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REMARKS ON TRANSMISSION, ANTITRANSMISSION AND ANTILOCAL PROPERTIES FOR SUMS OF STABLE GENERATORS

By

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§ 1. Introduction.

The antilocal property (to all directions) of an operator A states that, if $f=Af=0$ in a domain U, then f is identically zero on the whole space. (For the precise definition see § 2). This property is known to hold for the class of fractional powers of the Laplacian: $A=\Delta^{\lambda}$, where λ is a non-integral real number, by Goodman-Segal [**6**] when the space dimension is odd and by Murata [**14**] in general. Liess [**13**] showed this property for the fractional powers of the elliptic differential operator with analytic coefficients using the theory of pseudodifferential operators with analytic symbol.

As is well known, the fractional power $\Delta^{\alpha/2}$ is the generator of the isotropic stable process in case $0<\alpha<2$. In this case the antilocal property has an interesting application for the uniqueness problem of measures of Riesz potentials. See Kanda [**10**].

The class of generators for α-stable processes (called the stable generators) includes a class of operators which are different from $\Delta^{\alpha/2}$, for example, that of one-sided α-stable generators. For details see § 4. The author [**9**] showed that if A is a one-dimensional one-sided α-stable generator (to the right), A has a biased antilocal property in the sense: if $f=Af=0$ on $(0, \varepsilon)$, $f\in C_0^{\infty}(\boldsymbol{R})$, $\varepsilon>0$, then $f\equiv 0$ on $(0, +\infty)$ but not necessarily $f\equiv 0$ in $(-\infty, 0)$. The result has been shown for $\alpha\in(0, 1)$ and will be extended to $\alpha\in(0, 1)\cup(1, 2)$ in § 4, b), 2). [It is still open if the asymmetric Cauchy generators (the generalized case for $\alpha=1$) have the similar property.] The proof is carried out on the similar line as in Liess [**13**], and so Sato-Kashiwara-Kawai theory is essential in the proof.

The first aim of this note is to show what type of antilocal property holds for a finite sum of one-dimensional stable generators, for example, the sum of a one-sided α-stable generator to the right and a one-sided β-stable generator to the left. The solution will be given in § 4, b), 3). In fact the example cited

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just before has the antilocal property to all directions.

Secondly, the antilocal property has a localized version, which the author calls the antitransmission property. This property turns out to be equivalent to "not" transmission property in our simple case. These terminologies would make our proof clearer than the earlier ones. Indeed we first prove the anti-transmission properties and then the antilocal property follows.

We restrict our study to the one-dimensional simplest class of operators in the class of analytic pseudodifferential operators. Some results are valid for more general class of operators in one-dimensional case. As is mentioned before, operators with the antilocal property to all directions were discovered even in higher dimensional case. The author hope that the method in this paper would apply to the higher dimensional case in the future.

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§ 2. Notation and main theorem.

In this section we introduce a class of operators and some properties connected with it which we shall study in this paper. Throughout this section, Γ denotes an open cone in $\boldsymbol{R}\setminus\{0\}$. Setting $\Gamma_{\varepsilon,\delta}\equiv\{\zeta\in\boldsymbol{C}\,;\mathrm{Re}\,\zeta\in\Gamma,\ |\zeta|>\delta,\ |\mathrm{Im}\,\zeta|<\varepsilon|\mathrm{Re}\,\zeta|\}$ for $\varepsilon>0$, $\delta>0$, we define $S_A^s(\Gamma)$ as the set of all functions $a(\xi)\in C^\infty(\Gamma)$ such that there are $\varepsilon>0$, $\delta>0$ and $c>0$ for which $a(\xi)$ extends to an analytic function on $\Gamma_{\varepsilon,\delta}$ which satisfies $|a(\zeta)|\leqq c(1+|\zeta|)^s$ on $\Gamma_{\varepsilon,\delta}$. We also denote the class of symbols $a\in S_{1,0}^s(\boldsymbol{R})$ such that the restriction of $a(\xi)$ to Γ belongs to $S_A^s(\Gamma)$ by $S_A^s(\boldsymbol{R},\Gamma)$, where $S_{1,0}^s(\boldsymbol{R})$ is the class of classical symbols $a(x,\xi)=a(\xi)$ of pseudodifferential operators with constant coefficients of order s of type $(1,0)$ (see [8] Chap. VII, XVIII). By the definition of $S_{1,0}^s(\boldsymbol{R})$, a symbol in $S_A^s(\boldsymbol{R},\Gamma)$ is required to be of class $C^\infty(\boldsymbol{R})$, so we need the following

