(n-1)(n-2)/2+2$*-dimensional isomery group with compact isotropy subgroup at every point* $(n\geqq 6)$. *Then* $(M, \langle\ ,\ \rangle)$ *must be one of the following*:

(1) $(\boldsymbol{L}^2\times V^{n-1},\ ds_L^2+ds_V^2)$;

(2) $(\boldsymbol{L}^2\times \boldsymbol{E}^{n-1},\ -dt^2+ds^2+\exp(-2c_0t-2c_1s)ds_E^2)$ (c_0 *and* c_1 *are some constants such that* $c_0\neq 0$ or $c_1\neq 0$);

(3) $(U^2\times V^{n-2},\ ds_0^2+ds_V^2)$;

(4) $(U^2\times \boldsymbol{E}^{n-2},\ ds_0^2+f^2ds_E^2)$ ($f=y^{-c_2}$, c_2 *is a non-zero constant*);

(5) $(U^2\times V^{n-2},\ ds_\kappa^2/\alpha^2+ds_V^2)$ (α *is a non-zero constant*);

(6) $(U^2\times \boldsymbol{E}^{n-2},\ ds_\kappa^2/\beta^2+h^2ds_E^2)$ ($h=(\beta y)^{-c_3}$, c_3 *and* β *are non-zero constants*);

If $n=9$, *then the following additional case is possible*:

(7) $(\boldsymbol{R}\times \boldsymbol{E}^8,\ -dt^2+\exp(-2c_4t)ds_E^2)$ ($c_4>0$: *a constant*).

Here $(\boldsymbol{L}^2, ds_L^2)$ is the 2-dimensional Minkowski space, $(\boldsymbol{E}^m, ds_E^2)$ the m-dimensional Euclidean space and (V^{n-2}, ds_V^2) the simply connected $(n-2)$-dimensional Riemannian space of constant curvature. Further, (U^2, ds_κ^2) is the upper half-space $U^2=\{(x, y);\ y>0\}$ with metric $-2dxdy/y^2$ (when $\kappa=0$) $\kappa(dx^2-dy^2)/y^2$ (when $\kappa=1$ or -1).

REMARK 0.1. The space (6) with $c_3=1$ is the upper half-space $U^n=\{(x_1, \cdots, x_n);\ x_n>0\}$ with constant curvature 1 or -1 according to $\kappa=1$ or -1 respectively. The space (7) is isometric to the 9-dimensional upper-half space with constant curvature c_4^2 by the transformation

$$\boldsymbol{R}\times\boldsymbol{E}^8\ni(t, x_1, \cdots, x_8)\longrightarrow(x_1, \cdots, x_8, e^{c_4t}/c_4)\in U^9.$$

For these spaces, see [4] and [8].

The space (4) with $c_2=1$ is the upper half-space with constant curvature 0.

Throughout this note, we shall be in C^∞-category and manifolds shall be connected, unless otherwise stated.

1. Preliminaries.

Let $(M, \langle\ ,\ \rangle)$ be an n-dimensional Lorentz manifold with metric $\langle\ ,\ \rangle$ of signature $(-, +, \cdots, +)$. Let G be a connected isometry group of $(M, \langle\ ,\ \rangle)$, H_0 the isotropy subgroup of G at a point $o\in M$ and $G(o)$ the G-orbit of o. Then the linear isotropy subgroup $\tilde{H}_0=\{dh;\ h\in H_0\}$ acting on T_oM is a closed subgroup of $O(1, n-1)=\{A\in GL(n, \boldsymbol{R});\ {}^tASA=S\}$, where S is the matrix

$$\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

If H_o is compact, $\tilde{H}_o$ is conjugate to a subgroup of $O(1)\times O(n-1)$ (c.f., [10, p. 335]).

LEMMA 1.2. *If* $\dim H_o=(n-1)(n-2)/2$ *and* H_o *is compact, then* $\dim G(o)\leqq 1$ *or* $\geqq n-1$ *for* $n\geqq 3$.

PROOF. Since $\tilde{H}_o$ is compact and of dimension $(n-1)(n-2)/2=\dim(O(1)\times O(n-1))$, $\tilde{H}_o$ contains the connected component $1\times SO(n-1)$ of $O(1)\times O(n-1)$. Thus T_oM is naturally decomposed into the direct sum of 1-dimendional and $(n-1)$-dimensional subspaces which are $\tilde{H}_o$-invariant and irreducible. On the other hand, $T_o(G(o))$ is also $\tilde{H}_o$-invariant. Therefore we have $\dim T_o(G(o))\leqq 1$ or $\geqq n-1$.

PROOF OF PROPOSITION 1.1. Let G be a connected isometry group of dimension r. Assume that $(n-1)(n-2)/2+3\leqq r\leqq n(n-1)/2-1$. Then, $\dim H_o=\dim G-\dim(G/H_o)=\dim G-\dim G(o)\geqq(n-2)(n-3)/2+1$. Since H_o is compact, we can regard $\tilde{H}_o$ as a subgroup of $O(1)\times O(n-1)$. If $n-1\neq 4$, there is no k-dimensional subgroup of $O(n-1)$ for $(n-2)(n-3)/2<k<(n-1)(n-2)/2$. Therefore $\dim H_o=(n-1)(n-2)/2$ so that we have $3\leqq\dim G(o)\leqq n-2$. This contradicts Lemma 1.2.

REMARK 1.3. There exist 5-dimensional Lorentz manifolds M admitting a $9(=(5-1)(5-2)/2+3)$-dimensional isometry group G with compact isotropy subgroup. For example, let M be a product manifold $\boldsymbol{R}\times\boldsymbol{C}^2$ with metric $-dt^2+ds_E^2$ and $G=\boldsymbol{R}\times G'$ where ds_E^2 is the Euclidean metric of $\boldsymbol{C}^2$ and G' is the matrix group consisting of all matrices of the form

$$\begin{bmatrix} A & \tau \\ 0 & 1 \end{bmatrix}, \text{ where } A\in U(2),\ \tau\in\boldsymbol{C}^2.$$

Then $\dim G=9$ and the isotropy subgroup at the origin is $U(2)$ which is compact.

