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THE SHRINKING PROPERTY OF Σ-PRODUCTS

By

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1. Introduction.

Concerning the study of the normality of Σ-products, the following results have been proved in order:

(A) A Σ-product of metric spaces is normal (by Gul'ko [6] and Rudin [15] in 1977).

(B) A Σ-product of paracompact p-spaces is normal iff it has countable tightness (by Kombarov [8] in 1978).

(C) A Σ-product of paracompact Σ-spaces is normal if it has countable tightness (by the author [17] in 1984).

On the other hand, the shrinking property is between paracompactness and normality. Rudin [16] in 1983 began to study the shrinking property of Σ-products and LeDonne [10] in 1985 extended her results. That is, they respectively proved the following:

(A′) A Σ-product of metric spaces is shrinking.

(B′) A Σ-product of paracompact p-spaces is shrinking iff it is normal.

The main purpose of the present paper is to prove the further extension, according to (C), as follows:

(C′) A Σ-product of strong Σ-spaces is shrinking iff it is normal. Moreover, we prove that the "strong Σ-spaces" in (C′) can be replaced by "semi-metric spaces". This gives another generalization of (A′).

The weak $\mathcal{B}$-property is weaker than the shrinking one. Chiba [2] proved that a Σ-product of compact spaces has the weak $\mathcal{B}$-property. So she asked in [3] whether a Σ-product of paracompact M-spaces ($=p$-spaces) has the weak $\mathcal{B}$-property. Here, we give an affirmative answer to this question.

All results proved here were early announced in [19] as a report.

All spaces are assumed to be regular T_1. The letters n, m, k, i, j and l denote non-negative integers.

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2. The shrinking and subshrinking properties.

Let S be a space. Let $\mathcal{G}=\{G_\gamma : \gamma\in\Gamma\}$ be an open cover of S. We say that $\{H_\gamma : \gamma\in\Gamma\}$ is a (*regular*) *shrinking* of $\mathcal{G}$ if it is a (an open) cover of S such that $\bar{H}_\gamma\subset G_\gamma$ for each $\gamma\in\Gamma$. Moreover, we say that $\{H_{\gamma,n} : \gamma\in\Gamma$ and $n\geqq 1\}$ is a (*regular*) *σ-shrinking* of $\mathcal{G}$ if it is a (an open) cover of S and $\bar{H}_{\gamma,n}\subset G_\gamma$ for each $\gamma\in\Gamma$ and $n\geqq 1$. A space S is said to be *shrinking* if every open cover of S has a (regular) shrinking. A space S is said to be *subshrinking* if every open cover of S has a σ-shrinking. The following diagram is true:

$$\begin{array}{ccccc} \textbf{paracompact} & \longrightarrow & \textbf{shrinking} & \longrightarrow & \textbf{normal} \\ \downarrow & & \downarrow & & \downarrow \\ \textbf{subparacompact} & \longrightarrow & \textbf{subshrinking} & \longrightarrow & \textbf{subnormal}^{*)} \end{array}$$

We say that a space S has the *weak $\mathcal{B}$-property* [21] if every monotone increasing open cover $\{U_\gamma : \gamma<\kappa\}$ (that is, $U_\gamma\subset U_{\gamma'}$, if $\gamma<\gamma'<\kappa$) has a regular skrinking. This property is between shrinking one and countable paracompactness.

PROPOSITION 1. ([1, Corollary 3.2]). *The following are equivalent for a space S:*

(a) *S is shrinking.*

(b) *S is normal and subshrinking.*

(c) *Every open cover of S has a regular σ-shrinking.*

Observe that subparacompact spaces and perfect spaces (each closed set is G_δ) are subshrinking. It follows from Proposition 1 that normal subparacompact spaces and perfectly normal spaces are shrinking (cf. [22, Theorems 3 and 4]).

Let S be a set. A collection $\mathcal{A}$ of subsets of S is said to be *directed* if for any $A_1, A_2\in\mathcal{A}$ there is some $A_3\in\mathcal{A}$ such that $A_1\cup A_2\subset A_3$.

Since a countable increasing cover of a space is directed, the proof of [1, Corollary 3.2] also shows

PROPOSITION 2. *If every directed open cover of a space S has a regular σ-shrinking, then every directed open cover of S has a regular shrinking.*

Fixing an open cover of a normal space, we have

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*) A space S is said to be *subnormal* if for any disjoint closed sets A and B there are disjoint G_δ-sets G and H such that $A\subset G$ and $B\subset H$.

PROPOSITION 3. *Let S be a normal space and $\mathcal{G}$ an open cover of S. If $\mathcal{G}$ has a σ-shrinking, then it has a shrinking.*

This was kindly pointed out by Yasui. Indeed, it follows from

PROPOSITION 4 (The proof of [22, Theorem 4]). *Let S be a space and $\mathcal{G}=\{G_\gamma : \gamma\in\Gamma\}$ an open cover of S. If there is a regular σ-shrinking $\{U_{\gamma,n} : \gamma\in\Gamma$ and $n\geqq 1\}$ of $\mathcal{G}$ such that $\bar{U}_{\gamma,n}\subset U_{\gamma,n+1}$ for each $\gamma\in\Gamma$ and $n\geqq 1$, then $\mathcal{G}$ has a shrinking.*

3. Theorems and corollaries.

As Σ-products are well-known, they are dealt with not here but in the next section.

A space X is called a *strong Σ-space* (*Σ-space*) [13] if there are a σ-locally finite closed cover $\mathcal{F}$ of X and a cover $\mathcal{K}$ of X by (countably) compact sets such that, whenever $K\in\mathcal{K}$ and U is open in X with $K\subset U$, $K\subset F\subset U$ for some $F\in\mathcal{F}$.

Strong Σ-spaces and subparacompact Σ-spaces are coincident. The class of (strong) Σ-spaces is broad in the sense that it contains the classes of σ-spaces and (paracompact) M-spaces below.

Our main theorem is as follows:

THEOREM 1. *A Σ-product of strong Σ-spaces is shrinking iff it is normal.*

By Theorem 1 and [18, Theorem 1], we have

COROLLARY 1. *Let Σ be a Σ-product of paracompact Σ-spaces. Then the following are equivalent:*

(a) *Σ is collectionwise normal.*

(b) *Σ is normal.*

(c) *Σ is shrinking.*

Recall that a *paracompact M-space* (*$=p$-space*) [11] means the inverse image of a metric space by a perfect map.

THEOREM 2. *Let Σ be a Σ-product of paracompact M-spaces. Then every directed open cover of Σ has a regular shrinking.*

This result immediately gives an affirmative answer to the question in [3]. That is, COROLLARY 2. *A Σ-product of paracompact M-spaces has the weak $\mathscr{B}$-property.*

In particular, we have

COROLLARY 3. *A Σ-product of paracompact M-spaces is countably paracompact.*

Recall that a *σ-space* [14] is a space with a σ-locally finite (closed) net.

