TSUKUBA J. MATH.
Vol. 13 No. 1 (1989). 13—22

UNIVERSAL TRANSITIVITY OF CERTAIN CLASSES OF REDUCTIVE PREHOMOGENEOUS VECTOR SPACES

By

Shin-ichi KASAI

Introduction.

Let k be a field of characteristic zero. Let $\tilde{G}$ be a connected k-split linear algebraic group, $\rho: \tilde{G} \rightarrow GL(X)$ with $X = \mathrm{Aff}^n$ a k-homomorphism. If there exists a Zariski-dense $\rho(\tilde{G})$-orbit Y, we say that $(\tilde{G}, \rho, X)$ is a *prehomogeneous vector space* (abbrev. *P. V.*). When each irreducible component is castling equivalent to a non-trivial reduced irreducible prehomogeneous vector space or each irreducible component is a regular prehomogeneous vector space, we have completed a classification of reductive prehomogeneous vector spaces over a complex number field **C** (see [4], [5]).

Put $G = \rho(\tilde{G})$. Let l be the number of $G(k)$-orbits in $Y(k)$, i.e., $l = |G(k) \backslash Y(k)|$. We say that Y is a *universally transitive open orbit* if $l = |G(k) \backslash Y(k)| = 1$ for all k satisfying $H^1(k, \mathrm{Aut}(SL_2)) \neq 0$, i.e., there exists a nonsplit quaternion k-algebra. This condition is satisfied by every local field k other than **C**. Actually our classification depends on the transitivity of $G(k)$ on $Y(k)$ for *just one* k satisfying $H^1(k, \mathrm{Aut}(SL_2)) \neq 0$ (see Remarks 2.13 and 3.5). In [1] and [2], all irreducible regular prehomogeneous vector spaces with universally transitive open orbits are classified. In [10], we have classified simple or 2-simple prehomogeneous vector spaces with universally transitive open orbits.

In this paper, we shall classify reductive prehomogeneous vector spaces with universally transitive open orbits when each irreducible component is castling equivalent to a non-trivial reduced irreducible prehomogeneous vector space or each irreducible component is a regular prehomogeneous vector space.

This paper consists of the following three sections.

§1. Preliminaries.

§2. Reductive P. V.'s with universally transitive open orbits: the case I.

§3. Reductive P. V.'s with universally transitive open orbits: the case II.

The results are given in Theorems 2.11, 3.4 and Corollary 2.12.

Received June 9, 1987. Revised October 23, 1987.

§ 1. Preliminaries.

PROPOSITION 1.1. *We have* $l=1$ *for* $(\tilde{G}, \rho_1 \oplus \rho_2, X_1 \oplus X_2)$ *if and only if* (1) $l=1$ *for* $(\tilde{G}, \rho_1, X_1)$ *and* (2) $l=1$ *for* $(H, \rho_2|_H, X_2)$ *where* H *is a generic isotropy subgroup of* $(\tilde{G}, \rho_1, X_1)$.

PROOF. See Proposition 1.5 in [10]. Q.E.D.

COROLLARY 1.2. *Assume that* $l=1$ *for* $(\tilde{G}, \rho_1, X_1)$ *and* $(H^0, \rho_2|_{H^0}, X_2)$ *where* H^0 *is the connected component of a generic isotropy subgroup* H *of* $(\tilde{G}, \rho_1, X_1)$. *Then we we have* $l=1$ *for* $(\tilde{G}, \rho_1 \oplus \rho_2, X_1 \oplus X_2)$.

REMARK 1.3. Assume that $l=1$ for (G, ρ, X). Then $l=1$ for $(\tilde{G}, \tilde{\rho}, X)$ with $\tilde{\rho}(\tilde{G}) \supset \rho(G)$.

PROPOSITION 1.4. *The number* $l=|G(k) \backslash Y(k)|$ *is invariant under a castling transformation.*

PROOF. See [2]. Q.E.D.

THEOREM 1.5. ([1], [2]) *A regular irreducible P. V. has a universally transitive open orbit (i.e.* $l=1$*) if and only if it is castling equivalent to one of the following P. V.'s.*

(1.1) $(H \times GL(n), \rho \otimes \Lambda_1)$ *where* ρ *is an n-dimensional irreducible representation of* H.

(1.2) $(GL(2m), \Lambda_2)$ *with* $m \geqq 2$.

(1.3) $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1)$ *with* $n \geqq 2m$.

(1.4) $(GL(1) \times SO(2n), \Lambda_1 \otimes \Lambda_1)$ *with* $n \geqq 2$.