2. The case where dim $G=n(n-1)/2$.

Let G be a connected isometry group of dimension $n(n-1)/2$ with compact isotropy subgroup H_x at every point $x\in M$. Then $\tilde{H}_x$ is conjugate to a subgroup of $O(1)\times O(n-1)$, so that we have $\dim H_x \leqq (n-1)(n-2)/2$. On the other hand, $\dim H_x \geqq \dim G - \dim M = (n-1)(n-2)/2-1$. Thus we have $\dim H_x = (n-1)(n-2)/2$ or $(n-1)(n-2)/2-1$. For $n-1 \neq 4$, $O(n-1)$ contains no proper closed subgroup of dimension$>(n-2)(n-3)/2$ other than $SO(n-1)$ (c.f., [2, p. 48]). Thus, when $n-1\neq 4$, $\dim H_x=(n-1)(n-2)/2$. For $n-1=4$, $O(n-1)$ contains no subgroups of dimension $5=(5-1)(5-2)/2-1$ (c.f., [1, p. 347]). Thus, for $n\geqq 4$, we have $\dim H_x=(n-1)(n-2)/2$, so $\tilde{H}_x$ contains the connected component $1\times SO(n-1)$ of $O(1)\times O(n-1)$. Therefore, T_xM is naturally decomposed into the direct sum of 1-dimensional and $(n-1)$-dimensional subspaces which are $\tilde{H}_x$-invariant and irreducible. On the other hand, $T_x(G(x))$ is $\tilde{H}_x$-invariant and of dimension $n-1$. Thus we have irreducible decomposision $T_1(x)+T_x(G(x))$ of T_xM by the linear isotropy representation of H_x on T_xM. Since H_x is compact, the restriction η of the metric of M to $T_x(G(x))$ is positive definite, zero or negative definite by the Schur's lemma. Since $n-1\geqq 3$, η must be positive definite. Therefore we have

LEMMA 2.1. *Each orbit $G(x)$ $(x\in M)$ is a spacelike hypersurface.*

Since $\tilde{H}_x$ contains $1\times SO(n-1)$, we have $\langle T_1(x), T_x(G(x))\rangle=0$ so that $T_1(x)$ is timelike. Let $\xi(x)$ be a unit timelike vector belonging to $T_1(x)$.

LEMMA 2.2. *If M is time-orientable, then the vector field $\xi(p):=dg(\xi(x))$ $(p=gx,\ g\in G)$ is well-defined on $G(x)$ and G-invariant and it is extended to the vector field on M.*

PROOF. The first part of this Lemma is proved by the same method as the proof of Lemma 2 in [6]. Since M is time orientable, there exists a unit timelike vector field ζ on M. Then we can extend ξ on M so as to be $\langle\xi, \zeta\rangle<0$.

From now on, we assume that M is time-orientable. We note that G acts effectively on $G(x)$. In fact, if $g\in G$ acts on $G(x)$ trivially, we have $dg|T_xG(x) =id.$ and $dg(\xi(x))=\xi(x)$, so that $dg=id.$ on $T_xM=\boldsymbol{R}\{\xi(x)\}+T_xG(x)$. Therefore $g=id.$ on M. Furthermore we note that each G-orbit $G(x)$ is isometric to $\boldsymbol{E}^{n-1}$, S^{n-1}, $\boldsymbol{P}^{n-1}$ or $\boldsymbol{H}^{n-1}$, because the $(n-1)$-dimensional Riemannian manifold $G(x)$ admits an isometry group G of maximum dimension $n(n-1)/2$.

LEMMA 2.3. *Each integral curve oj ξ is a geodesic.*

PROOF. Let X be an arbitrary fixed non-zero vector in T_xM such that $\langle\xi(x), X\rangle=0$. Since $\tilde{H}_x$ contains $1\times SO(n-1)$ and $n-1\geqq 3$, there exists $h\in H_x$

such that $dh(X)=-X$ and $dh(\xi(x))=\xi(x)$. We have $\langle\nabla_\xi\xi, X\rangle=\langle dh(\nabla_\xi\xi), dh(X)\rangle =-\langle\nabla_\xi\xi, X\rangle$ so that we have $\langle\nabla_\xi\xi, X\rangle=0$. Since X is an arbitrary vector orthogonal to ξ and $\langle\nabla_\xi\xi, \xi\rangle=(1/2)\, \xi\langle\xi, \xi\rangle=0$, we have $\nabla_\xi\xi=0$. Thus each integral curve of ξ is a geodesic.

LEMMA 2.4. *$\nabla_X\xi=\lambda(\pi(X))X$ for any X such that $\langle X, \xi\rangle=0$ where π is the natural projection of the tangent bundle: $TM\rightarrow M$ and λ is a function on M which is constant on each G-orbit.*

The proof of Lemma 2.4 is similar to that of Lemma 8 in [6].

LEMMA 2.5. *The 1-form ω defined by $\omega(X)=\langle X, \xi\rangle$ is closed.*

PROOF. The 1-form ω is G-invariant and so $d\omega$ is G-invariant (especially, H_x-invariant). Since $\tilde{H}_x$ contains $1\times SO(n-1)$ and the linear isotropy representation of H_x on $T_x(G(x))$ is irreducible, we have $d\omega=0$.

PROOF OF THEOREM A. M is time-orientable, because M is simply connected. Since ω is a closed 1-form from Lemma 2.5, there exists a smooth function $f: M\rightarrow \boldsymbol{R}$ such that $df=\omega$. Let $\gamma_p(t)$ be an integral curve of ξ such that $\gamma_p(0)=p$. Then we can see $f(\gamma_p(t))=-t+f(p)$. We may assume that $f(M)$ is some open interval containing $0\in\boldsymbol{R}$. Let N be a connected component of $f^{-1}(0)$. Then we have $N=G(o)$ for some $o\in N$. For each $x\in N$, let I_x be the domain of γ_x. Since ξ is G-invariant on $N=G(o)$, for any $p, q\in N$, we have $I_p=I_q$ which is denoted by I. Then the Theorem A will follow immediately from the next Lemma 2.6 and Lemma 2.7.

LEMMA 2.6. *The map $F: I\times N\rightarrow M$ defined by*

$$F(t, x)=\operatorname{Exp} t\xi(x)=\gamma_x(t)$$

is a diffeomorphism.

LEMMA 2.7. *The map $F: (I\times N, -dt^2+\phi(t)ds_N^2)\rightarrow(M, \langle\ ,\ \rangle)$ is an isometry, where the metric ds_N^2 on N induced from $\langle\ ,\ \rangle$ and $\phi(t)=\exp 2(\int_0^t\lambda(s)ds)$.*

The proof of Lemmas 2.6 and 2.7 is similar to that of Lemmas 5 and 9 in [6].

3. The case where $\dim G=(n-1)(n-2)/2+2$.

We assume that $\dim G=(n-1)(n-2)/2+2$ and H_x is compact for every point $x\in M$.

PROPOSITION 3.1. *G acts transitively on M for $n\geqq 4$ and $n\neq 5$.*

PROOF. Assume that G does not act transitively on M. Then $\dim G(o)\leqq n-1$ for some $o\in M$. Hence $\dim H_o\geqq\dim G-(n-1)=(n-2)(n-3)/2+1$. By the same method as in the proof of Proposition 1.1, we can see that $\dim H_o=(n-1)(n-2)/2$. Hence $\dim G(o)=2$ which contradicts the Lemma 1.2.