REMARK 2.1. For $a\in S_A^s(\Gamma)$, there exists $a'\in S_A^s(\boldsymbol{R},\Gamma)$ such that $a'-a$ has compact support if a is defined to be zero outside Γ.

Indeed, choosing a nonnegative $\phi\in C^\infty(\boldsymbol{R})$ such that $\phi=1$ in $\Gamma\cap\{|\xi|\geqq 1\}$ and $\phi=0$ in $(\boldsymbol{R}\setminus\Gamma)\cup\{|\xi|<1/2\}$, the function $a'(\xi)\equiv\phi(\xi)a(\xi)$ is in $S_A^s(\boldsymbol{R},\Gamma)$ and $a'(\xi)-a(\xi)$ has compact support.

By Remark 2.1 we define an operator A corresponding to $a(\xi)\in S_A^s(\Gamma)$ by

$$Af(x)=\int e^{ix\xi}a'(\xi)\hat{f}(\xi)d\!\!{}^-\xi+\int e^{ix\xi}(a(\xi)-a'(\xi))\hat{f}(\xi)d\!\!{}^-\xi\,, \tag{2.1}$$

where $\hat{f}$ denotes the Fourier transform of f and $d\!\!{}^-\xi=(1/2\pi)d\xi$. Note that the second term is real analytic since $a(\xi)-a'(\xi)$ has compact support. We call the operator A defined by (2.1) the operator corresponding to the symbol $a\in S_A^s(\Gamma)$. For a finite sum $a(\xi)=\sum_{j=0}^{N} a_j(\xi)$ of $a_j(\xi)\in S_A^{s_j}(\Gamma)$, we define an operator in an obvious way and call it the operator corresponding to a.

The operators which we study are those corresponding to the symbol $a(\xi)=\sum_{j=0}^{N} a_j(\xi)$ satisfying

(A1) $a=\sum_{j=0}^{N} a_j, \quad a_j\in S_A^{s-\mu_j}(\boldsymbol{R}_+)\cap S_A^{s-\mu_j}(\boldsymbol{R}_-)$, where $0=\mu_0<\mu_1<\cdots<\mu_N$;

(A2) $a_j(t\xi)=t^{s-\mu_j}a_j(\xi)$ for $|\xi|>0$, $t>0$.

In what follows we fix an open interval D in $\boldsymbol{R}$.

A real function g on an open neighborhood of x is said to have an analytic extension to the right (resp. to the left) at x if there is a real analytic function h on $\{x'; |x'-x|<\varepsilon\}$ for some $\varepsilon>0$ such that $g-h=0$ in $(x-\varepsilon, x)$ (resp. $(x, x+\varepsilon)$).

Now we shall introduce the key notation.

DEFINITION 2.2. Let A be a linear operator: $C_0^\infty(D)\to C^\infty(D)$.

(i) A has the *transmission property* to the right (resp. to the left) at $x\in D$ if, for every $f\in C_0^\infty(D)$ such that $f=0$ on $(-\infty, x)\cap D$ (resp. $(x, +\infty)\cap D$), Af has an analytic extension to the right (resp. to the left) at x.

In this case we simply say that A has $[T]_x-R$ (resp. $[T]_x-L$).

(ii) A has the *antitransmission property* to the right (resp. to the left) at x if, for every $f\in C_0^\infty(D)$ such that $f=0$ on $(-\infty, x)\cap D$ (resp. $(x, +\infty)\cap D$) but $x\in A$-sing supp f, Af does not have analytic extension to the right (resp. to the left) at x. Here A-sing supp f denotes the analytic singular support of f.