(1.5) $(GL(1) \times Spin(7), \Lambda_1 \otimes \text{spin rep.})$.

(1.6) $(GL(1) \times Spin(9), \Lambda_1 \otimes \text{spin rep.})$.

(1.7) $(Spin(10) \times GL(2), \text{half-spin rep.} \otimes \Lambda_1)$.

(1.8) $(GL(1) \times E_6, \Lambda_1 \otimes \Lambda_1)$.

THEOREM 1.6. *Any non-regular irreducible P. V., which is not castling equivalent to* $(Sp(n) \times GL(2), \Lambda_1 \otimes 2\Lambda_1)$, *has the universally transitive open orbit.*

PROOF. See Corollary 3.22 in [10]. Q.E.D.

§ 2. Reductive P.V.'s with unveirsally transitive open orbits: the case I.

In this section, we shall consider the case when each irreducible component is castling equivalent to a non-trivial reduced irreducible P. V.

THEOREM 2.1. ([4]) *Let (G, ρ, V) be an indecomposable reductive P. V. Assume that each irreducible component of (G, ρ, V) is castling equivalent to a non-trivial reduced irreducible P. V. If (G, ρ, V) does not contain an irreducible P. V. with $l \geqq 2$, then it is castling equivalent to one of the following P. V.'s.*

(i)

(2.1) $(GL(1)^2 \times SL(2m+1), \Lambda_2 + \Lambda_2)$ *with* $m \geqq 2$.

(2.2) $(GL(1)^2 \times Spin(8)$, *half-spin rep.+vector rep.*).

(2.3) $(GL(1)^2 \times Spin(10)$, *half-spin rep.+vector rep.*).

(2.4) $(GL(1)^2 \times Spin(10)$, *even half-spin rep.+even half-spin rep.*).

(2.5) $(GL(1)^2 \times Sp(n) \times SL(m), \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_1)$ *with* $n \geqq m \geqq 1$.

(2.6) $(GL(1)^2 \times Sp(n) \times SL(2m+1), \Lambda^1 \otimes \Lambda_1 + 1 \otimes \Lambda_2)$ *with* $2n > 2m+1 \geqq 5$.

(2.7) $(GL(1)^3 \times Sp(n) \times SL(2m+1), (\Lambda_1 + \Lambda_1) \otimes 1 + \Lambda_1 \otimes \Lambda_1)$ *with* $n \geqq 2m+1 \geqq 1$.

(ii)

(2.8) $(GL(1)^2 \times Sp(n) \times SL(2m+1) \times Sp(n'), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1)$ *with* $2n, 2n' > 2m+1 \geqq 5$.

(2.9) $(GL(1)^2 \times Sp(n) \times SL(3) \times Sp(n'), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)} \otimes \Lambda_1)$.

(2.10) $(GL(1)^3 \times Sp(n) \times SL(3) \times Sp(n'), \Lambda_1 \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)} \otimes \Lambda_1)$.

(2.11) $(GL(1)^3 \times Sp(n) \times Sp(n') \times SL(3) \times Sp(m)$, $(\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^{(*)} \otimes \Lambda_1)$.

(2.12) $(GL(1)^4 \times Sp(n) \times Sp(n') \times SL(3) \times Sp(m)$, $(\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes (\Lambda_1^{(*)} \otimes \Lambda_1 + 1 \otimes \Lambda_1))$.

(2.13) $(GL(1)^4 \times Sp(n) \times Sp(n') \times Sp(n'') \times SL(3) \times Sp(m)$, $(\Lambda_1 \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1) \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \Lambda_1^{(*)} \otimes \Lambda_1)$.

(iii)

(2.14) $(GL(1)^2 \times SL(2m) \times Sp(n) \times SL(2m'+1), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1)$ *with* $n \geqq 2m$, $2m'+1 \geqq 2$.

(2.15) $(GL(1)^3\times SL(2m)\times Sp(n)\times SL(3)\times Sp(n'),$
$\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes 1\otimes\Lambda_1\otimes\Lambda_1)$ *with* $n\geqq 2m\geqq 2$.

(2.16) $(GL(1)^4\times SL(2m+1)\times Sp(n)\times SL(2)\times Sp(n')\times SL(2m'+1),$
$\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1+1\otimes 1\otimes\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes 1\otimes 1\otimes\Lambda_1\otimes\Lambda_1)$
with $n\geqq 2m+1\geqq 1$ *and* $n'\geqq 2m'+1\geqq 1$.