REMARK 3.2. In the Proposition 3.1, we cannot remove the condition that the isotropy subgroup at every point is compact. In fact, let M be the Lorentz manifold $\boldsymbol{R}\times N$ with metric $dt^2+ds_N^2$, where (N, ds_N^2) is the $(n-1)$-dimensional de-Sitter space and G be the group $\boldsymbol{R}\times G'$ where G' is the matrix group of the form

$$\begin{bmatrix}(1+a^2+|\chi|^2)/(2a) & \chi & (1-a^2+|\chi|^2)/(2a)\\ (1/a)A{}^t\chi & A & (1/a)A{}^t\chi\\ (1-a^2-|\chi|^2)/(2a) & -\chi & (1+a^2-|\chi|^2)/(2a)\end{bmatrix}\quad \begin{array}{l}a>0,\ \chi\in\boldsymbol{R}^{n-2},\\ A\in SO(n-2).\end{array}$$

G' is the connected subgroup of the proper Lorentz group $SO^+(1, n-1)$ acting on N (c.f. [7]). Then G is an $(n-1)(n-2)/2+2$-dimensional isometry group which has noncompact isotropy subgroups and does not act on M transitively (see § 4).

REMARK 3.3. There exists a 5-dimensional Lorentz manifold M aditting an $8(=(5-1)(5-2)/2+2)$-dimensional isometry group G with compact isotropy subgroup such that G does not acts transitively on M. In fact, take the space in Remark 1.3 as M and set $G=1\times G'$ (G' is the same as in Remark 1.3). Then G is not transitive on M.

From now on, we assume $n\geqq 6$. Set $H=H_o$ for some $o\in M$. By Proposition 3.1, we have $\dim H=(n-2)(n-3)/2$. Since H is compact and connected, $\tilde{H}$ is conjugate to a subgroup of $SO(1)\times SO(n-1)$ so that we can regard $\tilde{H}$ as an $(n-2)(n-3)/2$-dimensional subgroup of $SO(n-1)$. In the case $n-1\neq 8$, a $(n-2)(n-3)/2$-dimensional subgroup $\tilde{H}$ of $SO(n-1)$ leaves one and only one 1-dimensional subspace of $\boldsymbol{R}^{n-1}$ invariant. In the case $n-1=8$, we have either $\tilde{H}=SO(7)$ (which leaves one and only one 1-dimensional subspace of $\boldsymbol{R}^8$ invariant) or $\tilde{H}=Spin(7)$ with spin representation (see Kobayoshi [2, p. 49]).

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. By the use of an $Ad(H)$-invariant positive definite inner product on $\mathfrak{g}$ whose existence is guaranteed by the compactness of H, we have a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ (direct sum) of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{m}]\subset\mathfrak{m}$. Let $\pi: G\rightarrow G/H$ be the natural projection. We identify the tangent space T_oM and $\mathfrak{m}$ by $d\pi$. The Lorentz inner product on T_oM induces the Lorentz inner product $\langle\ ,\ \rangle_{\mathfrak{m}}$ on $\mathfrak{m}$ so that $d\pi: \mathfrak{m}\rightarrow T_oM$ is a linear isometry. Then the linear isotropy group $\tilde{H}$ acting on T_oM corresponds to $Ad(H)$ on $\mathfrak{m}$ by means of $d\pi$. We note that the inner product $\langle\ ,\ \rangle_{\mathfrak{m}}$ is $Ad(H)$-invariant. We define the Lorentz inner product B on $\mathfrak{g}$ so that

$$B(\mathfrak{h}, \mathfrak{m})=0, \quad B|_{\mathfrak{m}}=\langle\ ,\ \rangle_{\mathfrak{m}}$$

and $B|_{\mathfrak{h}}$ is positive definite. We extend B to the G-left invariant Lorentz metric on G which is denoted by the same letter B. Then (G, B) is a Lorentz manifold and $\pi: G\rightarrow G/H=M$ is the semi-Riemannian submersion (for the definition of the semi-Riemannian submersion, see O'Neill [9, p. 212]).

The structure of $\mathfrak{g}$ *for* $n-1\neq 8$. We assume $n-1\neq 8$. Since $Ad(H)$ is compact and $\dim Ad(H)=(n-2)(n-3)/2$, $Ad(H)$ acts on $\mathfrak{m}$ as $I_2\times SO(n-2)$. Then $\mathfrak{m}$ decomposes naturally into 2-dimensional subspace $\mathfrak{m}_2$ and $(n-2)$-dimensional subspace $\mathfrak{m}_1$ such that $Ad(H)|_{\mathfrak{m}_2}=id.$ and $Ad(H)|_{\mathfrak{m}_1}=SO(n-2)$. Using Schur's lemma, we have that $\mathfrak{m}_1$ is spacelike. Furthermore, we have $\langle\mathfrak{m}_1, \mathfrak{m}_2\rangle_{\mathfrak{m}}=0$ so that $\mathfrak{m}_2$ is timelike. Thus we have a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}_1+\mathfrak{m}_2$ such that

$$[\mathfrak{h}, \mathfrak{m}_1]\subset\mathfrak{m}_1, \quad [\mathfrak{h}, \mathfrak{m}_2]=\{0\}.$$

LEMMA 3.4. $[\mathfrak{m}_2, \mathfrak{m}_1]$ *is either* $\{0\}$ *or* $\mathfrak{m}_1$. *More precisely, there exists a linear map* $L: \mathfrak{m}_2\rightarrow \boldsymbol{R}$ *such that* $[A, X]=L(A)X$ *for any* $A\in\mathfrak{m}_2$ *and any* $X\in\mathfrak{m}_1$. *Here* L *is either zero or onto map.*

PROOF. For any fixed $A\in\mathfrak{m}_2$, we define a linear map $f_A: \mathfrak{m}_1\rightarrow\mathfrak{g}$ by $f_A(X)=[A, X]$ $(X\in\mathfrak{m}_1)$. Let p_0, p_1 and p_2 be orthogonal projection from $\mathfrak{g}$ to $\mathfrak{h}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. Since $\mathfrak{h}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are $Ad(H)$-invariant and $Ad(h)f_A=f_AAd(h)$ for any $h\in H$, we have

$$(*) \qquad p_if_AAd(h)=Ad(h)p_if_A \quad \text{for any} \quad h\in H (i=0, 1, 2).$$

Step 1. We claim $[\mathfrak{m}_2, \mathfrak{m}_1]\subset\mathfrak{h}+\mathfrak{m}_1$. Since $Ker(p_2f_A)$ is $Ad(H)$-invariant by $(*)$ and the adjoint representation of H on $\mathfrak{m}_1$ is irreducible, we have $Ker(p_2f_A)=\{0\}$ or $\mathfrak{m}_1$. Suppose $Ker(p_2f_A)=\{0\}$ for some $A\in\mathfrak{m}_2$. Then $p_2f_A: \mathfrak{m}_1\rightarrow\mathfrak{m}_2$ is injective so that $\dim Im(p_2f_A)=n-2>2=\dim\mathfrak{m}_2$. Hence we have $Ker(p_2f_A)=\mathfrak{m}_1$

for any $A\in\mathfrak{m}_2$, that is, $[\mathfrak{m}_2, \mathfrak{m}_1]\subset\mathfrak{h}+\mathfrak{m}_1$.