In this case we simply say that A has $[AT]_x-R$ (resp. $[AT]_x-L$).

(iii) A has the *antilocality* to the right (resp. to the left) if the following holds: if $f=Af=0$ in an open non-empty set $U\subset D$ and $f\in C_0^\infty(D)$ then $f=0$ in $(U+\boldsymbol{R}_+)\cap D$ (resp. $(U+\boldsymbol{R}_-)\cap D$).

In this case we simply say that A has $[AL]-R$ (resp. $[AL]-L$). A has the antilocality if A has both $[AL]-R$ and $[AL]-L$, and in this case we simply say that A has $[AL]$.

It is immediate that if A has $[T]_x-R$ then A does *not* have $[AT]_x-R$, $x\in D$. It also follows immediately from definition that the sum of two operators, one has $[T]_x-R$ and the other has $[AT]_x-R$, has $[AT]_x-R$, $x\in D$.

Then we have our main theorem:

THEOREM 2.3. *Let A be the operator corresponding to a symbol* $a(\xi)=\sum_{j=0}^{N} a_j(\xi)$ *satisfying* (A1) *and* (A2).

(i) *If, for every j,*

$$a_j(-1)-e^{-i\pi(s-\mu_j)}a_j(1)=0 \tag{2.2}$$

(resp. (2.3) $a_j(-1)-e^{i\pi(s-\mu_j)}a_j(1)=0$*), then A has $[T]_x-R$ (resp. $[T]_x-L$) for every $x\in D$.*

(ii) *If, for some j, the condition* (2.2) *(resp.* (2.3)*) does not hold, then A has $[AT]_x-R$ for every $x\in D$ and $[AL]-R$ (resp. $[AT]_x-L$ for every $x\in D$ and $[AL]-L$).*

In § 3 we shall prove Theorem. The properties $[AT]_x$ and $[AL]$ has an intimate relation. Indeed, $[AT]_x-R$ for every $x\in D$ implies $[AL]-R$, which will be shown in Lemma 3.5.

§ 3. Proof of Theorem.

In this section we shall show Theorem. For the proof we prepare some important results and simplified versions of Sato-Kashiwara-Kawai's result. The precise explanation of the notation below needs somewhat long sequence of terminologies. So we omit it here. For reference consult Kashiwara-Kawai-Kimura [**12**, Chap. 4], Hörmander [8, Chap. XVIII] or Liess [**13**]. First we note that

LEMMA 3.1. ([**13**, theorem 2.6]) *The operator A with symbol $a(\xi)$ in $S^s_A(\boldsymbol{R}, \Gamma)$ has Γ-analytic pseudolacal property (Γ-$[APL]$ for short), that is, $SS\ Af\cap(D\times\Gamma)\subset SS\ f$. In particular, if $a(\xi)$ is in $S^s_A(\boldsymbol{R}, \boldsymbol{R}_-)\cap S^s_A(\boldsymbol{R}, \boldsymbol{R}_+)$ then A has the analytic pseudolocal property (simply written as "A has $[APL]$"), that is, $SS\ Af\subset SS\ f$, where $SS\ f$ denotes the singular spectrum of f.*

(Note that $SS\ Af\subset SS\ f$ implies A-sing supp $Af\subset A$-sing supp f.)