THEOREM 3. *A Σ-product of σ-spaces is subshrinking.*

A space X is said to be *semi-metric* (cf. [7]) if it has a function g of $X\times\{n: n\geqq 1\}$ into the topology of X, satisfying

(i) $\{g(x, n): n\geqq 1\}$ is a neighborhood (=nbd) base of x for each $x\in X$,

(ii) $y\in\bigcap_{n=1}^{\infty} g(x_n, n)$ implies that $\{x_n\}$ converges to y.

We call the function g a *semi-metric function* of X. Note that a space X is semi-metric iff it is first countable and semi-stratifiable.

THEOREM 4. *A Σ-product of semi-metric spaces is subshrinking.*

By Proposition 1 and Theorem 4, we have

COROLLARY 4. *A Σ-product of semi-metric spaces is shrinking iff it is normal.*

4. Notations for Σ-products.

Let $\{X_\lambda: \lambda\in\Lambda\}$ be a collection of spaces. Let $X=\prod_{\lambda\in\Lambda}X_\lambda$ be the product of X_λ, $\lambda\in\Lambda$. Take a point $0=(0_\lambda)\in X$. For each $x=(x_\lambda)\in X$, let $\mathrm{Supp}(x)=\{\lambda\in\Lambda: x_\lambda\neq 0_\lambda\}$. Then the subspace $\Sigma=\{x\in X: \mathrm{Supp}(x)$ is at most countable$\}$ of X is called a *Σ-product* [4] of spaces X_λ, $\lambda\in\Lambda$. Such a point $0=(0_\lambda)\in\Sigma$ is called the *base point* of Σ. Such a space Σ is called a *Σ-product of $\cdots$ spaces* if each X_λ is a $\cdots$ space.

Here we must prepare some notations of Σ-products for the proofs of our theorems.

For the index set Λ, we denote by Λ_ω the set of all non-empty countable subsets of Λ. For each $R\in\Lambda_\omega$, X_R and $\Sigma_{\Lambda\setminus R}$ denote the countable product $\prod_{\lambda\in R}X_\lambda$ and the Σ-product of X_λ, $\lambda\in\Lambda\setminus R$, with the base point $(0_\lambda)_{\lambda\in\Lambda\setminus R}$, respectively. Moreover, p_R and $p_{\Lambda\setminus R}$ denote the projections of Σ onto X_R and

$\Sigma_{\Lambda\setminus R}$, respectively.

Let Ξ be an index set such that one can assign $R_\xi\in\Lambda_\omega$ for each $\xi\in\Xi$. Then X_{R_ξ}, $\Sigma_{\Lambda\setminus R_\xi}$, p_{R_ξ} and $p_{\Lambda\setminus R_\xi}$ are abbreviated by X_ξ, $\Sigma_{\Lambda\setminus\xi}$, p_ξ and $p_{\Lambda\setminus\xi}$, respectively.

Note that strong Σ-spaces, σ-spaces and semi-metric spaces are subparacompact and that the three classes of these spaces and the class of paracompact M-spaces are all countably productive. So, in case of Σ being a countable product, all our theorems are trivial.

Henceforth, all Σ-products are assumed to be proper. That is, we assume without special mention that the index set Λ is uncountable and each space X_λ, $\lambda\in\Lambda$, contains the point 1_λ different from 0_λ.

For each $R\in\Lambda_\omega$ and finite $r\subset\Lambda$ with $R\cap r=\emptyset$, consider an open nbd W_r of $0_{\Lambda\setminus R}$ $(=(0_\lambda)_{\lambda\in\Lambda\setminus R})$ in $\Sigma_{\Lambda\setminus R}$. The open nbd W_r is said to be *r-basic* if

$$W_r=(\Pi\{X_\lambda : \lambda\in\Lambda\setminus(R\cup r)\}\times\Pi\{W_\lambda : \lambda\in r\})\cap\Sigma_{\Lambda\setminus R}\,,$$

where W_λ is an open nbd of 0_λ in X_λ with $1_\lambda\notin W_\lambda$ for each $\lambda\in r$.

For each $R\in\Lambda_\omega$, a subset E of Σ is said to be *R-cylindrically closed* in Σ (cf. [20]) if $p_R^{-1}p_R(E)=E$ and $p_R(E)$ is closed in X_R.

For two index sets Δ and Ξ, $\Delta\oplus\Xi$ denotes the disjoint sum of Δ and Ξ.

5. Basic lemmas.

Let Σ be the Σ-product of spaces X_λ, $\lambda\in\Lambda$, with the base point $0=(0_\lambda)\in\Sigma$. Let $\mathcal{G}=\{G_\gamma : \gamma\in\Gamma\}$ be an open cover of Σ.

For each subset of F of X_R, where $R\in\Lambda_\omega$, we put

$M^*(F)=\{r\subset\Lambda\setminus R : r$ is a non-empty finite set and there is an r-basic open nbd W_r of $0_{\Lambda\setminus R}$ such that $\bar{F}\times\overline{W}_r\subset G_{\gamma_1}\cup\cdots\cup G_{\gamma_m}$ for some finite $\gamma_1,\cdots,\gamma_m\in\Gamma\}$.

LEMMA 1. *Let $R\in\Lambda_\omega$. Let F be a non-empty subset of X_R. If*

$$p_R^{-1}(F)\subset\cup\{(p_{\Lambda\setminus R})^{-1}(W_r) : r\in M^*(F)\},$$

then there is a pairwise disjoint subcollection $\{r(\delta) : \delta<\omega_1\}$ of $M^*(F)$.

PROOF. The proof is essentially due to Rudin [16]. Take any $r(0)\in M^*(F)$. For each $\delta<\omega_1$, assume that there is a pairwise disjoint subcollection $\{r(\zeta) : \zeta<\delta\}$ of $M^*(F)$. Let $Q=\cup\{r(\zeta) : \zeta<\delta\}$. Then $Q\in\Lambda_\omega$ with $Q\cap R=\emptyset$. Let $N=\{r\in M^*(F) : r\cap Q\neq\emptyset\}$. It suffices to show that $\{(p_{\Lambda\setminus R})^{-1}(W_r) : r\in N\}$ does not cover $p_R^{-1}(F)$. Pick $x\in F$. We take the point $y=(y_\lambda)\in\Sigma$ defined by $p_R(y)=x$, $y_\lambda=1_\lambda$ for each $\lambda\in Q$ and $y_\lambda=0_\lambda$ for each $\lambda\in\Lambda\setminus(R\cup Q)$. Then we have $y\in p_R^{-1}(F)\setminus\cup\{(p_{\Lambda\setminus R})^{-1}(W_r) : r\in N\}$.

BASIC LEMMA I. *Let Σ, $\mathcal{G}$ and $M^*(\cdot)$ be the same ones as above. Assume that the Σ-product Σ is normal. If there is a σ-locally finite closed cover $\{E(\xi): \xi\in\Delta^+\}$ of Σ and for each $\xi\in\Delta^+$ one can assign $R_\xi\in\Lambda_\xi$ such that*

$$p_\xi^{-1}p_\xi(E(\xi))\subset\cup\{(p_{\Lambda\setminus\xi})^{-1}(W_r): r\in M^*(p_\xi(E(\xi)))\},$$

then $\mathcal{G}$ has a σ-shrinking.