Step 2. We claim $[\mathfrak{m}_2, \mathfrak{m}_1]\subset\mathfrak{m}_1$. By the same procedure as that of Step 1, we have $Ker(p_0 f_A)=\{0\}$ or $\mathfrak{m}_1$. Suppose $Ker(p_0 f_A)=\{0\}$ for some $A\in\mathfrak{m}_2$. Then $\dim p_0 f_A(\mathfrak{m}_1)=n-2$. We can verify easily that $p_0 f_A(\mathfrak{m}_1)$ is ideal in $\mathfrak{h}$. On the other hand, there is no ideal of dimension $n-2$ in $\mathfrak{h}=\mathfrak{so}(n-2)$. Hence we have $Ker(p_0 f_A)=\mathfrak{m}_1$ for any $A\in\mathfrak{m}_2$, that is $[\mathfrak{m}_2, \mathfrak{m}_1]\subset\mathfrak{m}_1$.

Step 3. By the above discussion, f_A is a linear map from $\mathfrak{m}_1$ into itself and commutes with the action of $Ad(H)=SO(n-2)$ on $\mathfrak{m}_1$. Hence there exists linear map $L: \mathfrak{m}_2\rightarrow\boldsymbol{R}$ such that $[A, X]=L(A)X$ $(A\in\mathfrak{m}_2, X\in\mathfrak{m}_1)$.

LEMMA 3.5. $[\mathfrak{m}_1, \mathfrak{m}_1]\subset\mathfrak{h}$.

PROOF. Let p_0, p_1 and p_2 be maps as in the proof of Lemma 3.4. Given orthonormal vectors X and Y in $\mathfrak{m}_1$, there exists $h\in H$ such that $Ad(h)=id.$ on $\mathfrak{m}_2$ and $Ad(h)X=-X$, $Ad(h)Y=Y$ (for, $n-2\geqq 4$). Then we have

$$p_2[X, Y]=Ad(h)p_2[X, Y]=p_2[Ad(h)X, Ad(h)Y]$$
$$=-p_2[X, Y]$$

which implies $p_2[X, Y]=0$. Hence $p_2[\mathfrak{m}_1, \mathfrak{m}_1]=\{0\}$. Let express $p_1[X, Y]$ as $aX+bY+cZ$, where Z is a unit vector orthogonal to X and Y. Since $n-2\geqq 4$, there exists $h'\in H$ such that $Ad(h')=id.$ on $\mathfrak{m}_2$ and $Ad(h')X=-X$, $Ad(h')Y=-Y$, $Ad(h')Z=-Z$. The equality $Ad(h')p_1[X, Y]=p_1 Ad(h')[X, Y]$ implies $p_1[X, Y]=0$. Thus we have $[\mathfrak{m}_1, \mathfrak{m}_1]\subset\mathfrak{h}$.

From the same method as in Kobayashi and Nagano [3, p. 212], we have

LEMMA 3.6. $[\mathfrak{m}_2, \mathfrak{m}_2]\subset\mathfrak{m}_2$.

From Lemma 3.6, there exists a basis $\{e_0, e_1\}$ of $\mathfrak{m}_2$ such that $B(e_0, e_0)=-1$, $B(e_1, e_1)=1$ and $B(e_0, e_1)=0$, and there exist constants a and b such that $[e_0, e_1]=ae_0+be_1$. Then there are the following four possibilities:

CASE I: $[e_0, e_1]$ is a zero vector (i.e., $\mathfrak{m}_2$ is commutative);

CASE II: $[e_0, e_1]$ is a non-zero null vector (i.e., $a\neq 0$, $b=\delta a$, where $\delta^2=1$);

CASE III: $[e_0, e_1]$ is a spacelike vector (i.e., $b^2-a^2=\alpha^2$, $\alpha>0$);

CASE IV: $[e_0, e_1]$ is a timelike vector (i.e., $b^2-a^2=-\alpha^2$, $\alpha>0$).

There exists a basis f_0, f_1 such that
in case II,

$$B(f_0, f_0)=0=B(f_1, f_1),\quad B(f_0, f_1)=-1,\quad [f_0, f_1]=f_1,$$

in case III,

$$B(f_0, f_0)=-1, \quad B(f_1, f_1)=1, \quad B(f_0, f_1)=0 \quad \text{and} \quad [f_0, f_1]=\alpha f,$$

in case IV,

$$B(f_0, f_0)=1, \quad B(f_1, f_1)=-1, \quad B(f_0, f_1)=0 \quad \text{and} \quad [f_0, f_1]=\alpha f_1.$$

In case I, we denote e_0 and e_1 by f_0 and f_1 respectively. Hereafter, in any cases, we consider f_0 and f_1 instead of e_0 and e_1. Furthermore, in any cases, we denote $L(f_0)$ and $L(f_1)$ by c_0 and c_1 respectively where L is the linear map in Lemma 3.4.

LEMMA 3.7. *In cases II, III, and IV, we have* $c_1=0$.

PROOF. Let X be a non-zero vector belonging to $\mathfrak{m}_1$. By the Jacobi's identity

$$[f_0, [f_1, X]]=[[f_0, f_1], X]+[f_1, [f_0, X]],$$

we have $c_0c_1X=\beta c_1X+c_0c_1X$ ($\beta=1$ or α) so that we have $c_1=0$.

Determination of M *for* $n-1\neq 8$. Since M is simply connected, H is connected so that $Ad(H)$ acts on $\mathfrak{m}_2$ as the identity transformation. Therefore we have

LEMMA 3.8. *For each* $f_u\in\mathfrak{m}_2$ $(u=0, 1)$, *the vector field* ξ_u *defined by*

$$\xi_u(p):=dgd\pi(f_u(e)) \qquad (p=g(o),\ g\in G)$$

is well-defined on M *and* G*-invariant where* e *is the identity in* G.

We have the following formulas (**) according to the above each case I~IV:

CASE I. $\nabla_{\xi_u}\xi_v=0$, $\nabla_X\xi_u=-c_uX$ $(u, v=0, 1)$;

CASE II. $\nabla_{\xi_0}\xi_0=-\xi_0$, $\nabla_{\xi_0}\xi_1=\xi_1$, $\nabla_{\xi_1}\xi_0=0$,

(**) $$\nabla_{\xi_1}\xi_1=0, \quad \nabla_X\xi_0=-c_0X, \quad \nabla_X\xi_1=0;$$

CASES III and IV. $\nabla_{\xi_0}\xi_0=0$, $\nabla_{\xi_0}\xi_1=0$, $\nabla_{\xi_1}\xi_0=-\alpha\xi_1$,

$$\nabla_{\xi_1}\xi_1=-\alpha\xi_0, \quad \nabla_X\xi_0=-c_0X, \quad \nabla_X\xi_1=0.$$

Here X is any vector field orthogonal to ξ_0 and ξ_1 and ∇ is the Levi-Civita connection of the Lorentz metric $\langle\ ,\ \rangle$ on M.