LEMMA 3.2. ([**11**, theorem 4.4.1] or [8, Chap. VIII]) *Let u be a distribution on a neighborhood of $x_0\in\boldsymbol{R}$ such that supp $u\subset\{x';\, x'\geqq x_0\}$ and that u is micro-analytic at either of the conormal points $(x_0, \pm idx\infty)$. Then u vanishes on a neighborhood of x_0.*

LEMMA 3.3. ([**11**, theorem 8.5.7] or [8, theorem 7.4.3]) *Suppose that $a(\xi)\in\mathcal{S}'(\boldsymbol{R})$ satisfies the following: there exists a function $a(\zeta)$ which is holomorphic in $\{\operatorname{Im}\zeta<0\}$ and such that, for each fixed $\varepsilon>0$, $|a(\zeta)|\leqq C(1+|\zeta|)^N$ uniformly* *on* $\{\operatorname{Im}\zeta\leqq-\varepsilon\}$ *for some* $N, C>0$ *and that* $a(\xi-i0)=a(\xi)$ *for* $\xi\neq 0$. *Then* $supp(\int e^{ix\xi}a(\xi)\hat{f}(\xi)d\xi)\subset[x_0, +\infty)$ *for every* $f\in C_0^\infty(\boldsymbol{R})$, *where* $x_0\equiv\inf[supp\, f]$.

Now we shall begin the proof of Theorem. First we shall study the operator connected with $a_j(\xi)$. Define a symbol $d_j(\zeta)$ on $\boldsymbol{C}\setminus\{it\,;\,t\geqq 0\}$ so that

$$d_j(\zeta)=a_j(1)\zeta^{s-\mu_j},$$

where the branch is chosen so that $\zeta^{s-\mu_j}=1$ at $\zeta=1$. Then d_j is holomorphic in $\boldsymbol{C}\setminus\{it\,;\,t\geqq 0\}$ and

$$d_j(\xi)=d_j(\xi-i0)=\begin{cases}a_j(1)e^{-i\pi(s-\mu_j)}|\xi|^{s-\mu_j}, & \xi<0\\ a_j(1)\xi^{s-\mu_j}, & \xi>0.\end{cases}$$

In particular $d_j(\xi)\in S_A^{s-\mu_j}(\boldsymbol{R}_-)\cap S_A^{s-\mu_j}(\boldsymbol{R}_+)$. Further it follows from Lemma 3.3 that, for $f\in \boldsymbol{C}_0^\infty(D)$ such that $f=0$ on $(-\infty, x)\cap D$ (then f can be regarded as a function on $\boldsymbol{R}$ with $f=0$ on $(-\infty, x)$),

$$\text{supp}(\int e^{ix\xi}d(\xi)\hat{f}(\xi)d\xi)\subset[x, +\infty). \tag{3.1}$$

If the condition (2.2) is satisfied, then $a_j(\xi)=d_j(\xi)$ for every j. Therefore $[T]_x-R$ follows easily. The proof of $[T]_x-L$ is similar.

Next we prove the statement (ii) of Theorem. In what follows we assume that f is a function in $C_0^\infty(D)$ such that $f=0$ in $D\cap(-\infty, x)$. First note that Af is real analytic on $D\cap(-\infty, x)$ by Lemma 3.1. Further, since $a_j(\xi)=d_j(\xi)$ for $\xi>0$ and every j, we have

$$(x, idx\infty)\notin SS(\int e^{ix\xi}(a(\xi)-d(\xi))\hat{f}(\xi)d\xi) \tag{3.2}$$

by the Paley-Wiener theorem (see [**11**, definition 1.6.1, corollary 8.5.6] or [**15**, corollary 3.3]), where $d(\xi)=\sum_{j=0}^{N}d_j(\xi)$. Assume that Af has an analytic extension to the right at x (in the present case this means that $Af=h$ in $(x-\varepsilon, x)$ for some $\varepsilon>0$ and some real analytic function h near x). Then it follows from (3.2) that

$$(x, idx\infty)\notin SS(\int e^{ix\xi}(a(\xi)-d(\xi))\hat{f}(\xi)d\xi-h)$$

and from (3.1) that $\text{supp}(Af-\int e^{ix\xi}d(\xi)\hat{f}(\xi)d\xi-h)\cap\{x'\,;\,|x'-x|<\varepsilon\}\subset[x, x+\varepsilon)$. Hence, by Lemma 3.2, $(Af-\int e^{ix\xi}d(\xi)\hat{f}(\xi)d\xi-h)$ vanishes near x, and so

$$Af-\int e^{ix\xi}d(\xi)\hat{f}(\xi)d\xi \text{ is real analytic near } x. \tag{3.3}$$

