

COMPLETE RIEMANNIAN MANIFOLD MINIMALLY IMMERSSED IN A UNIT SPHERE $S^{n+p}(1)$

By

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1. Introduction.

Let M^n be an n -dimensional Riemannian manifold which is minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$. If M^n is compact, then many authors studied them and obtained many beautiful results (for examples [1], [3], [4], [5] and [6]). In this paper, we make use of Yau's *maximum principle* to extend these results to complete manifolds with Ricci curvature bounded from below.

2. Preliminaries.

Let M^n be an n -dimensional Riemannian manifold which is minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$. Then the *second fundamental form* h of the immersion is given by $h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$ and it satisfies $h(X, Y) = h(Y, X)$, where $\tilde{\nabla}$ and ∇ denote the covariant differentiation on $S^{n+p}(1)$ and M^n respectively, X and Y are vector fields on M^n . We choose a local field of orthonormal frames $e_1, \dots, e_n, \dots, e_{n+p}$ in $S^{n+p}(1)$ such that, restricted to M^n , the vector e_1, \dots, e_n are tangent to M^n . We use the following convention on the range of indices unless otherwise stated: $A, B, C, \dots = 1, 2, \dots, n+p$; $i, j, k, \dots = 1, 2, \dots, n$; $\alpha, \beta, \dots = n+1, \dots, n+p$. And we agree that repeated indices under a summation sign without indication are summed over the respective range. With respect to the frame field of $S^{n+p}(1)$ chosen above, let $\tilde{\omega}_1, \dots, \tilde{\omega}_{n+p}$ be the dual frames. Then structure equations of $S^{n+p}(1)$ are given by

$$(2.1) \quad d\tilde{\omega}_A = \sum \tilde{\omega}_{AB} \wedge \tilde{\omega}_B, \quad \tilde{\omega}_{AB} + \tilde{\omega}_{BA} = 0,$$

$$(2.2) \quad d\tilde{\omega}_{AB} = \sum \tilde{\omega}_{AC} \wedge \tilde{\omega}_{CB} - \tilde{\omega}_A \wedge \tilde{\omega}_B.$$

Restricting these forms to M^n , we have the structure equations of the immersion:

$$(2.3) \quad \omega_\alpha = 0,$$

$$(2.4) \quad \omega_{i\alpha} = \sum h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha,$$

$$(2.5) \quad d\omega_i = \sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(2.6) \quad d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l,$$

$$(2.7) \quad R_{ijkl} = (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + \sum (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

$$(2.8) \quad d\omega_{\alpha\beta} = \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j,$$

$$(2.9) \quad R_{\alpha\beta ij} = \sum (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta).$$

Then, the *second fundamental form* h can be written as

$$(2.10) \quad h(e_i, e_j) = \sum h_{ij}^\alpha e_\alpha.$$

If we define h_{ijk}^α by

$$(2.11) \quad \sum h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum h_{ik}^\alpha \omega_{kj} + \sum h_{kj}^\alpha \omega_{ki} + \sum h_{ij}^\beta \omega_{\beta\alpha},$$

then, from (2.2), (2.3) and (2.4), we have $h_{ijk}^\alpha = h_{ikj}^\alpha$.

Let K_N be the square of the length of curvature tensor of the *normal bundle*, that is,

$$(2.12) \quad K_N = \sum \left(\sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta) \right)^2$$

Setting

$$(2.13) \quad L_N = \sum \left(\sum_{ij} h_{ij}^\alpha h_{ij}^\beta \right)^2.$$

In this paper, we used the notations in [2].

LEMMA 1 ([4] or [6]). *If M^n is an n -dimensional Riemannian manifold minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$. Then, Simons' equation*

$$(2.14) \quad \begin{aligned} \frac{1}{2} \Delta \|h\|^2 &= \sum (h_{ijk}^\alpha)^2 + \sum (h_{ij}^\alpha h_{kl}^\alpha R_{lijk} + h_{ij}^\alpha h_{il}^\alpha R_{lkjk}) - \frac{1}{2} K_N \\ &= \sum (h_{ijk}^\alpha)^2 - K_N - L_N + n \|h\|^2 \end{aligned}$$

holds good and

$$(2.15) \quad \|h\|^2 = n(n-1) - R,$$

where $\|h\|$ denotes the length of the second fundamental form h such that $\|h\|^2 = \sum (h_{ij}^\alpha)^2$ and R is the scalar curvature of M^n .

LEMMA 2 ([7]). *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2 -function bounded from above on M^n , then for all $\epsilon > 0$, there exists a point x in M^n at which*

$$(2.16) \quad \sup f - \varepsilon < f(x),$$

$$(2.17) \quad \|\nabla f\| < \varepsilon,$$

$$(2.18) \quad \Delta f < \varepsilon.$$

3. Main results.

THEOREM 1. *Let M^n be an n -dimensional complete Riemannian manifold minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$ with Ricci curvature bounded from below. Then either M^n is totally geodesic, in this case, $M^n = S^n(1)$ holds locally or $\inf R \leq n(n-1) - n/(2-1/p)$.*

PROOF. According to Lemma 1, we have

$$(3.1) \quad \frac{1}{2} \Delta \|h\|^2 = \sum (h_{ij}^\alpha)^2 - K_N - L_N + n \|h\|^2,$$