By the G-invariance of ξ_u and the above formulas, we have

LEMMA 3.9. (1) *In the cases I and II, the integral curve of* ξ_1 *is a complete geodesic.*

(2) *In the cases I, III and IV, the integral curve of* ξ_0 *is a complete geodesic.*

By the similar way as the proof of Lemma 2.5, we have

LEMMA 3.10. (1) *In the cases I, III and IV, the* 1*-form* ω_0 *on* M *defined by*

$$\omega_0(X):=\langle X, \xi_0\rangle$$

is G-invariant and closed.

(2) *In the cases I and II, the* 1*-form* ω_1 *on* M *defined by*

$$\omega_1(X):=\langle X, \xi_1\rangle$$

is G-invariant and closed.

Now we will determine $G/H=M$ in each cases I, II, III and IV.

CASE I. Lemma 3.10 implies that there exist smooth functions ψ_0 and ψ_1 such that $d\psi_u=\omega_u$ $(u=0, 1)$. Since ξ_0 and ξ_1 are G-invariant, there exist 1-parameter groups of transformation ϕ_t^0 and ϕ_s^1 generated by ξ_0 and ξ_1 respectively. We can verify easily that for $p\in M$,

$$(\#)\qquad \begin{cases}\psi_0(\phi_t^0(p))=-t+\psi_0(p), & \psi_0(\phi_s^1(p))=\psi_0(p),\\ \psi_1(\phi_t^0(p))=\psi_1(p), & \psi_1(\phi_s^1(p))=s+\psi_1(p).\end{cases}$$

Let M_1^0 be a connected component of $M_1=\{p\in M; \psi_0(p)=\psi_1(p)=0\}$. Then M_1^0 is a connected $(n-2)$-dimensional closed submanifold of M. Furthermore M_1^0 is spacelike, because ξ_0 and ξ_1 are orthogonal to M_1.

LEMMA 3.11. *The map* $F: \boldsymbol{R}\times\boldsymbol{R}\times M_1^0\to M$ *defined by*

$$F(t, s, x)=\phi_t^0(\phi_s^1(x))$$

is a diffeomorphism, and $M_1=M_1^0$ *is simply connected.*

PROOF. Suppose that $F(t, s, x)=F(t', s', x')$. Then, from (#), we have $t=t'$ and $s=s'$. Therefore we have $\phi_t^0(\phi_s^1(x))=\phi_t^0(\phi_t^1(x'))$ so that we have $x=x'$. Thus F is injective. It is clear that F is smooth. Setting $N=F(\boldsymbol{R}\times\boldsymbol{R}\times M_1^0)$, then N is open in M. It remains to be shown that N is closed in M. Suppose that $\{F(t_k, s_k, x_k)=p_k\}$ is a sequence converging some point q in M. Since $t_k=-\psi_0(p_k)$ and $s_k=\psi_1(p_k)$, we have $t_k\to t_0:=-\psi_0(q)$ and $s_k\to s_0:=\psi_1(q)$ as $k\to\infty$. Since $x_k=\phi_{-s_k}^1(\phi_{-t_k}^0(p_k))$ converges $x_0:=\phi_{-s_0}^1(\phi_{-t_0}^0(q))$ as $k\to\infty$ and M_1^0 is closed, x_0 belongs to M_1^0 so that $q=\phi_{t_0}^0(\phi_{s_0}^1(x_0))$ belongs to N. Thus N is closed. Thus we have $N=F(\boldsymbol{R}\times\boldsymbol{R}\times M_1^0)$.

REMARK 3.12. For each $(a, b)\in \boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b):=\{p\in M;\ \phi_0(p)=a, \phi_1(p)=b\}$ is a simply connected $(n-2)$-dimensional spacelike submanifold of M.

LEMMA 3.13. *For each $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b)$ is congruent to $M_1=M_1(0, 0)$ in M.*

PROOF. Since G acts on M transitively, for some point p in $M_1(a, b)$ there exists $g\in G$ such that $g(o)=p(o\in M_1)$. Then we have $g(M_1)\subset M_1(a, b)$. In fact, for each point $q\in g(M_1)$, there exists a smooth curve $\tilde{c}:[0, 1]\to g(M_1)$ such that $\tilde{c}(0)=p$ and $\tilde{c}(1)=q$. Put $c:=g^{-1}\tilde{c}$. Then c is a smooth curve in M_1, so we have $\phi_0(c(s))=0=\phi_1(c(s))$ for any $s\in[0, 1]$. Therefore we have

$$(d\phi_u/ds)(\tilde{c}(s))=\langle\xi_u(\tilde{c}(s)),\ \dot{\tilde{c}}(s)\rangle=\langle dg\xi_u(c(s)),\ dg\dot{c}(s)\rangle$$

$$=\langle\xi_u(c(s)),\ \dot{c}(s)\rangle=(d\phi_u/ds)(c(s))=0 \qquad (u=0, 1).$$

Thus we have $\phi_0(q)=a$ and $\phi_1(q)=b$ so that we have $g(M_1)\subset M_1(a, b)$. Since $g(M_1)$ is open and closed in $M_1(a, b)$, we have $g(M_1)=M_1(a, b)$.

LEMMA 3.14. *M_1 is a homogeneous Riemannian manifold.*

PROOF. For any $p, q\in M_1$, there exists $g\in G$ such that $g(p)=q$. By the same method as in the proof of Lemma 3.13, we can see that $g|_{M_1}$ is an isometric transformation of M_1.

Set $G_1:=\{g\in G;\ gM_1=M_1\}$. Then G_1 is a Lie subgroup of G. We can verify that H is included in G_1 by the same discussion as in the proof of Lemma 3.13. Furthermore, G_1 acts on M_1 effectively. Thus $\dim G_1=\dim M_1+\dim H=(n-1)(n-2)/2$. Therefore the simply connected $(n-2)$-dimensional Riemannian manifold M_1 admitting an isometry group G_1 of maximum dimension $(n-1)(n-2)/2$ is isometric to S^{n-2}, $\boldsymbol{H}^{n-2}$ or $\boldsymbol{E}^{n-2}$.

LEMMA 3.15. *The map*

$$F:(\boldsymbol{R}\times\boldsymbol{R}\times M_1,\ -dt^2+ds^2+\exp(-2c_0t-2c_1s)ds_{M_1}^2)\longrightarrow(M, \langle\ ,\ \rangle)$$

is an isometry where $ds_{M_1}^2$ is the metric of M_1.