(2.17) $(GL(1)^5\times SL(2m+1)\times Sp(n)\times SL(2)\times Sp(n')\times SL(3)\times Sp(n''),$
$\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1\otimes 1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1\otimes 1+1\otimes 1\otimes\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1$
$+1\otimes 1\otimes 1\otimes\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes 1\otimes 1\otimes 1\otimes\Lambda_1\otimes\Lambda_1)$ *with* $n\geqq 2m+1\geqq 1$.

(2.18) $(GL(1)^6\times SL(2m+1)\times Sp(n)\times SL(2)\times Sp(n')\times Sp(n'')\times SL(2m'+1)$
$\times SL(2m''+1),$
$(\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1)\otimes 1\otimes 1\otimes 1\otimes 1+1\otimes 1\otimes\Lambda_1\otimes(\Lambda_1\otimes 1+1\otimes\Lambda_1)\otimes 1\otimes 1$
$+1\otimes 1\otimes 1\otimes(\Lambda_1\otimes 1\otimes\Lambda_1\otimes 1+1\otimes\Lambda_1\otimes 1\otimes\Lambda_1))$
with $n\geqq 2m+1\geqq 1$, $n'\geqq 2m'+1\geqq 1$ *and* $n''\geqq 2m''+1\geqq 1$.

(2.19) $(GL(1)^i\times SL(2)\times H_1\times\cdots\times H_i,$
$\Lambda_1\otimes(\tau_1\otimes 1\otimes\cdots\otimes 1+1\otimes\tau_2\otimes 1\otimes\cdots\otimes 1+\cdots+1\otimes\cdots\otimes 1\otimes\tau_i))$
where $(H_j, \tau_j)(1\leqq j\leqq i)$ *is one of* $(SL(2n+1), \Lambda_2)$ $(n\geqq 2)$, $(Sp(n), \Lambda_1)(n\geqq 2)$
and $(Spin(10)$, *half-spin rep.)*.

(2.20) $(GL(1)^{k+i}\times(G'\times SL(2))\times(H_1\times\cdots\times H_i),$
$(\rho_1+\cdots+\rho_k)\otimes 1\otimes\cdots\otimes 1+1\otimes\Lambda_1\otimes(\tau_1\otimes 1\otimes\cdots\otimes 1+\cdots+1\otimes\cdots\otimes 1\otimes\tau_i))$
where $(GL(1)^k\times G'\times SL(2), \rho_1+\cdots+\rho_k)$ *is one of* (2.14) *with* $m=1$, (2.15) *with* $m=1$, (2.16), (2.17) *and* (2.18), *and* $(H_j, \tau_j)(1\leqq j\leqq i)$ *is one of* $(SL(2n+1), \Lambda_2)(n\geqq 2)$, $(Sp(n), \Lambda_1)(n\geqq 2)$ *and* $(Spin(10)$, *half-spin rep.)*.

PROPOSITION 2.2. *We have* $l=1$ *for P.V.'s in* (i), *i.e.*, (2.1)~(2.7) *in Theorem* 2.1.

PROOF. By [10], we obtain our assertion. Q.E.D.

PROPOSITION 2.3. *We have* $l=1$ *for P.V.'s in* (ii), *i.e.*, (2.8)~(2.13) *in Theorem* 2.1.

PROOF. By Corollary 3.8 in [10], we have $l=1$ for (2.8) (resp. (2.9), (2.10), (2.11), (2.12), (2.13)) if and only if $l=1$ for $(GL(1)^2\times SL(2m+1), \Lambda_2+\Lambda_2)$ (resp. $(GL(1)^2\times SL(3), \Lambda_1^*+\Lambda_1^{(*)*})$, $(GL(1)^3\times SL(3), \Lambda_1^*+\Lambda_1+\Lambda_1^{(*)*})$, $(GL(1)^3\times SL(3),$

$\Lambda_1^*+\Lambda_1^*+\Lambda_1^{(*)*})$, $(GL(1)^4\times SL(3), \Lambda_1^*+\Lambda_1^*+(\Lambda_1^*+\Lambda_1)^{(*)})$, $(GL(1)^4\times SL(3), \Lambda_1^*+\Lambda_1^*+\Lambda_1^*+\Lambda_1^{(*)*}))$. Hence we obtain our assertion by [10]. Q. E. D.

PROPOSITION 2.4. *We have* $l=1$ *for a P. V.* (2.14) *in Theorem* 2.1.