Because

$$(3.2) \quad \sum_{ij} (\sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta))^2 \leq 2 \sum_{ij} (h_{ij}^\alpha)^2 \sum_{ij} (h_{ij}^\beta)^2,$$

we get

$$(3.3) \quad \begin{aligned} K_N &= \sum (\sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta))^2 \leq 2 \sum_{ij} (\sum_{ij} (h_{ij}^\alpha)^2 \sum (h_{ij}^\beta)^2) \\ &= 2 \|h\|^4 - 2 \sum (\sum_{ij} (h_{ij}^\alpha)^2)^2. \end{aligned}$$

(3.1) and (3.3) imply

$$(3.4) \quad \frac{1}{2} \Delta \|h\|^2 \geq \|h\|^2 \{n - (2-1/p) \|h\|^2\}. \quad (\text{see [1] or [4]})$$

If $\inf R > n(n-1) - n/(2-1/p)$, from Lemma 1, we have

$$(3.5) \quad \|h\|^2 = n(n-1) - R$$

Hence, $\|h\|^2$ is bounded. We define $f = \|h\|^2$, $F = (f+a)^{1/2}$ (where $a > 0$ is any positive constant number). F is bounded because $\|h\|^2$ is bounded.

$$dF = \frac{1}{2} (f+a)^{-1/2} df,$$

$$\begin{aligned} \Delta F &= \frac{1}{2} \left\{ -\frac{1}{2} (f+a)^{-3/2} \|df\|^2 + (f+a)^{-1/2} \Delta f \right\} \\ &= \frac{1}{2} \{ -2 \|dF\|^2 + \Delta f \} (f+a)^{-1/2} \\ &= \frac{1}{2F} \{ -2 \|dF\|^2 + \Delta f \}. \end{aligned}$$

Hence, $F\Delta F = -\|dF\|^2 + \frac{1}{2}\Delta f$, namely,

$$(3.6) \quad \frac{1}{2}\Delta f = F\Delta F + \|dF\|^2.$$

Applying the Lemma 2 to F , we have for all $\varepsilon > 0$, there exists a point x in M^n such that at x

$$(3.7) \quad \|dF(x)\| < \varepsilon.$$

$$(3.8) \quad \Delta F(x) < \varepsilon,$$

$$(3.9) \quad F(x) > \sup F - \varepsilon.$$

(3.6), (3.7) and (3.8) imply

$$(3.10) \quad \frac{1}{2}\Delta f < \varepsilon^2 + F\varepsilon = \varepsilon(\varepsilon + F) \quad (\text{from } F > 0).$$

We take the sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \rightarrow 0$ ($m \rightarrow \infty$) and for all m , there exists a point x_m in M^n such that (3.7), (3.8) and (3.9) hold good. Hence, $\varepsilon_m(\varepsilon_m + F(x_m)) \rightarrow 0$ ($m \rightarrow \infty$) (because F is bounded).

On the other hand, from (3.9),

$$F(x_m) > \sup F - \varepsilon_m.$$

Because F is bounded, $\{F(x_m)\}$ is a bounded sequence, we get

$$F(x_m) \longrightarrow F_0 \quad (\text{if necessary, we can choose subsequence}).$$

Hence,

$$F_0 \geq \sup F.$$

According to the properties of supremum, we have

$$(3.11) \quad F_0 = \sup F.$$

From the definition of F , we get

$$(3.12) \quad f(x_m) \longrightarrow f_0 = \sup f_0 \quad (\text{from } F_0 = \sup F)$$

From (3.4) and (3.10), we obtain

$$f[n - (2 - 1/p)f] \leq \frac{1}{2}\Delta f < \varepsilon^2 + \varepsilon F,$$

$$f(x_m)[n - (2 - 1/p)f(x_m)] < \varepsilon_m^2 + \varepsilon_m F(x_m) \leq \varepsilon_m^2 + \varepsilon_m F_0,$$

Let $m \rightarrow \infty$, we have $\varepsilon_m \rightarrow 0$, $f(x_m) \rightarrow f_0$. Hence,

$$f_0[n - (2 - 1/p)f_0] \leq 0$$

1) If $f_0=0$, we have $f=\|h\|^2\equiv 0$, hence M^n is totally geodesic, from [4], we know $M^n=S^n(1)$ holds locally.

2) If $f_0>0$, we have

$$\begin{aligned} n-(2-1/p)f_0 &\leq 0, \\ f_0 &\geq n/(2-1/p), \end{aligned}$$

that is, $\sup\|h\|^2\geq n/(2-1/p)$. From (2.15),

$$\inf R\leq n(n-1)-n/(2-1/p)$$

This completes the proof of Theorem 1.

THEOREM 2. *Let M^n be an n -dimensional complete Riemannian manifold with Ricci curvature bounded from below which is minimally immersed in a unit sphere $S^{n+p}(1)$ of dimension $n+p$. If $K_N=0$, then, either M^n is totally geodesic and $M^n=S^n(1)$ holds locally or $\inf R\leq n(n-2)$.*

PROOF. Because $K_N=0$ if and only if, for any α, β ,

$$\sum_{ij}(\sum_k(h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta))^2 = 0,$$

$$\begin{aligned} (3.13) \quad \frac{1}{2}\Delta\|h\|^2 &\geq -L_N + n\|h\|^2 \\ &\geq n\|h\|^2 - \|h\|^4 \\ &= \|h\|^2(n - \|h\|^2) \quad (\text{see [4]}) \end{aligned}$$

Hence, using the same arguments as Theorem 1, we have $\|h\|^2=0$ or $\sup\|h\|^2\geq n$. Thus, from Lemma 1, we know either M^n is totally geodesic and $M^n=S^n(1)$ holds locally or $\inf R\leq n(n-2)$.

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