PROOF. Let $(V, \Phi=(t_2, \cdots, t_{n-1}))$ be a local coordinate around a point p in M_1. Then $(\boldsymbol{R}\times\boldsymbol{R}\times V, id\times\Phi=(t, s, t_2, \cdots, t_{n-1}))$ is a local coordinate around (a, b, p) in $\boldsymbol{R}\times\boldsymbol{R}\times M_1$. Put $\tilde{V}:=F(\boldsymbol{R}\times\boldsymbol{R}\times M_1)$ and define $\tilde{\Phi}:\tilde{V}\to\boldsymbol{R}^n$ by $(id\times\Phi)\circ F^{-1}$. Then $(\tilde{V}, \tilde{\Phi}=(x_0, x_1, \cdots, x_{n-1}))$ is a local coordinate around $\tilde{p}=F(a, b, p)$ in M. Since $[\xi_0, \xi_1]=0$, we can see $dF(\partial/\partial t)=\partial/\partial x_0=\xi_0$ and $dF(\partial/\partial s)=\partial/\partial x_1=\xi_1$. Furthermore we have $dF(\partial/\partial t_j)=\partial/\partial x_j$ $(j=2, \cdots, n-1)$. We can also see that $\langle\partial/\partial x_u, \partial/\partial x_j\rangle=0$ $(u=0, 1)$. In fact

$$\langle\partial/\partial x_u, \partial/\partial x_j\rangle=\langle\xi_u, dF(\partial/\partial t_j)\rangle=(\partial/\partial t_j)(\phi_u(F(t, s, x))$$

$$=\begin{cases}(\partial/\partial t_j)(-t)=0 & (u=0)\\ (\partial/\partial t_j)(s)=0 & (u=1)\end{cases}.$$

Since $\nabla_X\xi_u=-c_uX$ $(u=0, 1)$ for any X orthogonal to ξ_0 and ξ_1, we have

$$\partial/\partial t\langle\partial/\partial t_i, \partial/\partial t_j\rangle=-2c_0\langle\partial/\partial t_i, \partial/\partial t_j\rangle$$

and

$$\partial/\partial s\langle\partial/\partial t_i, \partial/\partial t_j\rangle=-2c_1\langle\partial/\partial t_i, \partial/\partial t_j\rangle$$

so that we have

$$\langle\partial/\partial t_i, \partial/\partial t_j\rangle=\exp(-2c_0t-2c_1s)g_{ij}(t_2, \cdots, t_{n-1}).$$

Thus we have

$$F^*\langle\ ,\ \rangle=-dt^2+ds^2+\exp(-2c_0t-2c_1s)ds_{M_1}^2.$$

LEMMA 3.16. *If M_1 is $\boldsymbol{H}^{n-2}$ or S^{n-2}, then $c_0=c_1=0$, i.e., the metric of $\boldsymbol{R}\times\boldsymbol{R}\times M_1$ is a product metric.*

PROOF. Since, for each $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b)$ is isometric to M_1 by Lemma 3.13, the scalar curvature $S(a, b)$ of $M_1(a, b)$ coincides with the scalar curvature $S(0, 0)$ of M_1 which is non-zero. On the other hand, we have $S(a, b)=\exp(-2c_0a-2c_1b)\times S(0, 0)$ by Lemma 3.15. Since a and b are arbitrary, we have $c_0=c_1=0$.

We notice that, in the case $M_1=\boldsymbol{E}^{n-2}$, there are two cases (1) $c_0=c_1=0$ and (2) $c_0\neq 0$ or $c_1\neq 0$.

Summing up, in the case I, $(M, \langle\ ,\ \rangle)$ must be one of the following:

(i) $(\boldsymbol{L}^2\times M_1, ds_L^2+ds_{M_1}^2)$ where $(\boldsymbol{L}^2, ds_L^2)$ is the 2-dimensional Minkowski space and $(M_1, ds_{M_1}^2)$ is a simply connected $(n-2)$-dimensional Riemannian manifold of constant curvature;

(ii) $(\boldsymbol{R}^2\times\boldsymbol{E}^{n-2}, -dt^2+ds^2+\exp(-2c_0t-2c_1s)ds_E^2)$ where $c_0\neq 0$ or $c_1\neq 0$.

CASE II. Since ω_1 is closed, there exists a smooth function $\phi_1: M\to\boldsymbol{R}$ such that $d\phi_1=\omega_1$. Define the vector field η on M by $\eta(p):=\exp(-\phi_1(p))\xi_0(p)$ $(p\in M)$.

LEMMA 3.17. *The 1-form $\tilde{\omega}_0$ defined by $\tilde{\omega}_0(X):=\langle\eta, X\rangle$ is closed so that there exists smooth function $\tilde{\phi}_0: M\to\boldsymbol{R}$ such that $d\tilde{\phi}_0=\tilde{\omega}_0$.*

PROOF. Since $d\tilde{\omega}_0(X, Y)=\langle\nabla_X\eta, Y\rangle-\langle\nabla_Y\eta, X\rangle$ for any vector fields X and Y, we can verify that $\tilde{\omega}_0$ is closed by formulas (**).

Since ξ_0 is G-invariant, there exists the 1-parameter group of transformations ϕ_s^0 generated by ξ_0. Let $c_p(t)$ be the integral curve of ξ_1 through a point $p\in M$. From the G-invariance of ξ_1, $c_p(t)$ is defined for any $t\in\boldsymbol{R}$. Define the vector field ζ on M by $\zeta(q)=\exp(\psi_1(q))\xi_1(q)$ $(q\in M)$. Let ϕ_t^1 be the 1-parameter group of transformations generated by ζ. Then we have $\phi_t^1(p)=c_p(\exp(\psi_1(p))t)$ so that ϕ_t^1 is complete. Noting that $[\xi_0, \zeta]=0$, we have $\phi_s^0\phi_t^1=\phi_t^1\phi_s^0$. We can verify the following:

$$\tilde{\psi}_0(\phi_s^0(p))=\tilde{\psi}_0(p), \quad \tilde{\psi}_0(\phi_t^1(p))=-t+\tilde{\psi}_0(p),$$
$$\psi_1(\phi_s^0(p))=-s+\psi_1(p), \quad \psi_1(\phi_t^1(p))=\psi_1(p) \quad \text{for } p\in M.$$

Let M_1^0 be a connected component of $M_1:=\{p\in M; \tilde{\psi}_0(p)=\psi_1(p)=0\}$. Then M_1^0 is an $(n-1)$-dimensional closed submanifold of M. Furthermore M_1^0 is spacelike, because ξ_0 and ξ_1 are orthogonal to M_1^0.

LEMMA 3.18. *The map* $F: \boldsymbol{R}\times\boldsymbol{R}\times M_1^0\to M$ *defined by*

$$F(t, s, x)=\phi_t^1\phi_s^0(x) \quad \textit{for} \quad (t, s, x)\in\boldsymbol{R}\times\boldsymbol{R}\times M_1^0$$

is a diffeomorphism, and $M_1=M_1^0$ *is simply connected.*

The proof is similar to that of Lemma 3.11.