PROOF. By Corollary 3.8 in [10], we have $l=1$ for a P. V. $(2.14)\cong(Sp(n)\times(GL(2m)\times GL(2m'+1)), \Lambda_1\otimes(\Lambda_1\otimes1+1\otimes\Lambda_1))$ if and only if $l=1$ for $(GL(2m)\times GL(2m'+1), \Lambda_2\otimes1+\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)$. Since an irreducible P. V. $(GL(2m), \Lambda_2)$ has a universally transitive open orbit and a generic isotropy subgroup of $(GL(2m), \Lambda_2)$ is isomorphic to $Sp(m)$, it is enough to show that $(Sp(m)\times GL(2m'+1), \Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)$ has a universally transitive open orbit. By Proposition 4.4 in [10], we may assume that $2m>2m'+1$. Then, by Corollary 3.8 in [10], it has a universally transitive open orbit if and only if $(GL(2m'+1), \Lambda_2+\Lambda_2)$ has a universally transitive open orbit. By the proof of Proposition 2.15 in [10], we have $l=1$ for $(GL(2m'+1), \Lambda_2+\Lambda_2)$. Q. E. D.

PROPOSITION 2.5. *We have* $l=1$ *for a P. V.* (2.15) *in Theorem* 2.1.

PROOF. By Corollary 3.8 in [10], we have $l=1$ for a P. V. (2.15) if and only if $l=1$ for $(GL(1)^3\times SL(2m)\times Sp(n)\times SL(3), \Lambda_1\otimes\Lambda_1\otimes1+1\otimes\Lambda_1\otimes\Lambda_1+1\otimes1\otimes\Lambda_2)$. By the same argument as in the proof of Proposition 2.4, it is equivalent to say that $l=1$ for a P. V. $(GL(3)\times GL(1), (\Lambda_2+\Lambda_2)\otimes1+\Lambda_2\otimes\Lambda_1)\cong(GL(3)\times GL(1), \Lambda_1^*\otimes(1+1+\Lambda_1))$ (if $m\geqq2$), $(SL(2)\times GL(3)\times GL(1), (\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)\otimes1+1\otimes\Lambda_1^*\otimes\Lambda_1)$ (if $m=1$ and $n\geqq3$), $(GL(1)\times SL(2)\times Sp(2)\times GL(3)\times GL(1), \Lambda_1\otimes\Lambda_1\otimes\Lambda_1\otimes1\otimes1+1\otimes1\otimes\Lambda_1\otimes\Lambda_1\otimes1+1\otimes1\otimes1\otimes\Lambda_1^*\otimes\Lambda_1)$ (if $m=1$ and $n=2$). By Lemma 2.2 in [10], we have $l=1$ for $(GL(3)\times GL(1), \Lambda_1^*\otimes(1+1+\Lambda_1))$. By the proof of Proposition 2.4, $(SL(2)\times GL(3), \Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)$ has a universally transitive open orbit, and the $GL(3)$-part of its generic isotropy subgroup is locally isomorphic to $\left\{\begin{bmatrix}-\alpha, & \beta, & \gamma\\ 0, & \alpha, & 0\\ 0, & 0, & \alpha\end{bmatrix}\right\}$. Thus we have $l=1$ for $(SL(2)\times GL(3)\times GL(1), (\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)\otimes1+1\otimes\Lambda_1^*\otimes\Lambda_1)$. Also, by Proposition 2.4, we have $l=1$ for $(GL(1)\times SL(2)\times Sp(2)\times GL(3), \Lambda_1\otimes\Lambda_1\otimes\Lambda_1\otimes1+1\otimes1\otimes\Lambda_1\otimes\Lambda_1)$, and the $GL(3)$-part of its generic isotropy subgroup is locally isomorphic to $\left\{\begin{bmatrix}-\alpha, & \beta, & \gamma\\ 0, & \alpha, & 0\\ 0, & 0, & \alpha\end{bmatrix}\right\}$. Thus we have $l=1$ for $(GL(1)\times SL(2)\times Sp(2)\times GL(3)\times GL(1), \Lambda_1\otimes\Lambda_1\otimes\Lambda_1\otimes1\otimes1+1\otimes1\otimes\Lambda_1\otimes\Lambda_1\otimes1+1\otimes1\otimes1\otimes\Lambda_1^*\otimes\Lambda_1)$. Q. E. D.