Now we use the assumption posed in (ii). Let j_0 be the minimum of numbers j for which $a_j(-1)-e^{-i\pi(s-\mu_j)}a_j(1)\neq 0$. Then the restriction of a $(\xi)-d(\xi)$ to $\boldsymbol{R}_-$ defines a non-zero symbol in $S_A^{s-\mu_{j0}}(\boldsymbol{R}_-)$, where the modification obtained in Remark 2.1 is denoted by $c(\xi)\in S_A^{s-\mu_{j0}}(\boldsymbol{R}, \boldsymbol{R}_-)$. Then, since $a(\xi)=d(\xi)$ for $\xi\in\boldsymbol{R}_+$, it follows from (3.3) that

$$\int e^{ix\xi}c(\xi)\hat{f}(\xi)d\xi \text{ is real analytic near } x. \tag{3.4}$$

Since $c(\xi)$ is of the form

$$c(\xi)=\psi(\xi)(c_{j_0}|\xi|^{s-\mu_{j0}}+c_{j_0+1}|\xi|^{s-\mu_{j_0+1}}+\cdots c_N|\xi|^{s-\mu_N})$$

with $s-\mu_{j_0}>s-\mu_{j_0+1}>\cdots>s-\mu_N$ and $\psi\equiv 1$ on $\{\xi<-1\}$, we see that $c(\xi)$ does not vanish for $\xi<-M$ for sufficiently large $M>0$. Put $R(\xi)\equiv\psi(\xi/2M)\cdot(c(\xi))^{-1}$. Then $R(\xi)$ is in $S_A^{-s+\mu_{j0}}(\boldsymbol{R}, \boldsymbol{R}_-)$ and satisfies $R(\xi)\cdot c(\xi)=\psi(\xi/2M)$.

By $\boldsymbol{R}_-$-$[APL]$ for the operator with symbol $R(\xi)$ (Lemma 3.1) and by (3.4),

$$(x, -idx\infty)\notin SS(\int e^{ix\xi}\psi(\xi/2M)\hat{f}(\xi)d\xi). \tag{3.5}$$

Since $\psi(-\xi/2M)$ has its support in $\{\xi\geqq 2M\}$, it follows from the Paley-Wiener theorem that

$$(x, -idx\infty)\notin SS(\int e^{ix\xi}\psi(-\xi/2M)\hat{f}(\xi)d\xi). \tag{3.6}$$

Combining (3.5), (3.6) we have

$$(x, -idx\infty)\notin SS(\int e^{ix\xi}(\psi(\xi/2M)+\psi(-\xi/2M))\hat{f}(\xi)d\xi)=SS\,f. \tag{3.7}$$

The last equality holds since $1-(\psi(\xi/2M)+\psi(-\xi/2M))$ has compact support.

Using Lemma 3.2 with the assumption that $\operatorname{supp} f\subset[x, +\infty)\cap D$, we have $x\notin\operatorname{supp} f$, in particular $x\notin A$-sing $\operatorname{supp} f$. This proves $[AT]_x-R$ for A. The proof for $[AT]_x-L$ is similar.

REMARK 3.4. In view of the conditions (2.2), (2.3) and Definition 2.2, we can easily see that these conditions are also necessary.

For the proof of second part of (ii), we note that the operator A has $[APL]$, since each symbol $a_j(\xi)$ satisfies the assumption (A1) and so their modifications are in $S_A^{s-\mu}(\boldsymbol{R}, \boldsymbol{R}_-)\cap S_A^{s-\mu}(\boldsymbol{R}, \boldsymbol{R}_+)$. Hence the proof follows directly from the following

LEMMA 3.5. *If a linear operator* $A: C_0^\infty(D)\rightarrow C^\infty(D)$ *has the properties* $[APL]$ *and* $[AT]_x-R$ *(resp.* $[AT]_x-L$*) for all* $x\in D$, *then* A *has* $[AL]-R$ *(resp.*

$[AL]-L)$ *on* D.