REMARK 3.19. For each $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b):=\{p\in M; \tilde{\psi}_0(p)=a, \psi_1(p)=b\}$ is a simply connected $(n-2)$-dimensional spacelike submanifold of M.

The following two Lemma 3.20 and 3.21 are proved by the same method as in Lemma 3.13 and 3.14 respectively.

LEMMA 3.20. *For each* $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b)$ *is congruent to* M_1 *in* M.

LEMMA 3.21. M_1 *is a homogeneous Riemannian manifold.*

Set $G_1:=\{g\in G; g(M_1)=M_1\}$. Then we also have that G_1 is a closed Lie subgroup of G and includes H. G_1 acts effectively on M_1 so that M_1 is S^{n-2}, $\boldsymbol{H}^{n-2}$ or $\boldsymbol{E}^{n-2}$.

LEMMA 3.22. *The map* $F: (\boldsymbol{R}\times\boldsymbol{R}\times M_1, -2\exp(-s)dtds+\exp(-2c_0s)ds_{M_1}^2)\to (M, \langle\ ,\ \rangle)$ *is an isometry.*

PROOF. As in the proof of Lemma 3.15, we take a local coordinate $(V, \Phi=(t_2, \cdots, t_{n-2}))$ around a point p in M_1 and a local coordinate $(\tilde{V}, \tilde{\Phi}=(x_0, x_1, \cdots, x_{n-1}))$ around a point $F(a, b, p)$ in M. Then we can see $dF(\partial/\partial t)=\partial/\partial x_0=\exp(-s)\xi_1$, $dF(\partial/\partial s)=\partial/\partial x_1=\xi_0$ and $dF(\partial/\partial t_i)=\partial/\partial x_i$ $(i=2, \cdots, n-1)$ at $(t, s, p) \in \boldsymbol{R}\times\boldsymbol{R}\times M_1$. Furthermore, we can see

$$\partial/\partial s\langle\partial/\partial t_i, \partial/\partial t_j\rangle=-2c_0\langle\partial/\partial t_i, \partial/\partial t_j\rangle$$

and

$$\partial/\partial t\langle\partial/\partial t_i, \partial/\partial t_j\rangle=0 \quad (i, j=2, \cdots, n-1)$$

so that we have

$$\langle\partial/\partial t_i, \partial/\partial t_j\rangle=\exp(-2c_0 s)g_{ij}(t_2, \cdots, t_{n-1}).$$

Uhus we have

$$F^*\langle\ ,\ \rangle=-2\exp(-s)dtds+\exp(-2c_0 s)ds_{M_1}^2.$$

We also have the following Lemma 3.23 by the same method as in the case I.

LEMMA 3.23. *If M_1 is S^{n-2} or $\boldsymbol{H}^{n-2}$, then $c_0=0$.*

We note that the space $(\boldsymbol{R}\times\boldsymbol{R}, -2\exp(-s)dtds)$ is isometric to the upper half-space $U^2=\{(x, y); y>0\}$ with flat metric $-2dxdy/y^2$ by the transformation $(t, s)\rightarrow(x, y)=(t, \exp(s))$.

Thus, in case II, M must be one of the following:

(iii) $(U^2\times M_1, -2dxdy/y^2+ds_{M_1}^2)$ where $(M_1, ds_{M_1}^2)$ is a simply connected $(n-2)$-dimensional Riemannian manifold of constant curvature;

(iv) $(U^2\times \boldsymbol{E}^{n-2}, -2dxdy/y^2+(1/y)^{2c_0}ds_E^2)$.

REMARK 3.24. When $c_0=1$, the space (iv) is the n-dimensional upper half-space $U^2=\{(x_1, \cdots, x_n): x_n>0\}$ with flat metric

$$(1/x_n^2)(-2dx_{n-1}dx_n+dx_1^2+\cdots+dx_{n-2}^2).$$

CASE III and IV. Since ω_0 is closed by Lemma 3.10, there exists a smooth function $\psi_0: M\rightarrow\boldsymbol{R}$ with $d\psi_0=\omega_0$. Put $\eta(p)=\exp(-\kappa\alpha\psi_0(p))\xi_1(p)$ where $\kappa=\langle\xi_1, \xi_1\rangle$ (i.e., $\kappa=1, -1$ in the cases III, IV respectively). Define a 1-form $\tilde{\omega}_1$ by $\tilde{\omega}_1(X)=\langle X, \eta\rangle$. Then we have the following Lemma by the same method as in Lemma 3.17.

LEMMA 3.25. *$\tilde{\omega}_1$ is a closed 1-form so that there exists a smooth function $\tilde{\psi}_1: M\rightarrow\boldsymbol{R}$ with $d\tilde{\psi}_1=\tilde{\omega}_1$.*

Since ξ_0 is G-invariant, there exists the 1-parameter group of transformations ϕ_t^0 generated by ξ_0. Let $c_p(s)$ be an integral curve of ξ_1 through a point $p\in M$. Then, for each point $p\in M$, $c_p(t)$ is defined for any $t\in \boldsymbol{R}$, because of the G-invariance of ξ_1. Define the vector field ζ on M by $\zeta(p)=\exp(\kappa\alpha\psi_0(p))\xi_1(p)$ $(p\in M)$. Let ϕ_s^1 be the 1-parameter group of transformations generated by ζ. Then we have $\phi_s^1(p)=c_p(\exp(\kappa\alpha\psi_0(p))s)$ so that ϕ_s^1 is complete. Noting $[\xi_0, \zeta]=0$, we have $\phi_t^0\phi_s^1=\phi_s^1\phi_t^0$. We can verify the following:

$$\psi_0(\phi_t^0(p))-\kappa t+\psi_0(p), \quad \psi_0(\phi_s^1(p))=\psi_0(p)$$
$$\tilde{\psi}_1(\phi_t^0(p))=\tilde{\psi}_1(p), \quad \tilde{\psi}_1(\phi_s^1(p))=\kappa s+\tilde{\psi}_1(p).$$

Let M_1^0 be a connected component of $M_1:=\{p\in M; \psi_0(p)=\tilde{\psi}_1(p)=0\}$. Then by the same procedure as in the case II, we have Lemmas 3.26, 3.27, 3.29, 3.30 and Remark 3.28.

LEMMA 3.26. *M_1^0 is a connected $(n-2)$-dimensional spacelike closed submanifold of M.*

LEMMA 3.27. *The map $F: \boldsymbol{R}\times\boldsymbol{R}\times M_1^0\to M$ defined by*

$$F(t, s, x)=\phi_s^1\phi_t^0(x) \qquad \text{for} \quad (t, s, x)\in\boldsymbol{R}\times\boldsymbol{R}\times M_1^0$$

is a diffeomorphism, and $M_1=M_1^0$ is simply connected.