PROOF. Pick $\xi\in\Delta^+$. Let $F_\xi=p_\xi(E(\xi))$. It follows from Lemma 1 that there is a pairwise disjoint subcollection $\{r(\beta): \beta<\omega_1\}$ of $M^*(F_\xi)$. For each $\beta<\omega_1$, we can choose a finite subset $\phi(\xi, \beta)$ of Γ such that $\bar{F}_\xi\times\overline{W}_{r(\beta)}\subset\cup\{G_\gamma: \gamma\in\phi(\xi, \beta)\}$. It follows from the Δ-system lemma (for example, see [9, p. 49]) that there is a Δ-system $\{\phi(\xi, \beta_\delta): \delta<\omega_1\}$ with the root $\theta(\xi)$. We may rewrite $\{\beta_\delta: \delta<\omega_1\}$ by $\{\delta: \delta<\omega_1\}$ for brevity. Then it satisfies

(i) $\{r(\delta): \delta<\omega_1\}$ is pairwise disjoint collection of finite subsets of $\Lambda\setminus R_\xi$,

(ii) $\bar{F}_\xi\times\overline{W}_{r(\delta)}\subset\cup\{G_\gamma: \gamma\in\phi(\xi, \delta)\}$ and $\phi(\xi, \delta)$ is a finite subset of Γ,

(iii) $\phi(\xi, \delta)\cap\phi(\xi, \delta')=\theta(\xi)$ for each $\delta, \delta'<\omega_1$ with $\delta\neq\delta'$.

By the normality of Σ and (ii), for each $\delta<\omega_1$ there is a finite collection $\{U(\xi, \delta, \gamma): \gamma\in\phi(\xi, \delta)\}$ of open sets in Σ such that

$$\bar{F}_\xi\times\overline{W}_{r(\delta)}\subset\cup\{U(\xi, \delta, \gamma): \gamma\in\phi(\xi, \delta)\},$$

$$\overline{U(\xi, \delta, \gamma)}\subset G_\gamma \text{ whenever } \gamma\in\phi(\xi, \delta).$$

It should be noted by (i) that the Σ-product $\Sigma_{\Lambda\setminus\xi}$ (see Section 4) is covered by $\{W_{r(\delta)}: \delta<\omega_1\}$. By (iii), we have

$$E(\xi)\subset p_\xi^{-1}(F_\xi)=F_\xi\times\Sigma_{\Lambda\setminus\xi}=\cup\{F_\xi\times W_{r(\delta)}: \delta<\omega_1\}$$

$$\subset\cup\{U(\xi, \delta, \gamma): \gamma\in\phi(\xi, \delta) \text{ and } \delta<\omega_1\}$$

$$\subset(\cup\{U(\xi, \delta, \gamma): \gamma\in\phi(\xi, \delta)\setminus\theta(\xi) \text{ and } \delta<\omega_1\})\cup(\cup\{G_\gamma: \gamma\in\theta(\xi)\}).$$

Again by the normality of Σ, there is a finite collection $\{E(\xi, \gamma): \gamma\in\theta(\xi)\}$ of closed sets in Σ such that $E(\xi)$ is covered by

$$\{U(\xi, \delta, \gamma): \gamma\in\phi(\xi, \delta)\setminus\theta(\xi) \text{ and } \delta<\omega_1\}\cup\{E(\xi, \gamma): \gamma\in\theta(\xi)\}$$

and $E(\xi, \gamma)\subset G_\gamma\cap E(\xi)$ for each $\gamma\in\theta(\xi)$. Put $E(\xi, \delta, \gamma)=\overline{U(\xi, \delta, \gamma)}\cap E(\xi)$ for each $\gamma\in\phi(\xi, \delta)\setminus\theta(\xi)$ and $\delta<\omega_1$. Then

$$\{E(\xi, \delta, \gamma): \gamma\in\phi(\xi, \delta)\setminus\theta(\xi) \text{ and } \delta<\omega_1\}\cup\{E(\xi, \gamma): \gamma\in\theta(\xi)\}$$

is a collection of closed sets in Σ such that it covers $E(\xi)$, $E(\xi, \gamma)\subset G_\gamma$ for each $\gamma\in\theta(\xi)$ and $E(\xi, \delta, \gamma)\subset G_\gamma$ for each $\gamma\in\phi(\xi, \delta)\setminus\theta(\xi)$ and $\delta<\omega_1$.

We represent $\Delta^+=\cup_{n=1}^\infty\Delta_n^+$ such that $\{E(\xi): \xi\in\Delta_n^+\}$ is locally finite in Σ

for each $n \geqq 1$. Now, we put

$$H_{\gamma,n}=(\cup\{E(\xi,\delta,\gamma):\xi\in\Delta_n{}^+,\ \delta<\omega_1 \text{ and } \gamma\in\phi(\xi,\delta)\backslash\theta(\xi)\})$$
$$\cup(\cup\{E(\xi,\gamma):\xi\in\Delta_n{}^+ \text{ and } \gamma\in\theta(\xi)\})$$

for each $\gamma\in\Gamma$ and $n\geqq 1$. It is easy to see that $\{H_{\gamma,n}:\gamma\in\Gamma \text{ and } n\geqq 1\}$ is a cover of Σ such that $H_{\gamma,n}\subset G_\gamma$ for each $\gamma\in\Gamma$ and $n\geqq 1$. We show that each $H_{\gamma,n}$ is closed in Σ. Pcik any $\gamma_0\in\Gamma$ and $n_0\geqq 1$. By $E(\xi,\gamma_0)\subset E(\xi)$, $\{E(\xi,\gamma_0):\xi\in\Delta_{n_0}^+ \text{ with } \gamma_0\in\theta(\xi)\}$ is a locally finite collection of closed sets in Σ. It follows from (iii) that

$$\{E(\xi,\delta,\gamma_0):\delta<\omega_1 \text{ with } \gamma_0\in\phi(\xi,\delta)\backslash\theta(\xi)\}$$

consists of at most one member for each $\xi\in\Delta_{n_0}^+$. So, by $E(\xi,\delta,\gamma_0)\subset E(\xi)$,

$$\{E(\xi,\delta,\gamma_0):\xi\in\Delta_{n_0}^+ \text{ and } \delta<\omega_1 \text{ with } \gamma_0\in\phi(\xi,\delta)\backslash\theta(\xi)\}$$

is a locally finite collection of closed sets in Σ. By the choice of H_{γ_0,n_0}, it is closed in Σ. Therefore $\{H_{\gamma,n}:\gamma\in\Gamma \text{ and } n\geqq 1\}$ is a σ-shrinking of $\mathcal{G}$. □

Next, for each subset F of X_R, where $R\in\Lambda_\omega$, we put

$M(F)=\{r\subset\Lambda\backslash R: r$ is a non-empty finite set and there is an r-basic open nbd W_r of $0_{\Lambda\backslash R}$ in $\Sigma_{\Lambda\backslash R}$ such that $\bar{F}\times\overline{W}_r\subset G_\gamma$ for some $\gamma\in\Gamma\}$.

Note that Lemma 1 is also true for the $M(F)$ instead of $M^*(F)$.