PROOF. We only prove the case $[AL]-R$.

Let U be an open set in D. Let $\varLambda\equiv\{x\in D; f=Af=0 \text{ in } (U+\boldsymbol{R}_+)\cap D\cap(-\infty, x)\}$. Clearly $\varLambda\supset U$ so $\varLambda\neq\emptyset$. We show that $\varLambda$ is open and closed in $(U+\boldsymbol{R}_+)\cap D$.

(i) $\varLambda$ *is closed.*

Since $x\in\varLambda$ and $y<x$ implies $y\in\varLambda$ for $x, y\in D\cap(U+\boldsymbol{R}_+)$, we may show $\sup\varLambda\in\varLambda$. Let $x_0\equiv\sup\varLambda$. However since f and Af are C^∞-functions in D, it follows that $f(x_0)=Af(x_0)=0$. Hence $x_0\in\varLambda$.

(ii) $\varLambda$ *is open.*

Take any $y\in\varLambda$. Then $\operatorname{supp} Af\subset D\cap[y, +\infty)$. Since Af has analytic extension $h\equiv 0$ to the right at y, it follows that $y\notin A$-sing supp$f$ by $[AT]-R$. Since $\operatorname{supp} f\subset D\cap[y, +\infty)$, it implies that $y\notin\operatorname{supp} f$. By $[APL]$, $(y, \pm idx\infty)\notin SS\ Af$. Since $\operatorname{supp} Af\subset D\cap[y, +\infty)$, then $y\notin\operatorname{supp} Af$ by Lemma 3.2. Hence for some neighborhood σ of y, $f=Af=0$ in σ. Hence $y+\eta\in\varLambda$ for some $\eta>0$.

q. e. d.

§ 4. Examples.

a) $[T]$.

Let $D\subset\boldsymbol{R}$ be a domain. Let A be a differential operator with constant coefficients on D:

$$A=\sum_{k=0}^{l} c_{l-k}D_x^k,\quad \left(D_x=\frac{1}{i}\frac{d}{dx}\right).$$

Then A has $[T]_x-R, L$ for $x\in D$. It is clear that A has $[T]_x-R, L$ by definition. However we may confirm it in terms of the symbol.

Indeed, its symbol is $a(\xi)=\sum_{j=0}^{l} c_j\xi^{l-j}$ and this clearly satisfies (A1), (A2) with $s=1$, $\mu_j=j$. (2.2), (2.3) are checked since $e^{\pm i\pi(l-j)}a_j(1)=(-1)^{l-j}c_j=a_j(-1)$.

Hence A has $[T]_x-R, L$ $(x\in D)$ by Theorem.

b) $[AT]$ and $[AL]$.

1) Let A be the Hilbert transform on $D=\boldsymbol{R}$,

$$Af(x)=\int_{-\infty}^{+\infty}\frac{f(x-y)}{y}dy,\quad f\in C_0^\infty(\boldsymbol{R}).$$

Then A has $[AT]_x-R, L$ $(x\in D)$ and $[AL]$.

Indeed A may be rewritten $Af(x)=\mathcal{F}^{-1}[-\pi\operatorname{isgn}(\xi)\hat{f}(\xi)](x)$. Hence the symbol of A is $a(\xi)=-\pi\operatorname{isgn}(\xi)$. Then $a(\xi)$ is certainly homogeneous of order 0 and satisfies (A1), (A2).

We see $a(-1)=\pi i\neq-\pi i=e^0 a(1)$. Hence A has $[AT]_x-R,\ L$ $(x\in D)$ and hence $[AL]$ by Theorem.

