TSUKUBA J. MATH.
Vol. 12 No. 2 (1988). 441—449

TILTING MODULES, DOMINANT DIMENSION AND EXACTNESS OF DUALITY FUNCTORS

By

R.R. COLBY

Dedicated to Professor Hiroyuki Tachikawa for his sixtieth birthday

Let R and S be rings and let ${}_RW_S$ be a bimodule. We shall denote both the functors $\mathrm{Hom}_R(\_, W): R\text{-Mod}\to\text{Mod-}S$ and $\mathrm{Hom}_S(\_, W): \text{Mod-}S\to R\text{-Mod}$ by Δ_W and the composition of the two, in either order, by Δ_W^2. Recall that (for fixed W) there is a natural transformation $\delta: 1_{R\text{-Mod}}\to\Delta_W^2$, defined via the usual evaluation maps $\delta_M: M\to\Delta_W^2(M)$. An R-module M is called *W-reflexive* (*W-torsionless*) in case δ_M is an isomorphism (a monomorphism). Then, an R-module M is W-torsionless if and only if it is isomorphic to a submodule of a direct product of copies of ${}_RW$. Also recall that ${}_RW_S$ is *balanced* in case $R\cong\mathrm{End}_S(W)$ and $S\cong\mathrm{End}_R(W)^{op}$ canonically, and that ${}_RW_S$ defines a *Morita Duality* if it is balanced and both ${}_RW$ and W_S are injective cogenerators (see [1], [3] or [10], for an account of Morita Duality).

We begin by studying exactness properties of the functor Δ_W^2. The case $W=R$ has been extensively studied in ([4], [5], [6] and [7]) and Theorem 1, Lemma 2, Proposition 3 and Proposition 4 are generalizations of results obtained there. A finite dimensional algebra R of positive dominant dimension possesses (what we consider to be) a canonical pair of tilting left and right modules ${}_RU$ and V_R. Associated with these are the endomorphism rings $S=\mathrm{End}_R(U)^{op}$ and $T=\mathrm{End}_R(V)$, and the bimodule ${}_TW_S={}_T(V\otimes_R U)_S$. We relate exactness properties of the functors Δ_U, Δ_V, Δ_W and their squares to dominant dimension. For these canonically chosen tilting modules we show that

1) if dom. dim. $R\geqq 2$ then Δ_U^2 preserves monomorphisms both in Mod-S and in R-Mod;

2) if dom. dim. $R\geqq 3$ then Δ_U^2 is left exact on Mod-S and the functors Δ_W^2 preserve monomorphisms in Mod-S and in T-Mod. In this case, if $\Delta_W: T\text{-Mod}\leftrightarrow\text{Mod-}S: \Delta_W$ defines a Morita Duality, then R is QF (and conversely);

3) if dom. dim. $R\geqq 4$ then the functors Δ_W^2 are left exact on both Mod-S and on T-Mod.

We shall denote the injective envelope of a module M by $E(M)$ and, if M is an R-module, we denote the annihilator in M of a subset I of R by $\mathrm{Ann}_M(I)$.

Received September 1, 1987.

THEOREM 1. *Let* $_RW_S$ *be a bimodule with* $R=\mathrm{End}_S(W)$. *Let* I *denote the ideal of all endomorphisms in* R *which factor through injective S-modules. The following are equivalent:*

1) W_S *cogenerates* $E(W_S)$.
2) *If* $M\in\mathrm{Mod}\text{-}S$ *is W-reflexive then* $E(M_S)$ *is W-torsionless.*
3) $\Delta_W^2:\mathrm{Mod}\text{-}S\to\mathrm{Mod}\text{-}S$ *preserves monomorphisms.*
4) $\mathrm{Ann}_W(I)=0$.