BASIC LEMMA II. *Let Σ, $\mathcal{G}$ and $M(\cdot)$ be the same ones as above. If there is a σ-locally finite closed (open) cover $\{E(\xi):\xi\in\Delta^+\}$ of Σ and for each $\xi\in\Delta^+$ one can assign $R_\xi\in\Lambda_\omega$ such that*

$$p_\xi{}^{-1}p_\xi(E(\xi))\subset\cup\{(p_{\Lambda\backslash\xi})^{-1}(W_r):r\in M(p_\xi(E(\xi)))\},$$

then $\mathcal{G}$ has a (regular) σ-shrinking.

PROOF. The proof is simpler than the previous one. Let $F_\xi=p_\xi(E(\xi))$ for each $\xi\in\Delta^+$. It follows from Lemma 1 for $M(\cdot)$ that there is a pairwise disjoint subcollection $\{r(\delta):\delta<\omega_1\}$ of $M(F_\xi)$. We can choose some $\gamma(\xi,\delta)\in\Gamma$ such that $\bar{F}_\xi\times\overline{W}_{r(\delta)}\subset G_{\gamma(\xi,\delta)}$ for each $\xi\in\Delta^+$ and $\delta<\omega_1$. Without loss of generality, we may assume that all $\gamma(\xi,\delta)$, $\delta<\omega_1$, are the same or different. So we put

$$\Delta^1=\{\xi\in\Delta^+: \text{All } \gamma(\xi,\delta),\ \delta<\omega_1, \text{ are the same}\},$$

$$\Delta^2=\{\xi\in\Delta^+: \text{All } \gamma(\xi,\delta),\ \delta<\omega_1, \text{ are different}\}.$$

Then $\Delta^+=\Delta^1\oplus\Delta^2$. Moreover, we may put $\gamma_\xi=\gamma(\xi,\delta)$ for each $\xi\in\Delta^1$ and $\delta<\omega_1$.

Similarly, we can check that $\overline{E(\xi)}\subset G_\gamma$ for each $\xi\in\Delta^1$.

Let $\Delta^+=\bigcup_{n=1}^\infty\Delta_n^+$ such that $\{E(\xi): \xi\in\Delta_n^+\}$ is locally finite in Σ for each $n\geqq 1$. Here, we put

$$H_{\gamma,n}=(\bigcup\{E(\xi): \xi\in\Delta_n^+\cap\Delta^1 \text{ with } \gamma_\xi=\gamma\})\cup$$
$$(\bigcup\{(F_\xi\times W_{\gamma(\delta)})\cap E(\xi): \xi\in\Delta_n^+\cap\Delta^2 \text{ and } \delta<\omega_1 \text{ with } \gamma(\xi, \delta)=\gamma\})$$

for each $\gamma\in\Gamma$ and $n\geqq 1$. Then $\{H_{\gamma,n}: \gamma\in\Gamma$ and $n\geqq 1\}$ is a (an open) cover of Σ. Moreover, we can show that $\overline{H}_{\gamma,n}\subset G_\gamma$ for $\gamma\in\Gamma$ and $n\geqq 1$. This verification is similar to the previous one. Therefore $\{H_{\gamma,n}: \gamma\in\Gamma$ and $n\geqq 1\}$ is a (regular) σ-shrinking of $\mathcal{G}$. □

Basic Lemmas I and II are necessary for the proofs of Theorem 1 and others, respectively.

6. Proof of Theorem 1.

LEMMA 2 ([13, Lemma 1]). *Let X be a strong Σ-space. Then there is a sequence $\{\mathcal{F}_n\}$ of locally finite closed covers of X, satesfying*

(a) $\mathcal{F}_n=\{F(\alpha_1\cdots\alpha_n): \alpha_1, \cdots, \alpha_n\in\Omega\{$ *for each* $n\geqq 1$,

(b) $F(\alpha_1\cdots\alpha_n)=\bigcup\{F(\alpha_1\cdots\alpha_n\alpha_{n+1}): \alpha_{n+1}\in\Omega\}$ *for each* $\alpha_1, \cdots, \alpha_n\in\Omega$,

(c) *for each $x\in X$, there is a sequence $\alpha_1, \alpha_2, \cdots\in\Omega$ such that*

(i) $\bigcap_{n=1}^\infty F(\alpha_1\cdots\alpha_n)$ *is a compact set containing* x,

(ii) *if $\{D_n\}$ is a decreasing sequence of non-empty closed sets in X such that $D_n\subset F(\alpha_1\cdots\alpha_n)$ for each $n\geqq 1$, then $\bigcap_{n=1}^\infty D_n\neq\emptyset$.*

The above sequence $\{\mathcal{F}_n\}$ is a called a *spectral strong Σ-net* [13] of X. Moreover, the sequence $\{F(\alpha_1\cdots\alpha_n): n\geqq 1\}$ in (c) is called a *local Σ-net* at x.

Lemma 2 was used in [17, 18].

For an $n\times n$ matrix $\xi=(\alpha_{ij})_{i,j\leq n}$ and $1\leqq k\leqq n$, the $k\times k$ matrix $(\alpha_{ij})_{i,j\leq k}$ is denoted by $\xi|k$. In particular, $\xi|n-1$ is often abbreviated by ξ_- and $\xi|0$ implies the 0×0 matrix $(\emptyset)$.

PROOF OF THEOREM 1. Let Σ be the Σ-product of strong Σ-spaces X_λ, $\lambda\in\Lambda$, with the base point $0=(0_\lambda)\in\Sigma$, and assume that Σ is normal. Let $\mathcal{G}=\{G_\gamma: \gamma\in\Gamma\}$ be any open cover of Σ. We use the notation $M^*(\cdot)$ defined in the previous section.

For each $n\geqq 0$, we construct an index set $\Delta_n=\Delta_n^+\oplus\Xi_n$ of $n\times n$ matrices such that for each $\xi\in\Delta_n$ one can assign $E(\xi)\subset\Sigma$ and for each $\xi\in\Xi_n$ one can assign $x_\xi\in\Sigma$ and $R_\xi\in\Lambda_\omega$, satisfying the following conditions (1)-(6) for each

$n\geqq 1$:

(1) For each $\mu\in\Xi_{n-1}$, $\{F(\alpha_{n1}\cdots\alpha_{nk}):\alpha_{n1},\cdots,\alpha_{nk}\in\Omega(\mu)\}$, $k\geqq 1$, is a spectral strong Σ-net of X_{μ}.

(2) $\Delta_n=\{\xi=(\alpha_{ij})_{i,j\leqq n}:\xi_-\in\Xi_{n-1}$ and $\alpha_{ij}\in\Omega(\xi|i-1)$ for $1\leqq i,\ j\leqq n\}$ and $\Delta_0=\{(\emptyset)\}$.

(3) For each $\xi=(\alpha_{ij})_{i,j\leqq n}\in\Delta_n$,

$$E(\xi)=\bigcap_{i=1}^{n}(p_{\xi|i-1})^{-1}(F(\alpha_{i1}\cdots\alpha_{in}))$$

and $E(\emptyset)=\Sigma$.

(4) $\Delta_n{}^+=\{\xi\in\Delta_n: E(\xi)\subset\bigcup\{(p_{\Lambda\setminus\xi_-})^{-1}(W_r): r\in M^*(p_{\xi_-}(E(\xi)))\}\}$.

