TSUKUBA J. MATH.
Vol. 13 No. 1 (1989). 99—105

A GENERALIZATION OF A RESULT OF K. R. JOHNSON

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In [2], Grytczuk showed that

$$\sum_{d\mid k}|c_d(n)|=z^{\omega(k/(k,n))}(k, n),$$

where $c_k(n)$ denotes Ramanujan's trigonometric sum and $\omega(m)$ counts the number of distinct prime divisors of m. In [3], Johnson evaluated the sum

$$\sum_{d\mid n}|c_k(d)|.$$

In [4], I generalized the result of Grytczuk to a larger class of functions. In this paper I generalize the result of Johnson.

If h is an arithmetic function, we define the arithmetic function H_k by

$$H_k(n)=\sum_{d\mid(k,n)}\mu(k/d)h(d). \tag{1}$$

In [4] it is shown that $H_1(n)=h(1)$, if $a\geqq 1$,

$$H_{p^a}(n)=\begin{cases}h(p^a)-h(p^{a-1}) & \text{if } p^a\mid n\\ -h(p^{a-1}) & \text{if } p^{a-1}\parallel n\\ 0 & \text{if } p^{a-1}\nmid n\end{cases} \tag{2}$$

and that $H_k(n)$ is a multiplicative function of k. In [4], we investigated the sum

$$\sum_{d\mid k}|H_d(n)|.$$

and in this paper we shall investigate the sum

$$\sum_{d\mid n}|H_k(d)|.$$

Since $H_k(n)$ is not a multiplicative function of n, this task is a little more difficult. We shall assume throughout this paper that h is a multiplicative function.

To make our generalization of Johnson's result as clear as possible we shall follow his notation as closely as possible. In particular, for a given positive integer k we denote by $\bar{k}$ the core of k, that is, the largest square-free divisor of k, and we denote by k^* the integer $k/\bar{k}$.

Received March 11, 1988.

LEMMA 1. *Let k be a square-free integer. Then*

$$\mu(k)H_k(n)=\sum_{d\mid(n,k)} h(d)\mu(d). \tag{3}$$

PROOF. Since h and μ are multiplicative functions we see that the right-hand side of (3) is a multiplicative function of n. Suppose $(m, n)=1$. Then

$$\begin{aligned}&\mu(k)H_k(m)\mu(k)H_k(n)\\&=\mu^2(k)\sum_{d\mid(m,k)} h(d)\mu(k/d)\sum_{d\mid(n,k)} h(d)\mu(k/d)\\&=\mu^2(k)\sum_{\substack{d_1\mid(m,k)\\ d_2\mid(n,k)}} h(d_1)h(d_2)\mu(k/d_1)\mu(k/d_2).\end{aligned}$$

If $(m, n)=1$, then $(d_1, d_2)=1$. Thus $h(d_1)h(d_2)=h(d_1d_2)\mu(k/d_1)\mu(k/d_2)=\mu(k)\mu(k/d_1d_2)$. (The latter result is easily proved from the definition of the Möbius function, μ.) Also $(m, n)=1$ implies that $(m, k)(n, k)=(mn, k)$, and so if $d\mid(mn, k)$, we can write $d=d_1d_2$, $(d_1, d_2)=1$, so that $d_1\mid(m, k)$ and $d_2(n, k)$. Conversely, if $d_1\mid(m, k)$ and $d_2\mid(n, k)$, then $d_1d_2\mid(mn, k)$. Thus

$$\begin{aligned}\mu(k)H_k(m)\mu(k)H_k(n)&=\mu^2(k)\sum_{d\mid(mn,k)} h(d)\mu(k)\mu(k/d)\\&=\mu^2(k)\mu(k)H_k(mn).\end{aligned} \tag{4}$$

If $\mu(k)=0$, then both sides of (4) equal zero and so are equal to each other. If $\mu(k)\neq 0$, then $\mu^2(k)=1$ and (4) can be written

$$\mu(k)H_k(m)\mu(k)H_k(n)=\mu(k)H_k(mn),$$

that is, $\mu(k)H_k(n)$ is a multiplicative function of n, whether k is square-free or not.