LEMMA 2.6. *We have* $l=1$ *for a P. V.* $(GL(1)^2\times H\times SL(2), \rho\otimes\tau+1\otimes\Lambda_1)$ *if and only if* $l=1$ *for a P. V.* $(GL(1)^3\times H\times SL(2)\times Sp(n)\times SL(2m+1), \rho\otimes\tau\otimes1\otimes1+1\otimes\Lambda_1\otimes\Lambda_1\otimes1+1\otimes1\otimes\Lambda_1\otimes\Lambda_1)$ $(n>m\geqq0)$.

PROOF. By calculations similar to the proof of Lemma 2.7 in [10], we obtain that a generic isotropy subgroup of $(Sp(n)\times GL(2m), \Lambda_1\otimes\Lambda_1)$ is isomorphic to $\left\{\left(\begin{bmatrix} A, & 0 \\ 0, & B \end{bmatrix}, {}^tA^{-1}\right); A\in Sp(m), B\in Sp(n-m)\right\}\cong Sp(m)\times Sp(n-m)$ (cf. p. 101 in [13]). Thus, by Proposition 1.1, we have $l=1$ for a P. V. $(GL(1)^3\times H\times SL(2)\times Sp(n)\times SL(2m+1), \rho\otimes\tau\otimes 1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes 1\otimes\Lambda_1\otimes\Lambda_1)$ if and only if $l=1$ for a P. V. $(GL(1)\times H\times SL(2)\times Sp(n-1)\times GL(2m+1), \Lambda_1\otimes\rho\otimes\tau\otimes 1\otimes 1+1\otimes 1\otimes(\Lambda_1\otimes 1+1\otimes\Lambda_1)\otimes\Lambda_1)$. By Corollary 3.8 and Lemma 4.3 in [10], a P. V. $(GL(1)\times H\times SL(2)\times GL(2m+1)\times Sp(n-1), \Lambda_1\otimes\rho\otimes\tau\otimes 1\otimes 1+1\otimes 1\otimes(\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1))$ has a universally transitive open orbit if and only if a P. V. $(GL(1)\times H\times SL(2)\times GL(2m+1), \Lambda_1\otimes\rho\otimes\tau\otimes 1+1\otimes 1\otimes\Lambda_1\otimes\Lambda_1+1\otimes 1\otimes 1\otimes\Lambda_2)$ has a universally transitive open orbit. By Proposition 4.4 in [10], it is equivalent to say that a P. V. $(GL(1)^2\times H\times SL(2), \rho\otimes\tau+1\otimes\Lambda_1)$ has a universally transitive open orbit. Hence we obtain our assertion. Q. E. D.

PROPOSITION 2.7. *We have* $l=1$ *for a P. V.'s* (2.16) *and* (2.18) *in Theorem* 2.1.

PROOF. By Theorem 2.19 in [10], a P. V. $(GL(1)^2\times SL(2), \Lambda_1+\Lambda_1)$ (resp. $(GL(1)^3\times SL(2), \Lambda_1+\Lambda_1+\Lambda_1)$) has a universally transitive open orbit, and hence, by Lemma 2.6, we have $l=1$ for (2.16) (resp. (2.18)). Q. E. D.

PROPOSITION 2.8. *We have* $l=1$ *for a P. V.* (2.17) *in Theorem* 2.1.

PROOF. By Lemma 2.6 and Corollary 3.8 in [10], we have $l=1$ for a P. V. (2.17) if and only if $l=1$ for $(GL(1)^4\times SL(2)\times Sp(n')\times SL(3), \Lambda_1\otimes 1\otimes 1+\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes\Lambda_1\otimes\Lambda_1+1\otimes 1\otimes\Lambda_1^*)$. By the proof of Lemma 2.6, it is enough to show that $l=1$ for a P. V. $(GL(1)\times SL(2)\times GL(3)\times GL(1), \Lambda_1\otimes\Lambda_1\otimes 1\otimes 1+1\otimes(\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)\otimes 1+1\otimes 1\otimes\Lambda_1^*\otimes\Lambda_1)$ (if $n\geqq 3$), $(GL(1)\times GL(1)\times SL(2)\times Sp(2)\times GL(3)\times GL(1), \Lambda_1\otimes 1\otimes\Lambda_1\otimes 1\otimes 1\otimes 1+1\otimes(\Lambda_1\otimes\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1+1\otimes 1\otimes\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes 1\otimes 1\otimes\Lambda_1^*\otimes\Lambda_1))$ (if $n=2$). By the proof of Proposition 2.5, $(SL(2)\times GL(3)\times GL(1), (\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)\otimes 1+1\otimes\Lambda_1^*\otimes\Lambda_1)$ and $(GL(1)\times SL(2)\times Sp(2)\times GL(3)\times GL(1), \Lambda_1\otimes\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1+1\otimes 1\otimes\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes 1\otimes 1\otimes\Lambda_1^*\otimes\Lambda_1)$ have universally transitive open orbits, and the $SL(2)$-parts of its generic isotropy subgroups are locally isomorphic to $SO(2)$. Thus we have $l=1$ for $(GL(1)\times SL(2)\times GL(3)\times GL(1), \Lambda_1\otimes\Lambda_1\otimes 1\otimes 1+1\otimes(\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)\otimes 1+1\otimes 1\otimes\Lambda_1^*\otimes\Lambda_1)$ and $(GL(1)\times GL(1)\times SL(2)\times Sp(2)\times GL(3)\times GL(1), \Lambda_1\otimes 1\otimes\Lambda_1\otimes 1\otimes 1\otimes 1+1\otimes(\Lambda_1\otimes\Lambda_1\otimes\Lambda_1\otimes 1\otimes 1+1\otimes 1\otimes\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes 1\otimes 1\otimes\Lambda_1^*\otimes\Lambda_1))$. Q. E. D.