(5) For each $\xi\in\Xi_n$, $x_\xi\in E(\xi)\setminus\bigcup\{(p_{\Lambda\setminus\xi_-})^{-1}(W_r): r\in M^*(p_{\xi_-}(E(\xi)))\}$.

(6) For each $\xi\in\Xi_n$, $R_\xi=R_{\xi_-}\cup\mathrm{Supp}\,(x_\xi)$.

Using Lemma 2, this construction is easily performed. Note that $E(\xi)$ is an R_{ξ_-}-cylindrically closed set in Σ (see Section 4) such that $E(\xi)\subset E(\xi_-)$ for each $\xi\in\Delta_n$ and $n\geqq 1$. It is verified that $\{E(\xi):\xi\in\Delta_n\}$ is locally finite in Σ for each $n\geqq 1$. Let $\Delta^+=\bigcup_{n=1}^{\infty}\Delta_n{}^+$. Considering R_{ξ_-} instead of R_ξ, the σ-locally finite collection $\{E(\xi):\xi\in\Delta^+\}$ of closed sets in Σ satisfies the conditions of Basic Lemma I except the following:

LEMMA 3. *$\{E(\xi):\xi\in\Delta^+\}$ covers Σ.*

PROOF. Assuming the contrary, pick some $y\in\Sigma\setminus\bigcup\{E(\xi):\xi\in\Delta^+\}$. By (1) and the choice of y, we can inductively choose a sequence $\{\alpha_{ij}: i,\ j\geqq 1\}$ such that for each $n\geqq 1$ $\xi(n)=(\alpha_{ij})_{i,j\leqq n}\in\Xi_n$ and $\{F(\alpha_{n1}\cdots\alpha_{nk}): k\geqq 1\}$ is a local Σ-net at $p_{\xi(n-1)}(y)$ in $X_{\xi(n-1)}$, where $\alpha_{nk}\in\Omega(\xi(n-1))$ and $\xi(0)=(\emptyset)$. Let $R=\bigcup_{n=1}^{\infty}R_{\xi(n)}$. Then $R\in\Lambda_\omega$. In this proof, $p_{\xi(n-1)}$ is abbreviated by p_{n-1}. Put $K_n=\bigcap_{k\geqq n}\overline{p_{n-1}(E(\xi(k)))}$ for each $n\geqq 1$. Since $p_{n-1}(E(\xi(k))$ is contained in $F(\alpha_{n1}\cdots\alpha_{nk})$ for each $k\geqq n$, it follows from (i) of (c) in Lemma 2 that K_n is compact. Since $y\in E(\xi(k))$ for each $k\geqq 1$, we have $p_{n-1}(y)\in K_n$. Note that $p_{n-1}^n(K_{n+1})\subset K_n$, where p_{n-1}^n is the projection of $X_{\xi(n)}$ onto $X_{\xi(n-1)}$. Hence $\{K_n,\ p_{n-1}^n|K_n\}$ is an inverse sequence of non-empty compact spaces. Then the limit $K=\varprojlim\{K_n,\ p_{n-1}^n|K_n\}$ is non-empty and compact. Since each p_{n-1}^n is the projection, we can consider that K is a subspace of X_R. So there are some finite $\gamma_1,\cdots,\gamma_m\in\Gamma$ such that $K\times\{0_{\Lambda\setminus R}\}\subset G_{\gamma_1}\cup\cdots\cup G_{\gamma_m}$. Take some open sets U and V in X_R and $\Sigma_{\Lambda\setminus R}$, respectively, such that $K\subset U$, $0_{\Lambda\setminus R}\in V$ and $U\times V\subset G_{\gamma_1}\cup\cdots\cup G_{\gamma_m}$.

CLAIM. *$p_{n-1}(E(\xi(n))\times X_{Q(n)}\subset U$ for some $n\geqq 1$, where $Q(k)=R\setminus R_{\xi(k-1)}$, $k\geqq 1$.*

PROOF. Assume the contrary. We can take some

$$u_n \in (p_{n-1}(E(\xi(n)) \times X_{Q(n)}) \setminus U$$

for each $n \geqq 1$. Pick $n \geqq 1$. Put $L_{nk} = \{p_{n-1}^{\infty}(u_k), p_{n-1}^{\infty}(u_{k+1}), \cdots\}$ for each $k \geqq n$, where p_{n-1}^{∞} is the projection of X_R onto $X_{\xi(n-1)}$. Since $L_{nk} \subset p_{n-1}(E(\xi(k)))$, we have

$$\bar{L}_{nk} \subset \overline{p_{n-1}(E(\xi(k)))} \subset F(\alpha_{n1} \cdots \alpha_{nk})$$

for each $k \geqq n$. Since $\{F(\alpha_{n1} \cdots \alpha_{nk}) : k \geqq 1\}$ is a local Σ-net at $p_{n-1}(y)$ in $X_{\xi(n-1)}$, it follows from (ii) of (c) in Lemma 2 that $\bigcap_{k \geqq n} \bar{L}_{nk}$ is non-empty. Let $L_n = \bigcap_{k \geqq n} \bar{L}_{nk}$. Then we have $L_n \subset K_n$. Moreover, by $p_{n-1}^{n}(L_{n+1k}) = L_{nk}$, we have $p_{n-1}^{n}(L_{n+1}) \subset L_n$. Hence $\{L_n, p_{n-1}^{n} | L_n\}$ is an inverse sequence of non-empty compact spaces. Then the limit $L = \varprojlim \{L_n, p_{n-1}^{n} | L_n\}$ is a non-empty subspace of K $(\subset X_R)$. Pick some $z \in L$. Since $p_{n-1}^{\infty}(z) \in L_n \subset \bar{L}_{nn}$ for each $n \geqq 1$, the z is a cluster point of $\{u_n\}$ in X_R. Since each u_n is not in U, z is not in U. On the other hand, we have $z \in L \subset K \subset U$. This is a contradiction. Claim has been proved.

Now, let $p_{n-1}(E(\xi(n))) \times X_{Q(n)} \subset U$. Since $X_{Q(n)} \times V$ is an open nbd of $0_{\Lambda \setminus \xi(n-1)}$ in $\Sigma_{\Lambda \setminus \xi(n-1)}$, there are some finite $q \subset \Lambda \setminus R$ and a q-basic open nbd W_q of $0_{\Lambda \setminus \xi(n-1)}$ in $\Sigma_{\Lambda \setminus \xi(n-1)}$ such that $\overline{W}_q \subset X_{Q(n)} \times V$. Then we have

$$\overline{p_{n-1}(E(\xi(n)))} \times \overline{W}_q = p_{n-1}(E(\xi(n))) \times \overline{W}_q \subset U \times V \subset G_{\gamma_1} \cup \cdots \cup G_{\gamma_m}.$$

Hence $q \in M^*(p_{n-1}(E(\xi(n))))$. Remember $\xi(n) \in \Xi_n$. By (5), $x_{\xi(n)} \notin p_{n-1}(E(\xi(n))) \times W_q$ is true. By (6), $\mathrm{Supp}(x_{\xi(n)}) \subset R_{\xi(n)} \subset R$. Since R and q are disjoint, we obtain

$$x_{\xi(n)} \in p_{n-1}(E(\xi(n))) \times X_{Q(n)} \times \{0_{\Lambda \setminus R}\} \subset p_{n-1}(E(\xi(n))) \times W_q.$$