2) Let A be the sum of the Laplacian and the stable generator with "drift" (a first order differential operator) on $D=\boldsymbol{R}$ (i.e., Lévy generator).

$$A=A_0+A_1+A_2,$$

$$A_0 f(x)=-a\frac{d^2 f}{dx^2}(x),\quad A_1 f(x)=b\frac{df}{dx}(x)\ \text{ and}$$

$A_2 f(x)=A_\alpha f(x)$, where A_α, $\alpha\in(0,2)$, is given by

$$A_\alpha f(x)=\begin{cases}\int_{\boldsymbol{R}^1}[f(x+y)-f(x)]N(y)dy, & (0<\alpha<1)\\ \int_{\boldsymbol{R}^1}[f(x+y)-f(x)-y\frac{df}{dx}(x)]N(y)dy, & (1<\alpha<2)\\ \int_{\boldsymbol{R}^1}[f(x+y)-f(x)-\frac{df}{dx}(x)\cdot\sin y]N(y)dy, & \alpha=1\end{cases}$$

where "the Lévy measure" $N(y)dy$ is of the form

$$N(y)dy=(p1_{\boldsymbol{R}_-}(y)+q1_{\boldsymbol{R}_+}(y))\frac{dy}{|y|^{1+\alpha}}.$$

Here $p,q\in[0,1]$, $p+q=1$, $1_{\boldsymbol{R}_\pm}(y)=1$ or 0 according as $y\in\boldsymbol{R}_\pm$ or not. In case $\alpha=1$ we restrict ourselves to the symmetric case, that is, $p=q=1/2$.

Then A has $[AL]-R$ (resp. $[AL]-L$) if and only if the support of the Lévy measure contains $\boldsymbol{R}_+$ (resp. $\boldsymbol{R}_-$).

Indeed, the symbol is of the form

$$a(\xi)=a_0(\xi)+a_1(\xi)+a_2(\xi)$$

$$a_0(\xi)=a\xi^2,\quad a_1(\xi)=ib\xi\ \text{ and}$$

$$a_2(\xi)=\begin{cases}-C_\alpha[pe^{i\,\mathrm{sgn}(\xi)\alpha\pi/2}+qe^{-i\,\mathrm{sgn}(\xi)\alpha\pi/2}]|\xi|^\alpha, & \alpha\in(0,1)\cup(1,2)\\ -C\pi/2|\xi|, & \alpha=1.\ \text{(See Feller [5] Chap. XVIII)}\end{cases}$$

Then (A1), (A2) are satisfied with $\mu_1=1\wedge(2-\alpha)$, $\mu_2=1\vee(2-\alpha)$.

For A_0 and A_1, they have $[T]_x-R,\ L$ $(x\in D)$ as above.

However for A_2, we have for $\alpha\in(0,1)\cup(1,2)$,

$$a_2(-1)=-C_\alpha[pe^{-i\alpha\pi/2}+qe^{i\alpha\pi/2}],$$

$$e^{-i\alpha\pi}a_2(1)=-C_\alpha[pe^{-i\alpha\pi/2}+qe^{-i3\alpha\pi/2}],\ \text{ and}$$

$$e^{i\alpha\pi}a_2(1)=-C_\alpha[pe^{i3\alpha\pi/2}+qe^{i\alpha\pi/2}].$$

And for $\alpha=1$,

$$a_2(-1)=-C\pi/2\neq e^{\pm i\pi}(-C\pi/2)=e^{\pm i\pi}a_2(1).$$

Hence, in case $\alpha\in(0, 1)\cup(1, 2)$ if $q>0$ then not (2.2) is checked and if $p>0$ then not (2.3) is checked. Therefore, if $q>0$ then A has $[AT]_x-R$ $(x\in D)$ and hence $[AL]-R$. If $p>0$ then A has $[AT]_x-L$ $(x\in D)$ and hence $[AL]-L$. In particular if $p:q>0$ then A has $[AL]$.

For the case $\alpha=1$, conditions in (ii) are clearly checked, and so A has $[AT]_x-R$, L $(x\in D)$ and hence $[AL]$ by Theorem.