REMARK 3.28. For each $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b):=\{p\in M; \psi_0(p)=a, \tilde{\psi}_1(p)=b\}$ is a simply connected $(n-2)$-dimensional spacelike submanifold of M.

LEMMA 3.29. *For each $(a, b)\in\boldsymbol{R}\times\boldsymbol{R}$, $M_1(a, b)$ is congruent to M_1 in M.*

LEMMA 3.30. *M_1 is a homogeneous Riemannian manifold.*

By the same method as in the case II, M_1 is isometric to S^{n-1}, $\boldsymbol{H}^{n-1}$ or $\boldsymbol{E}^{n-2}$. We also have following Lemmas 3.31 and 3.32.

LEMMA 3.31. *The map*

$$F: (\boldsymbol{R}\times\boldsymbol{R}\times M_1, -\kappa(dt^2-\exp(-2\alpha t)ds^2)+\exp(-2c_0t)ds_{M_1}^2)\to(M, \langle\ ,\ \rangle)$$

is an isometry.

LEMMA 3.32. *If $M_1=S^{n-2}$ or $\boldsymbol{H}^{n-2}$, then $c_0=0$.*

We note that $(\boldsymbol{R}\times\boldsymbol{R}, -\kappa(dt^2-\exp(-2\alpha t)ds^2)$ is isometric to $(U^2=\{(x, y); y>0\}, ds_\kappa^2=\kappa(dx^2-dy^2)/(\alpha y)^2)$ by the transformation $(t, s)\to(x=s, y=\exp(\alpha t)/\alpha)$.

Thus, in case III, $(M, \langle\ ,\ \rangle)$ must be one of the following:

(v) $(U^2\times M_1,\ ds_{+1}^2/\alpha^2+ds_{M_1}^2)$;

(vi) $(U^2\times \boldsymbol{E}^{n-2},\ ds_{+1}^2/\alpha^2+(1/\alpha y)^{2c/\alpha}ds_E^2)$,

and in case IV, $(M, \langle\ ,\ \rangle)$ must be one of the spaces

(vii) $(U^2\times M_1,\ ds_{-1}^2/\alpha^2+ds_{M_1}^2)$,

(viii) $(U^2\times \boldsymbol{E}^{n-2},\ ds_{-1}^2/\alpha^2+(1/\alpha y)^{2c/\alpha}ds_E^2)$,

where $(M_1,\ ds_{M_1}^2)$ is a simply connected $(n-2)$-dimensional Riemannian manifold of constant curvature.

The case $n=9$. When $n-1=8$, $\tilde{H}$ is isomorphic to $SO(7)$ or $Spin(7)$ which has a spin representation. When H is isomorphic to $SO(7)$, the argument is the same as in the case $n-1\neq 8$. Therefore it is enough to deal with the case that H is isomorphic of $Spin(7)$.

Since $\tilde{H}$ is conjugate to the subgroup $Spin(7)$ of $SO(8)$, there exists a timelike G-invariant vector field ξ on M with $\langle\xi, \xi\rangle=-1$.

By the same method as the proof of Lemma 2.5, we have

LEMMA 3.33. *The* 1*-form* ω *defined by* $\omega(X)=\langle\xi, X\rangle$ *is* G*-invariant and closed so that threre exists a smooth function* $f: M\rightarrow \boldsymbol{R}$ *with* $df=\omega$.

The G-invariance of ξ implies the completeness of ξ. There exists the 1-parameter group of transformations ϕ_t generated by ξ. Then we have $f(\phi_t(p))=-t+f(p)$ $(t\in \boldsymbol{R},\ p\in M)$. Put $N=\{p\in M;\ f(p)=0\}$. Then a connected component N^o of N is a connected closed 8-dimensional spacelike hypersurface of M. By the similar way as in the case I, N^o is a homogeneous Riemannian manifold admitting an isometry group $G':=\{g\in G;\ g(N^o)=N^o\}$ of dimension $8(8-1)/2+1=29$ which acts effectively on N^o and includes H. Then, by the theorem in [8], N^o is isometric to $\boldsymbol{E}^8$ and $G'=Spin(7)\boldsymbol{R}^8$ (a semi-direct product). We have $\nabla_X\xi=-cX$ for any X orthogonal to ξ where c is a constant. In fact, $Spin(7)$ acts transitively on $S^7:=\{Z\in T_xM;\ \langle Z, \xi\rangle=0,\ \langle Z, Z\rangle=1\}$ so that the proof is the same as in [6, Lemma 8]. We also have that the map $F: \boldsymbol{R}\times N^o\rightarrow M$ defined by $F(t, x)=\phi_t(x)$ for $(t, x)\in \boldsymbol{R}\times N^o$ is a diffeomorphism and the map $F: (\boldsymbol{R}\times N^o,\ -dt^2+\exp(-2c)ds_{N^o}^2\rightarrow(M, \langle\ ,\ \rangle)$ is an isometry.

4. Final Comment.

In connection with Remark 3.2, we must correct some parts in the previous paper [6]. There are some ambiguous stataments in [6]. In the Theorem, the statement "whose isotropy subgroup is compact" should be "whose isotropy subgroup *at every point* is compact". The statement "H is compact" that precedes Lemma 1 should be "H is compact *at every point*". We cannot remove the condition that the isotropy subgroup at every is compact, by the following example.

EXAMPLE. Let M be the n-dimensional de-Sitter space $S_1^n=\{(u_0, u_1, \cdots, u_n) \in \boldsymbol{R}^{n+1}; -u_0^2+u_1^2+\cdots+u_n^2=1\}$ and G the matrix group of the form

$$\begin{bmatrix} (1+a^2+|\chi|^2)/(2a) & \chi & (1-a^2+|\chi|^2)/(2a) \\ (1/a)A^t\chi & A & (1/a)A^t\chi \\ (1-a^2-|\chi|^2)/(2a) & -\chi & (1+a^2-|\chi|^2)/(2a) \end{bmatrix} \quad \begin{array}{l} a>0,\ \chi\in\boldsymbol{R}^{n-1} \\ A\in SO(n-1), \end{array}$$

(c.f., Remark 3.2). Then, for every point p in S_1^n such that $u_0+u_n>0$ (resp. <0), the G-orbit of p is $U^+=\{(v_0, \cdots, v_n)\in S_1^n; v_0+v_n>0\}$ (resp. $U^-=\{(v_0, \cdots, v_n) \in S_1^n: v_0+v_n<0\}$) and the isotropy subgroup at p is compact. But, for every point q in S_1^n such that $u_0+u_n=0$, the G-orbit of q is a lightlike hypersurface of S_1^n and the isotropy subgroup at q is non-compact.

Acknowledgement. The author would like to thank Professors H. Kitahara and S. Yorozu for their helpfull advice and encouragement. The author also would like to thank the referee for his kind and useful advice.

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