This is a contradiction. Lemma 3 has been proved. □

Thus, Basic Lemma 1 assures that $\mathcal{G}$ has a σ-shrinking. Since Σ is normal, it follows from Proposition 1 or 3 that $\mathcal{G}$ has a shrinking. The proof of Theorem 1 is completed. □

7. Proofs of other theorems.

LEMMA 4. *Let X be a M-space. Then there is a sequence $\{\mathcal{V}_n\}$ of locally finite open covers of X, satisfying*

(a) $\mathcal{V}_n = \{V(\alpha_1 \cdots \alpha_n) : \alpha_1, \cdots, \alpha_n \in \Omega\}$ *for each* $n \geqq 1$,

(b) $V(\alpha_1 \cdots \alpha_n) = \bigcup \{V(\alpha_1 \cdots \alpha_n \alpha_{n+1}) : \alpha_{n+1} \in \Omega\}$ *for each* $\alpha_1, \cdots, \alpha_n \in \Omega$,

(b) *if* $\bigcap_{n=1}^{\infty} V(\alpha_1 \cdots \alpha_n) \neq \emptyset$ *and* $\{D_n\}$ *is a decreasing sequence of non-empty* *closed sets in* X *such that* $D_n \subset \overline{V(\alpha_1 \cdots \alpha_n)}$ *for each* $n \geqq 1$, *then* $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$.

This lemma easily follows from the definition of M-spaces in [11]. The proof is similar to that of [12, Theorem 1] or [13, Lemma 1.4]. Note that the intersection $\bigcap_{n=1}^{\infty} \overline{V(\alpha_1 \cdots \alpha_n)}$ is compact if X is a paracompact M-space. The above sequence $\{\mathcal{V}_n\}$ is called a *spectral M-base* of X, for the sake of convenience.

Since Theorems 2 and 3 are obtained below by modifying the proof of Theorem 1, we also use the same notations as in it except $M(\cdot)$ instead of $M^*(\cdot)$.

PROOF OF THEOREM 2. Let Σ be the Σ-product of paracompact M-spaces X_λ, $\lambda \in \Lambda$, with the base point $0=(0_\lambda) \in \Sigma$. Let $\mathcal{G}=\{G_\lambda : \lambda \in \Lambda\}$ be any directed open cover of Σ.

For each $n \geqq 0$, we construct an index set $\Delta_n = \Delta_n^+ \oplus \Xi_n$ of $n \times n$ matrices such that for each $\xi \in \Delta_n$ one can assign $U(\xi) \subset \Sigma$ and for each $\xi \in \Xi_n$ one can assign $x_\xi \in \Sigma$ and $R_\xi \in \Lambda_\omega$, satisfying the conditions (1)–(6) in the proof of Theorem 1, where E, F, $M^*(\cdot)$ and "spectral strong Σ-net" should be replaced by U, V, $M(\cdot)$ and "spectral M-base", respectively.

Using Lemma 4, this construction is also easy. Let $\Delta^+ = \bigcup_{n=1}^{\infty} \Delta_n^+$. In the similar way to the proof of Lemma 3, we can show that $\{U(\xi) : \xi \in \Delta^+\}$ covers Σ. It should be noted there that the $G_{\gamma_1} \cup \cdots \cup G_{\gamma_m}$ can be replaced by some $G_\gamma \in \mathcal{G}$, because $\mathcal{G}$ is directed. So we may use $M(p_{n-1}(U(\xi(n))))$ instead of $M^*(p_{n-1}(E(\xi(n))))$. After all, $\{U(\xi) : \xi \in \Delta^+\}$ satisfies the conditions in the parenthetic part of Basic Lemma II. Hence $\mathcal{G}$ has a regular σ-shrinking. It follows from Proposition 2 that $\mathcal{G}$ has a regular shrinking. □

LEMMA 5 ([12, Theorem 1]). *Let* X *be a* σ*-space. Then there is a sequence* $\{\mathcal{F}_n\}$ *of locally finite closed covers of* X, *satisfying*

(a) $\mathcal{F}_n = \{F(\alpha_1 \cdots \alpha_n) : \alpha_1, \cdots, \alpha_n \in \Omega\}$ *for each* $n \geqq 1$,

(b) $F(\alpha_1 \cdots \alpha_n) = \bigcup \{F(\alpha_1 \cdots \alpha_n \alpha_{n+1}) : \alpha_{n+1} \in \Omega\}$ *for each* $\alpha_1, \cdots, \alpha_n \in \Omega$,

(c) *for each* $x \in X$, *there is a sequence* $\alpha_1, \alpha_2, \cdots \in \Omega$ *such that* $x \in \bigcap_{n=1}^{\infty} F(\alpha_1 \cdots \alpha_n)$ *and each open nbd of* x *contains some* $F(\alpha_1 \cdots \alpha_n)$.

The above sequence $\{\mathcal{F}_n\}$ is called a *spectral* σ*-net* of X and the sequence $\{F(\alpha_1 \cdots \alpha_n) : n \geqq 1\}$ in (c) is called a *local* σ*-net* at x.

PROOF OF THEOREM 3. Let Σ be the Σ-product of σ-spaces X_λ, $\lambda \in \Lambda$, with the base point $0=(0_\lambda) \in \Sigma$. Let $\mathcal{G}=\{G_\gamma : \gamma \in \Gamma\}$ be any open cover of Σ.

For each $n\geqq 0$, we construct the same $\Delta_n=\Delta_n^+\oplus\Xi_n$, $E(\xi)\subset\Sigma$, $x_\xi\in\Sigma$ and $R_\xi\in\Lambda_\omega$ as in the proof of Theorem 1. They also satisfy the same conditions (1)-(6) except that only "spectral strong Σ-net" in (1) is replaced by "spectral σ-net". Similarly, it suffices from Basic Lemma II to show the following:

LEMMA 6. *$\{E(\xi): \xi\in\Delta^+\}$ covers Σ, where $\Delta^+=\bigcup_{n=1}^\infty\Delta_n^+$.*

PROOF. Assume the contrary. Pick some $y\in\Sigma\setminus\bigcup\{E(\xi): \xi\in\Delta^+\}$. We can inductively choose a sequence $\{\alpha_{ij}: i, j\geqq 1\}$ such that for each $n\geqq 1$ $\xi(n)=(\alpha_{ij})_{i,j\leqq n}\in\Xi_n$ and $\{F(\alpha_{n1}\cdots\alpha_{nk}): k\geqq 1\}$ is a local σ-net at $p_{\xi(n-1)}(y)$ in $X_{\xi(n-1)}$, where $\alpha_{nk}\in\Omega(\xi(n-1))$ and $\xi(0)=(\emptyset)$. Let $R=\bigcup_{n=1}^\infty R_{\xi(n)}$. Abbreviate $p_{\xi(n-1)}$ with p_{n-1}. Pick the point $z\in\Sigma$ defined by $p_R(z)=p_R(y)$ and $p_{\Lambda\setminus R}(z)=0_{\Lambda\setminus R}$. Take some $\gamma_0\in\Gamma$ with $z\in G_{\gamma_0}$. Moreover, take an open nbd B of z in Σ such that $B\subset G_{\gamma_0}$ and

$$B=p_{m-1}(B)\times X_{R\setminus R_{\xi(m-1)}}\times p_{\Lambda\setminus R}(B)$$

for some $m\geqq 1$. By the choice of $\{F(\alpha_{i1}\cdots\alpha_{ik}): k\geqq 1\}$, for each $i\leqq m$ we can choose some $n_i\geqq 1$ such that

$$p_{i-1}(z)=p_{i-1}(y)\in F(\alpha_{i1}\cdots\alpha_{in_i})\subset p_{i-1}(B).$$