3) Let A be the $(N+1)$-sum of stable generators on $D=\boldsymbol{R}$.

$$A=\sum_{j=0}^{N}A_{\alpha_j},\quad \alpha_j\in(0, 2),\quad \alpha=\alpha_0>\alpha_1>\cdots>\alpha_N.$$

Here A_{α_j} is given in 2), each Lévy measure is of the form

$$N_j(y)dy=[p_j1_{\boldsymbol{R}_-}(y)+q_j1_{\boldsymbol{R}_+}(y)]\frac{dy}{|y|^{1+\alpha_j}}.$$

(In case that $\alpha_j=1$, we always assume $p_j=q_j=1/2$.)

Then A has $[AL]-R$ (resp. $[AL]-L$) if and only if the union of supports of the Lévy measures contains $\boldsymbol{R}_+$ (resp. $\boldsymbol{R}_-$). In particular, if $N=1$, $\alpha_0=\alpha$, $\alpha_1=\beta$ with α, $\beta\in(0, 2)$ and $q_0=1$, $p_1=1$, then A has $[AL]$.

Indeed, the symbol of A is $a(\xi)=\sum_{j=0}^{N}a_j(\xi)$,

$$a_j(\xi)=-C_\alpha[p_je^{i\,\mathrm{sgn}(\xi)\alpha_j\pi/2}+q_je^{-i\,\mathrm{sgn}(\xi)\alpha_j\pi/2}]|\xi|^{\alpha_j}.$$

Then (A1), (A2) are satisfied with $\mu_j=\alpha-\alpha_j$.

As in 2), we have for each $j=0, \cdots, N$, if $q_j>0$ then not (2.2) is checked and if $p_j>0$ then not (2.3) is checked.

Hence we have the following results:

	$q_0=\cdots=q_N=0$	$p_0=\cdots=p_N=0$	otherwise
	$[AL]-L$	$[AL]-R$	$[AL]$

REMARK 4.1. In case $q_0=\cdots=q_N=0$ (resp. $q_0=\cdots=q_N=1$), the fact that A does not have $[AL]-R$ (resp. $[AL]-L$) is known by the counter example in [**9**]. And so in this case $[AL]-R$ (resp. $[AL]-L$) implies $[AT]_x-R$ (resp. $[AT]_x-L$) for every $x\in D$ by Theorem.

Appendix.

Though we have treated the case of finite sum $a(\xi)=\sum_{j=0}^{N}a_j(\xi)$ alone, one may treat the case for formal analytic symbol with constant coefficients

$a(\xi)=\sum_{j\geq 0} a_j(\xi)$ with the assumptions

(B1) $a\in SF_A^s(\boldsymbol{R}_+)\cap SF_A^s(\boldsymbol{R}_-)$

(B2) a is polyhomogeneous in the sense that

$$a_j(t\xi)=t^{s-j}a_j(\xi) \quad \text{for}\,|\xi|>1,\ t>1.$$

(For the definition of the symbol class $SF_A^s(\boldsymbol{R}_\pm)$, see [**9**].)
One puts the condition that $a_j(\xi)$ is homogeneous of order $s-j$ to ensure convergence of the formal symbol (see [**1**], [**3**] and [**13**]).

For those symbols, discussions in § 3 remain valid with replacing μ_j with j and some modifications necessary.

Then we have

PROPOSITION. (i) *If, for every* j,

$(AP.1)$ $a_j(-1)-e^{-i\pi(s-j)}a_j(1)\neq 0$ *(resp.* $(AP.2)$ $a_j(-1)-e^{i\pi(s-j)}a_j(1)\neq 0)$,

then A *has the property* $[T]_x-R$ *(resp.* $[T]_x-L$*) for every* $x\in D$.

(ii) *If, for some* j, *the condition* $(AP.1)$ *(resp.* $(AP.2)$*) does not hold, then* A *has the property* $[AT]_x-R$ *for every* $x\in D$ *and* $[AL]-R$ *(resp.* $[AT]_x-L$ *for every* $x\in D$ *and* $[AL]-L$*).*

As in § 3, conditions are also necessary. Cf. [8] Chap. XVIII, section 2.

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