LEMMA 2.9. *We have* $l=1$ *for a P. V.* $(GL(1)^{k+i}\times(G'\times SL(2))\times(H_1\times\cdots\times H_i),$ $(\rho_1+\cdots+\rho_k)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\Lambda_1)\otimes(\tau_1\otimes 1\otimes\cdots\otimes 1+1\otimes\tau_2\otimes 1\otimes\cdots\otimes 1+\cdots+1\otimes$ $\cdots\otimes 1\otimes\tau_i))$ *if and only if* $l=1$ *for a P. V.* $(GL(1)^k\times G'\times SL(2),\ \rho_1+\cdots+\rho_k)$, *where* (H_j, τ_j) $(1\leqq j\leqq i)$ *is one of* $(SL(2n+1), \Lambda_2)$ $(n\geqq 2)$, $(Sp(n), \Lambda_1)$ $(n\geqq 2)$ *and* $(Spin(10)$, *half-spin rep.*).

PROOF. We have $l=1$ for irreducible P. V.'s $(SL(2n+1)\times GL(2), \Lambda_2\otimes\Lambda_1)$ $(Sp(n)\times GL(2), \Lambda_1\otimes\Lambda_1)$ and (Spin(10)$\times GL(2)$, half-spin rep.$\otimes\Lambda_1$), and each $GL(2)$-part of its generic isotropy subgroup contains $SL(2)$ (see [13]). Hence, by Proposition 1.1 and Corollary 1.2, we obtain our assertion. Q. E. D.

PROPOSITION 2.10. *We have* $l=1$ *for P. V.'s* (2.19) *and* (2.20)

PROOF. By Propositions 2.4, 2.5, 2.7, 2.8 and Lemma 2.9, we obtain our assertion. Q. E. D.

THEOREM 2.11. *Let* (G, ρ, V) *be an indecomposable reductive P. V. Assume that each irreducible component of* (G, ρ, V) *is castling equivalent to a non-trivial reduced irreducible P. V. Then we have* $l=1$ *for* (G, ρ, V) *if and only if* (G, ρ, V) *does not contain any irreducible P. V. with* $l\geqq 2$, *namely, it is castling equivalent to one of P. V.'s* (2.1)$\sim$(2.20) *in Theorem* 2.1.

PROOF. By Propositions 2.2, 2.3, 2.4, 2.5, 2.7, 2.8, 2.10 and Theorem 2.1, we obtain our assertion. Q. E. D.

COROLLARY 2.12. *Let* (G, ρ, V) *be a regular indecomposable reductive P. V. with a universally transitive open orbit. If each irreducible component of* (G, ρ, V) *is castling equivalent to a non-trivial reduced irreducible P. V., then* (G, ρ, V) *is castling equivalent to one of the following P. V.'s.*

(1) $(GL(1)^2\times Sp(n)\times SL(2m+1),\ \Lambda_1\otimes 1+\Lambda_1\otimes\Lambda_1)$ *with* $n\geqq 2m+1\geqq 1$.

(2) $(GL(1)^2\times Spin(8)$, *half-spin rep.+vector rep.*).

(3) $(GL(1)^2\times Spin(10)$, *half-spin rep.+vector rep.*).

(4) $(GL(1)^2\times Spin(10)$, *even half-spin rep.+even half-spin rep.*).