Let $l=\max\{n_1, \cdots, n_m, m\}$. Then we can easily verify $p_{l-1}(E(\xi(l)))\subset p_{l-1}(B)$. Let $Q=R\setminus R_{\xi(l-1)}$. Since $X_Q\times p_{\Lambda\setminus R}(B)$ is an open nbd of $0_{\Lambda\setminus\xi(l-1)}$ in $\Sigma_{\Lambda\setminus\xi(l-1)}$, there is some finite $q\subset\Lambda\setminus R$ and a q-basic open nbd W_q of $0_{\Lambda\setminus\xi(l-1)}$ in $\Sigma_{\Lambda\setminus\xi(l-1)}$ such that $\overline{W}_q\subset X_Q\times p_{\Lambda\setminus R}(B)$. Then we have

$$\begin{aligned}\overline{p_{l-1}(E(\xi(l)))\times W_q}&=p_{l-1}(E(\xi(l)))\times\overline{W}_q\subset p_{l-1}(B)\times X_Q\times p_{\Lambda\setminus R}(B)\\ &=B\subset G_{\gamma_0}.\end{aligned}$$

Hence $q\in M(p_{l-1}(E(\xi(l))))$. So we can obtain a contradiction in the same way as the last part of the proof of Lemma 3. Lemma 6 has been proved. Consequently, the proof of Theorem 3 is completed. □

Let Ξ be a set consisting of finite sequences and $(\emptyset)$. For each $\xi=(\alpha_1\cdots\alpha_{n-1}\alpha_n)\in\Xi$, ξ_- and $\xi\oplus\alpha$ denote $(\alpha_1\cdots\alpha_{n-1})$ and $(\alpha_1\cdots\alpha_n\alpha)$, respectively. The 0-tuple sequence is only $(\emptyset)$.

PROOF OF THEOREM 4. Let Σ be the Σ-product of semi-metric spaces X_λ, $\lambda\in\Lambda$, with the base point $0=(0_\lambda)\in\Sigma$. Let $\mathcal{G}=\{G_\gamma: \gamma\in\Gamma\}$ be any open cover of Σ.

For each $n\geqq 0$, we shall construct a collection $\mathcal{C}_n$ of closed sets in Σ and an index set Ξ_n of n-tuple sequences such that for each $\xi\in\Xi_n$ one can assign

$R_\xi \in \Lambda_\omega$, $E(\xi) \subset \Sigma$, $x_\xi \in X_{\xi_-}$, $\{y_{\xi,k}\} \subset \Sigma$ and a function g_ξ, satisfying the following conditions (1)-(7) for each $n \geqq 1$:

(1) $\mathcal{C}_n = \cup\{\mathcal{C}(\mu): \mu \in \Xi_{n-1}\}$ is σ-locally finite in Σ.

(2) Each $C \in \mathcal{C}(\mu)$, $\mu \in \Xi_{n-1}$, is an R_μ-cylindrically closed set in Σ such that $C \subset \cup\{(p_{\Lambda\setminus\mu})^{-1}(W_r): r \in M(p_\mu(C))\}$.

(3) $\xi \in \Xi_n$ implies $\xi_- \in \Xi_{n-1}$.

(4) $\{E(\xi): \xi \in \Xi_n\}$ is σ-locally finite in Σ, for each $\xi \in \Xi_n$ $E(\xi)$ is an R_{ξ_-}-cylindrically closed set in Σ and $E(\emptyset) = \Sigma$.

(5) For each $\mu \in \Xi_{n-1}$,

$$p_\mu(E(\mu)) \subset p_\mu(\cup\mathcal{C}(\mu)) \cup (\cup\{p_\mu(E(\xi)): \xi \in \Xi_n \text{ with } \xi_- = \mu\}).$$

(6) For each $\xi \in \Xi_n$, g_ξ is a semi-metric function of X_ξ such that

$$p_{\xi_-}^{\xi}(g_\xi(x, k)) \subset g_{\xi_-}(p_{\xi_-}^{\xi}(x), k)$$

for each $x \in X_\xi$ and $k \geqq 1$, where $p_{\xi_-}^{\xi}$ denotes the projection of X_ξ onto X_{ξ_-} and $g_\emptyset$ is a semi-metric function of $X_\emptyset$.

(7) For each $\xi \in \Xi_n$,

a) $p_{\xi_-}(E(\xi)) \subset g_{\xi_-}(x_\xi, n)$,

b) $y_{\xi,k} \in p_{\xi_-}^{-1}(g_{\xi_-}(x_\xi, k)) \setminus \cup\{(p_{\Lambda\setminus\xi_-})^{-1}(W_r): r \in M(g_{\xi_-}(x_\xi, k))\}$ for each $k \geqq 1$,

c) $R_\xi = R_{\xi_-} \cup (\cup\{\mathrm{Supp}(y_{\xi,k}): k \geqq 1\})$.

The basic idea of this construction is found in [20]. The case of $n=0$ is trivial. Assume that it has been already performed for no greater than n. Pick $\xi \in \Xi_n$ and fix it. Put

$$\mathcal{V} = \{V: V \text{ is a non-empty open set in } X_\xi \text{ such that}$$
$$p_\xi^{-1}(V) \subset \cup\{(p_{\Lambda\setminus\xi})^{-1}(W_r): r \in M(V)\}.$$

Let $D_\xi = p_\xi(E(\xi))$. Observe that $D_\xi = (p_{\xi_-}^{\xi})^{-1}(p_{\xi_-}(E(\xi)))$ if $n \geqq 1$ and $D_\emptyset = X_\emptyset$. So D_ξ is closed in X_ξ. Since D_ξ is subparacompact, there is a σ-locally finite closed cover $\mathcal{F}$ of D_ξ, which refines

$$\{V \cap D_\xi: V \in \mathcal{V}\} \cup \{g_\xi(x, n+1) \cap D_\xi: x \in D_\xi \setminus \cup\mathcal{V}\}.$$