Thus, to prove (3) we need only show that (3) holds when $n=p^r$, a prime power. Since k is square-free we have $(p^r, k)=(p, k)$. Thus we need only consider the case when $n=p$, a prime. We have, for the left-hand side of (3)

$$\begin{aligned}\mu(k)H_k(p)&=\mu(k)\sum_{d\mid(p,k)}\mu(k/d)h(d)\\&=\begin{cases}\mu^2(k)h(1) & \text{if } p\nmid k\\ \mu^2(k)h(1)+\mu(k)\mu(k/p)h(p) & \text{if } p\mid k\end{cases}\\&=\begin{cases}1 & \text{if } p\nmid k\\ 1-h(p) & \text{if } p\mid k,\end{cases}\end{aligned}$$

since $k=pp_1\cdots p_r$ being square-free implies that $\mu(k)\mu(k/p)=(-1)^{r+1}(-1)^r=-1$. The right-hand side of (3) is

$$\sum_{d\mid(p,k)} h(d)\mu(d)=\begin{cases} h(1)\mu(1) & \text{if} \quad p\nmid k \\ h(1)\mu(1)+\mu(p)h(p) & \text{if} \quad p\mid k \end{cases}$$

$$=\begin{cases} 1 & \text{if} \quad p\nmid k \\ 1-h(p) & \text{if} \quad p\mid k \end{cases}$$

Thus, both sides of (3) are equal in this case and the proof is complete.

LEMMA 2. *If h is a completely multiplicative function, we have*

$$H_k(nk^*)=h(k^*)H_{\bar{k}}(n),$$

where $\bar{k}$ and k^* are defined as above.

PROOF. Since $H_k(n)$ is a multiplicative function of k we have, by (2), that

$$H_k(n)=0 \qquad \text{if} \quad k^*\nmid n.$$

Since $k^*\bar{k}=k$ we have

$$\begin{aligned} H_k(nk^*)&=\sum_{d\mid(nk^*,\,k)} h(d)\mu(k/d) \\ &=\sum_{d\mid k^*(n,\,\bar{k})} h(d)\mu(k/d) \qquad (5)\\ &=\sum_{d\mid k^*(n,\,\bar{k})} h(d)\mu(k^*\bar{k}/d). \end{aligned}$$

Now $\mu(k/d)=0$ if $k^*\nmid d$. Thus, from (5), we have

$$\begin{aligned} H_k(nk^*)&=\sum_{d\mid(n,\,\bar{k})} h(k^*d)\mu(\bar{k}/d) \\ &=h(k^*)\sum_{d\mid(n,\,\bar{k})} h(d)\mu(\bar{k}/d) \\ &=h(k^*)H_{\bar{k}}(n), \end{aligned}$$

which was to be proved.

LEMMA 3. *If k and n are square-free, then*

$$\mu(k)H_k(n)=\mu(n)H_n(k).$$

PROOF. By Lemma 1 we have

$$\begin{aligned} \mu(k)H_k(n)&=\sum_{d\mid(k,\,n)} h(d)\mu(d) \\ &=\sum_{d\mid(n,\,k)} h(d)\mu(d) \\ &=\mu(n)H_n(k), \end{aligned}$$

which was to be proved.