(5) $(GL(1)^i\times SL(2)\times H_1\times\cdots\times H_i,\ \Lambda_1\otimes(\tau_1\otimes 1\otimes\cdots\otimes 1+1\otimes\tau_2\otimes 1\otimes\cdots\otimes 1+\cdots$ $+1\otimes\cdots\otimes 1\otimes\tau_i))$, *where* (H_j, τ_j) $(1\leqq j\leqq i)$ *is one of* $(Sp(n), \Lambda_1)$ $(n\geqq 2)$ *and* $(Spin(10)$, *half-spin rep.*).

REMARK 2.13. In [2], it is proved that, for $(GL(7), \Lambda_3)$, $Y(k)$ is $G(k)$-transitive for any local field k other than R. And, $(GL(7), \Lambda_3)$ is the only one irreducible P. V. which depends on the transitivity of $G(k)$ on $Y(k)$ for just one

k satisfying $H^1(k, \mathrm{Aut}(SL_2))\neq 0$. In our case I, there is not any P. V. satisfying such condition. Because, in our classification [4], no P. V. contains an irreducible component which is castling equivalent to $(GL(7), \Lambda_3)$.

§3. Reductive P.V.'s with universally transitive open orbits: the case II.

In this section, we shall consider the case when each irreducible component is a regular P. V.

PROPOSITION 3.1. *We have* $l=1$ *for a P. V.* $(GL(1)^k\times G'\times SL(2), \rho_1+\cdots+\rho_k)$ *if and only if* $l=1$ *for a P. V.* $(GL(1)^{k+1}\times(G'\times SL(2))\times H, (\rho_1+\cdots+\rho_k)\otimes 1+(1\otimes\Lambda_1)\otimes\tau)$, *where* (H, τ) *is one of* $(SL(2), \Lambda_1)$, $(Sp(n), \Lambda_1)$ $(n\geqq 2)$ *and* $(Spin(10)$, *half-spin rep.).*

PROOF. By Theorem 1.5, irreducible P. V.'s $(SL(2)\times GL(2), \Lambda_1\otimes\Lambda_1)$, $(Sp(n)\times GL(2), \Lambda_1\otimes\Lambda_1)$ and $(\mathrm{Spin}(10)\times GL(2)$, half-spin rep.$\otimes\Lambda_1)$ have universally transitive open orbits, and each $GL(2)$-part of its generic isotropy subgroup contains $SL(2)$ (see [13]). Hence, by Proposition 1.1 and Corollary 1.2, we obtain our assertion. Q. E. D.

PROPOSITION 3.2. *We have* $l=1$ *for a P. V.* $(GL(1)^k\times H, \rho_1+\cdots+\rho_k)$ *if and only if* $l=1$ *for a P. V.* $(GL(1)^{k+1}\times H\times SL(n), (\rho_1+\cdots+\rho_k)\otimes 1+\tau\otimes\Lambda_1)$, *where* τ *is an n-dimensional irreducible representation of a connected semi-simple algebraic group H satisfying the following condition: there is a simple normal algebraic subgroup K of H such that* $\tau|_K\neq 1$ *and* $\rho_i|_K\neq 1$ *for some* i $(1\leqq i\leqq k)$.

PROOF. Since an irreducible P. V. $(H\times GL(n), \tau\otimes\Lambda_1)$ $(\deg\tau=n)$ has a universally transitive open orbit and a generic isotropy subgroup of it is ismorphic to H, we obtain our assertion by Proposition 1.1 and Corollary 1.2. Q. E. D.

PROPOSITION 3.3. *We have* $l=1$ *for a P. V.* $(GL(1)^{s+1}\times H\times SL(n), (\rho_1+\cdots+\rho_s)\otimes 1+\tau\otimes\Lambda_1)$ *if and only if* $l=1$ *for a P. V.* $(GL(1)^{k+1}\times H\times SL(n), \sigma\otimes 1+\tau\otimes\Lambda_1)$, *where* $(GL(1)^k\times H, \sigma)$ *is a direct sum of* $(GL(1)^s\times H', \rho_1+\cdots+\rho_s)$ *and a P. V.* $(GL(1)^t\times H'', \rho_{s+1}+\cdots+\rho_k)$ $(t=k-s\geqq 1)$ *with* $l=1$ *and* τ *is an n-dimensional irreducible representation of H satisfying the following condition: there is a simple normal algebraic subgroup* K'' *of* H'' *such that* $\tau|_{K''}\neq 1$ *and* $e_i|_{K''}\neq 1$ *for some* i $(s+1\leqq i\leqq k)$.