Let $\mathcal{F}^+ = \{F \in \mathcal{F}: F \subset V \cap D_\xi$ for some $V \in \mathcal{V}\}$ and $\mathcal{F}^- = \mathcal{F} \setminus \mathcal{F}^+$. Put $\mathcal{C}(\xi) = \{C = p_\xi^{-1}(F): F \in \mathcal{F}^+\}$. Then each $C \in \mathcal{C}(\xi)$ satisfies (2) and $C \subset E(\xi)$. Let $\Xi(\xi)$ be an index set of $(n+1)$-tuple sequences such that $\mathcal{F}^- = \{F_{\xi\oplus\alpha}: \xi\oplus\alpha \in \Xi(\xi)\}$. Take any $\eta = \xi\oplus\alpha \in \Xi(\xi)$. Let $E(\eta) = p_\xi^{-1}(F_\eta)$. We can choose some $x_\eta \in D_\xi \setminus \cup\mathcal{V} (\subset X_\xi)$ such that $p_\xi(E(\eta)) = F_\eta \subset g_\xi(x_\eta, n+1) \cap D_\xi$. By $x_\eta \notin \cup\mathcal{V}$, we have $g_\xi(x_\eta, k) \notin \mathcal{V}$ for each $k \geqq 1$. So, we can find a sequence $\{y_{\eta,k}\}$ of points in Σ, satisfying (7b). Define R_η as in (7c). We can take a semi-metric function g_η of X_η which satisfies (6). Here, ranging ξ over $\Xi(\xi)$, we set

$$\mathcal{C}_{n+1}=\cup\{\mathcal{C}(\xi):\xi\in\Xi_n\} \quad\text{and}\quad \Xi_{n+1}=\oplus\{\Xi(\xi):\xi\in\Xi_n\}.$$

It is easy to check that the conditions (1)-(7) are satisfied for $n+1$.

By (1) and (2), $\cup_{n=1}^{\infty}\mathcal{C}_n$ satisfies the conditions of Basic Lemma II except that it covers Σ. So it suffices to show

LEMMA 7. *$\mathcal{C}=\cup_{n=1}^{\infty}\mathcal{C}_n$ covers Σ.*

PROOF. Assume the contrary, pick some $y\in\Sigma\setminus\cup\mathcal{C}$. Then there is a sequence $\{\xi(n): n\geqq 0\}$ of finite sequences such that $\xi(n)\in\Xi_n$, $\xi(n+1)_-=\xi(n)$ and $y\in E(\xi(n))$ for each $n\geqq 0$ (see Claim 1 in the proof of [20, Theorem 1]). For each $m\geqq 1$, the sequence $\{p_{m-1}^{n-1}(x_{\xi(n)}): n\geqq m\}$ of points converges to $p_{m-1}(y)$ in $X_{\xi(m-1)}$, where p_{m-1}^{n-1} and p_{m-1} denote the projections of $X_{\xi(n-1)}$ and Σ, respectively, onto $X_{\xi(m-1)}$ (see Claim 2 in the proof of [20, Theorem 1]). Let $R=\cup_{n=1}^{\infty}R_{\xi(n)}$. Pick the point $z\in\Sigma$ defined by $p_R(z)=p_R(y)$ and $p_{\Lambda\setminus R}(z)=0_{\Lambda\setminus R}$. Take some $\gamma_0\in\Gamma$ with $z\in G_{\gamma_0}$, and an open nbd B of z in Σ such that $B\subset G_{\gamma_0}$ and

$$B=p_{m-1}(B)\times X_{R\setminus R_{\xi(m-1)}}\times p_{\Lambda\setminus R}(B)$$

for some $m\geqq 1$. Since $p_{m-1}^{n-1}(x_{\xi(n)})\to p_{m-1}(y)$ $(n\to\infty)$, there is some $k\geqq m$ such that $p_{m-1}^{k-1}(x_{\xi(k)})\in p_{m-1}(B)$. Let $g_{k-1}=g_{\xi(k-1)}$. Since $p_{k-1}(B)$ is an open nbd of $x_{\xi(k)}$ and $\{g_{k-1}(x_{\xi(k)}, i): i\geqq 1\}$ is a nbd base of $x_{\xi(k)}$ in $X_{\xi(k-1)}$, we can choose some $l\geqq 1$ such that $\overline{g_{k-1}(x_{\xi(k)}, l)}\subset p_{k-1}(B)$. There is some finite $q\subset\Lambda\setminus R$ and a q-basic open nbd W_q of $0_{\Lambda\setminus\xi(k-1)}$ in $\Sigma_{\Lambda\setminus\xi(k-1)}$ such that $\overline{W}_q\subset X_Q\times p_{\Lambda\setminus R}(B)$, where $Q=R\setminus R_{\xi(k-1)}$. Then we have

$$\overline{g_{k-1}(x_{\xi(k)}, l)}\times\overline{W}_q\subset p_{k-1}(B)\times X_Q\times p_{\Lambda\setminus R}(B)=B\subset G_{\gamma_0}.$$

Hence $q\in M(g_{k-1}(x_{\xi(k)}, l))$. By (7b), $y_{\xi(k),l}\notin g_{k-1}(x_{\xi(k)}, l)\times W_q$ is true. On the other hand, by (7c), $\mathrm{Supp}(y_{\xi(k),l})\subset R_{\xi(k)}\subset R$. Since R and q are disjoint and $p_{k-1}(y_{\xi(k)}, l)\in g_{k-1}(x_{\xi(k)}, l)$, we have

$$y_{\xi(k),l}\in g_{k-1}(x_{\xi(k)}, l)\times X_Q\times\{0_{\Lambda\setminus R}\}\subset g_{k-1}(x_{\xi(k)}, l)\times W_q,$$

which is a contradiction. Lemma 7 has been proved. Therefore, the proof of Theorem 4 is completed. □

8. Questions.

The subshrinking property of Σ-products seems to be important for the study of the shrinking one of them. So we raise

QUESTION 1. If a Σ-product of strong Σ-spaces is subnormal, is it subshrinking?

We can obtain an extension of Theorem 1 if this is solved in the affirmative.

The referee of [20] asked to the author whether the results (A′) and (B′) in the introduction can be generalized to the semi-stratifiable case. Here, we state it more concretely.

QUESTION 2. Is a Σ-product of semi-stratifiable spaces subshrinkinng (if it has countable tightness)?

QUESTION 3. If a Σ-product of semi-stratifiable spaces is normal (and has countable tightness), is it shrinking?

Of course, if the answer to Question 2 is affirmative, then so is that of Question 3. Since σ-spaces and semi-metric spaces are semi-stratifiable, Theorems 3 and 4 are partial answers to Question 2. It is assured by [20, Theorem 3] that a Σ-product of semi-stratifiable spaces is at least subnormal.

Finally, we raise the following two questions concerning the normality of Σ-products of β-spaces. The definition of β-spaces is seen in [5, Definition 7.7].

QUESTION 4. Let Σ be a Σ-product such that each finite subproduct of it is a paracompact β-space and has countable tightness. Is then Σ normal?

QUESTION 5. Let Σ be a Σ-product such that each finite subproduct of it is a paracompact β-space. If Σ is normal, is it collectionwise normal?

Observe that both Σ-spaces and semi-stratifiable spaces are β-spaces (cf. [5, Theorem 7.8]). If Question 4 (Question 5) would be solved in the affirmative, we could obtain a nice extension of [17, Theorem 1] and [20, Theorem 1] ([18, Theorem 1] and [20, Theorem 2]).

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