LEMMA 4. *If h is a completely multiplicative function, $h(n)\neq 0$ and $h(k)\neq 0$ then for all k and n we have*

$$\frac{\mu(\bar{k})}{h(k^*)}H_k(nk^*)=\frac{\mu(\bar{n})}{h(n^*)}H_n(kn^*).$$

PROOF. Since $\bar{k}$ and $\bar{n}$ are square-free, we have, by Lemma 3,

$$\mu(\bar{k})H_{\bar{k}}(\bar{n})=\mu(\bar{n})H_{\bar{n}}(\bar{k}). \tag{6}$$

By (1), we have

$$\begin{aligned}H_{\bar{k}}(\bar{n})&=\sum_{d\mid(\bar{k},\bar{n})}h(d)\mu(\bar{k}/d)\\&=\sum_{d\mid(\bar{k},n)}h(d)\mu(\bar{k}/d)\\&=H_{\bar{k}}(n),\end{aligned}$$

Since $\bar{k}$ being square-free implies that $(\bar{k},\bar{n})=(\bar{k},n)$. Thus, by (6), we have

$$\mu(\bar{k})H_{\bar{k}}(n)=\mu(\bar{n})H_{\bar{n}}(k). \tag{7}$$

By Lemma 2 and (7), we have

$$\begin{aligned}\frac{\mu(\bar{k})}{h(k^*)}H_k(nk^*)&=\frac{\mu(\bar{k})}{h(k^*)}h(k^*)H_{\bar{k}}(n)\\&=\mu(\bar{k})H_{\bar{k}}(n)\\&=\mu(\bar{n})H_{\bar{n}}(k)\\&=\frac{\mu(\bar{n})}{h(n^*)}h(n^*)H_{\bar{n}}(k)\\&=\frac{\mu(\bar{n})}{h(n^*)}H_n(kn^*),\end{aligned}$$

which was to be proved.

LEMMA 5. *Let*

$$F_k(n)=\sum_{d\mid n}|H_{\bar{d}}(k)|.$$

Then $F_k(n)$ is a multiplicative function of n for any fixed k

PROOF. Let $(m,n)=1$. Then

$$\begin{aligned}F_k(mn)&=\sum_{d\mid mn}|H_{\bar{d}}(k)|\\&=\sum_{\substack{d_1\mid m\\ d_2\mid n}}|H_{\bar{d}_1\bar{d}_2}(k)|\end{aligned}$$

$$= \sum_{d_1 \mid m} |H_{\bar{d}_1}(k)| \sum_{d_2 \mid n} |H_{\bar{d}_2}(k)|$$

$$= F_k(m) F_k(n),$$

since $H_{\bar{d}}(k)$ is a multiplicative function of d for k fixed. This proves the result.

LEMMA 6. *If k is a fixed integer, then*

$$F_k(n) = \prod_{\substack{p^a \| n \\ p \nmid k}} (a+1) \prod_{\substack{p^a \| n \\ p \mid k}} (a|h(p)-1|+1).$$

PROOF. By (2), we have

$$H_p(k) = \begin{cases} h(p) - h(1) & \text{if } p \mid k \\ -h(1) & \text{if } p \nmid k \end{cases}$$

Since $F_k(n)$ is a multiplicative function of n, for fixed k, we need only evaluate $F_k(p^a)$. We have

$$F_k(p^a) = \sum_{d \mid p^a} a |H_{\bar{d}}(k)|$$

$$= |H_1(k)| + a |H_p(k)|$$

$$= \begin{cases} 1 + a & \text{if } p \nmid k \\ 1 + a|h(p)-1| & \text{if } p \mid k, \end{cases}$$

since h being a multiplicative function implies that $h(1)=1$. This proves the result.