PROOF. By Proposition 3.2, we obtain our assertion. Q. E. D.

THEOREM 3.4. *Let (G, ρ, V) be an indecomposable reductive P. V. with a universally transitive open orbit. If all irreducible components of (G, ρ, V) are regular P. V.'s, then (G, ρ, Λ) is obtained from the following P. V.'s* (1)$\sim$(9) *by a finite number of transformations in Propositions* 3.1, 3.2, 3.3 *and castling transformations* (*see Theorem* 2.5 *in* [5]).

(1) $(H\times GL(n), \rho\otimes\Lambda_1)$ *where ρ is an n-dimensional irreducible representation of H.*

(2) $(GL(2m), \Lambda_2)$ *with* $m\geqq 2$.

(3) $(Sp(n)\times GL(2m), \Lambda_1\otimes\Lambda_1)$ *with* $n\geqq 2m$.

(4) $(GL(1)\times SO(2n), \Lambda_1\otimes\Lambda_1)$ *with* $n\geqq 2$.

(5) $(GL(1)\times Spin(7), \Lambda_1\otimes$*spin rep.*).

(6) $(GL(1)\times Spin(9), \Lambda_1\otimes$*spin rep.*).

(7) $(Spin(10)\times GL(2)$, *half-spin rep.*$\otimes\Lambda_1)$.

(8) $(GL(1)\times E_6, \Lambda_1\otimes\Lambda_1)$.

(9) $(GL(1)^2\times Spin(8)$, *half-spin rep.+vector rep.*).

PROOF. By Theorem 2.5 in [5] and Theorem 1.5, Propositions 2.2, 3.1, 3.2, 3.3, we obtain our assertion. Q. E. D.

REMARK 3.5. In our case II, there are P. V.'s which depend on the transitivity of $G(k)$ on $Y(k)$ for just one k satisfying $H^1(k, \mathrm{Aut}(SL_2))\neq 0$. We can obtain a such P. V. from $(GL(7), \Lambda_3)$ and P. V.'s (1)$\sim$(9) in Theorem 3.4 by same procedures as in Theorem 3.4. Because $(GL(7), \Lambda_3)$ is the only one regular irreducible P. V. which depends on the transitivity of $G(k)$ on $Y(k)$ for just one k satisfying $H^1(k, \mathrm{Aut}(SL_2))\neq 0$.

References

[1] Igusa, J., On functional equations of complex powers, Invent. Math. **85** (1986), 1-29.

[2] Igusa, J., On a certain class of prehomogeneous vector spaces, J. Pure Appl. Algebra **47** (1987), 265-282.

[3] Kasai, S., A classification of reductive prehomogeneous vector spaces with two irreducible components, I, Jap. J. Math. **14** (1988), 385-418.

[4] Kasai, S., A classification of a certain class of reductive prehomogeneous vector spaces, to appear in Comm. Algebra.

[5] Kasai, S., A classification of a certain class of reductive prehomogeneous vector spaces, II, to appear in J. Algebra.

[6] Kimura, T., Kasai, S. and Yasukura, O., A classification of the representations of reductive algebraic groups which admit only a finite number of orbits, Amer. J. Math. **108** (1986), 643-691.

[7] Kimura, T. and Kasai, S., The orbital decomposition of some prehomogeneous

vector spaces, Adv. Stud. Pure Math. **6** (1985), 437-480.
[8] Kimura, T., Kasai, S., Inuzua, M. and Yasukura, O., A classification of 2-simple prehomogeneous vector spaces of type I, J. Algebra **114** (1988), 369-400.
[9] Kimura, T., Kasai, S., Taguchi, M. and Inuzuka, M., Some P. V.-equivalences and a classification of 2-simple prehomogeneous vector spaces of type II, Trans. Amer. Math. Soc. **308** (1988), 433-494.
[10] Kimura, T., Kasai, S. and Hosokawa, H., Universal transitivity of simple and 2 simple prehomogeneous vector spaces, Ann. Inst. Fourier **38** (1988), 11-41.
[11] Kimura, T., A classification of prehomogeneous vector spaces of simple algebraic groups with scalar multiplications, J. Algebra **83** (1983), 72-100.
[12] Kimura, T., A classification theory of prehomogeneous vector spaces, Adv. Stud. Pure Math. **14** (1988), 223-256.
[13] Sato, M. and Kimura, T., A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. **65** (1977), 1-155.

Institute of Mathematics
University of Tsukuba
Ibaraki, 305, Japan