THEOREM. *If h is a completely multiplicative function such that $h(n) \neq 0$, then*

$$\sum_{d \mid n} |H_k(d)| = \begin{cases} 0 & \text{if } k^* \nmid n \\ |h(k^*)| \prod_{\substack{p^a \| n/k^* \\ p \nmid k}} (a+1) \prod_{\substack{p^a \| n/k^* \\ p \mid k}} (a|h)p)-1|+1) & \text{if } k^* \mid n. \end{cases}$$

PROOF. If $k^* \nmid n$, then $k^* \nmid d$ for and $d \mid n$, and so $H_k(d)=0$ for all $d \mid n$. This gives the first result. Suppose $k^* \mid n$. Then, by Lemma 4,

$$\sum_{d \mid n} |H_k(d)| = \sum_{d \mid n/k^*} |H_k(dk^*)|$$

$$= \sum_{d \mid n/k^*} \left| \frac{H_d(kd^*) \mu(\bar{d}) h(k^*)}{h(d^*) \mu(\bar{k})} \right|$$

$$= |h(k^*)| \sum_{d \mid n/k^*} \left| \frac{H^d(kd^*)}{h(d^*)} \right|$$

$$=|h(k^*)|\sum_{d\mid n/k^*}|H_{\bar{d}}(k)|$$

$$=|h(k^*)|F_k(n/k^*)$$

by Lemma 2. The result follows from Lemma 6 and completes the proof.

We now give some examples.

1. Let r be a real number and let $h(n)=n^r$. Then h is a completely multiplicative function, and so, by the Theorem, we have

$$\sum_{d\mid n}|H_k(d)|=\begin{cases}0 & \text{if } k^*\nmid n\\ k^{*r}\prod_{\substack{p^a\| n/k^*\\ p\nmid k}}(a+1)\prod_{\substack{p^a\| n/k^*\\ p\mid k}}(a|p^r-1|+1) & \text{if } k^*\mid n.\end{cases}$$

The case $r=1$ gives the result of Johnson, [4]. If we take $r=0$ we have $h(n)=1$ for all n and (8) becomes

$$\sum_{d\mid n}|H_k(d)|=\begin{cases}0 & \text{if } k^*\nmid n\\ \prod_{\substack{p^a\| n/k^*\\ p\nmid k}}(a+1) & \text{if } k^*\mid n.\end{cases}$$

2. Let z be a complex number and let $h(n)=z^{\Omega(n)}$, where $\Omega(n)$ counts the total number of prime divisors of n. Then it is not hard to show from the definition of Ω to show that h is a completely multiplicative function. Thus

$$\sum_{d\mid n}|H_k(d)|=\begin{cases}0 & \text{if } k^*\nmid n\\ |z|^{\Omega(k^*)}\prod_{\substack{p^a\| n/k^*\\ p\nmid k}}(a+1)\prod_{\substack{p^a\| n/k^*\\ p\mid k}}(a|z-1|+1) & \text{if } k^*\mid n\end{cases}\tag{9}$$

If we take the special case $z=2$, then (9) becomes

$$\sum_{d\mid n}|H_k(d)|=\begin{cases}0 & \text{if } k^*\nmid n\\ 2^{\Omega(k^*)}\prod_{p^a\| n/k^*}(a+1) & \text{if } k^*\mid n\end{cases}$$

$$=\begin{cases}0 & \text{if } k^*\nmid n\\ 2^{\Omega(k^*)}d(n/k^*) & \text{if } k^*\mid n,\end{cases}$$

where $d(m)$ counts the number of divisors of m.

3. If X is a character modulo q and $h(n)=X(n)$, then $h(n)$ is a completely multiplicative function. If $(q, n)=1$, then

$$\sum_{d\mid n}|H_k(d)|=\begin{cases}0 & \text{if } k^*\nmid n\\ \prod_{\substack{p^a\| n/k^*\\ p\nmid k}}(a+1)\prod_{\substack{p^a\| n/k^*\\ p\mid k}}(a|X(p)-1|+1) & \text{if } k^*\mid n\end{cases}$$

One can further refine the sum by noting that if $(k, q)>1$, then $p\mid k$ implies that $X(p)=0$. Thus the two products combine to give $d(n/k^*)$.

Finally, we remark that like the result in [4] it seems likely that the result obtained in this paper can be further extended to the class of arithmetic functions considered by Anderson and Apostol in [1]. We hope to return to this in a later paper.

References

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