PROOF. We modify the proof of [7, Theorem 1]. Let $E_S=E(W_S)$ and denote the injection of W_S into E_S by i. We first prove that 1) implies 4). Suppose that W_S cogenerates E_S. There is a sequence

$$W \xrightarrow{\ i\ } E \xrightarrow{\ j\ } W^X$$

in Mod-S where j is a monomorphism. For $x\in X$ let $p_x: W^X\to W$ be the canonical projection and let $b_x=p_x\circ j\circ i\in I$. Then if $K=\sum\{Rb_x: x\in X\}$, $K\subseteqq I$ and note that $\mathrm{Ann}_W(K)=0$. Hence we also have $\mathrm{Ann}_W(I)=0$. Next, assume condition 4). Suppose $\alpha: M\to N$ is a monomorphism in Mod-S and consider the induced exact sequence

$$\Delta_W(N) \xrightarrow{\ \Delta_W(\alpha)\ } \Delta_W(M) \xrightarrow{\ \beta\ } \mathrm{Coker}\,\Delta_W(\alpha) \longrightarrow 0$$

in R-Mod. If $f\in\Delta_W(M)$ and $r\in I$ then $r\circ f$ factors through an injective so there exists $\bar{f}\in\mathrm{Hom}_S(N, W)$ such that $\bar{f}\circ\alpha=r\circ f$. That is $I\Delta_W(M)\subseteqq\mathrm{Im}\,\Delta_W(\alpha)=\mathrm{Ker}\,\beta$. Thus we have

$$I\,\mathrm{Coker}\,(\Delta_W(\alpha))=I\beta(\Delta_W(M))=\beta(I\Delta_W(M))=0.$$

Now let $\phi\in\Delta_W(\mathrm{Coker}\,\Delta_W(\alpha))$. Since

$$I\phi(\mathrm{Coker}\,\Delta_W(\alpha))=\phi(I\,\mathrm{Coker}\,(\Delta_W(\alpha))=0$$

and $\mathrm{Ann}_W(I)=0$ by 4) we obtain that $\phi=0$. Thus $\Delta_W(\mathrm{Coker}\,\Delta_W(\alpha))=0$ so $\Delta_W^2(\alpha)$ is a monomorphism. This completes the proof that 4) implies 3). That 3) implies 2) follows from the observation that a non-zero kernel of $\delta_{E(M_S)}$ would have to intersect M non-trivially, and it is clear that 2) implies 1).

Straightforward modification of the proof of [4, Theorem 2] provides a proof of the following lemma.

LEMMA 2. *Let* $_RW_S$ *be a balanced bimodule and assume that the functor* Δ_W^2 *preserves monomorphisms in* Mod-*S*. *Let*

$$0 \longrightarrow {}_RW \longrightarrow {}_RE_1 \longrightarrow {}_RE_2$$

be an injective copresentation of $_RW$. If $_RW$ cogenerates $_RE_1$ and $_RE_2$, then Δ_W^2: Mod-S→Mod-S is left exact.

In case $W=R$, the equivalence of conditions 1) and 3) of the following result was observed in [6, Remark (d)].

PROPOSITION 3. *Let $_RW_S$ be a balanced bimodule. The following are equivalent.*

1) *W_S is injective.*
2) *If α is a monomorphism in Mod-S then $\Delta_W^3(\alpha)$ is an epimorphism in R-Mod.*
3) *Δ_W^2: Mod-S→Mod-S preserves monomorphisms and Δ_W^2: R-Mod→R-Mod is right exact.*

In particular, if $_RW_S$ is a balanced bimodule, then both W_S and $_RW$ are injective if and only if both the Δ_W^2 functors are exact.

PROOF. If is clear that condition 1) implies condition 3). Assume condition 3) and let $\alpha: M\to N$ be a monomorphism in Mod-S. Since $\Delta_W^2(\alpha)$ is a monomorphism, we have $\Delta_W(\mathrm{Coker}\,(\Delta_W(\alpha))=0$. Hence $\Delta_W^2(\mathrm{Coker}\,(\Delta_W(\alpha))=0$ too so $\Delta_W^3(\alpha)$ is an epimorphism. Now assume condition 2). If $\alpha: M\to S$ is a monomorphism in Mod-S we obtain an exact sequence

$$\Delta_W(S) \xrightarrow{\Delta_W(\alpha)} \Delta_W(M) \longrightarrow \mathrm{Coker}\,\Delta_W(\alpha) \longrightarrow 0$$

in R-Mod. Using 2), the W-reflexivity of $\Delta_W(S)=W$, and the fact that $\Delta_W(M)$ is W-torsionless, the commutativity and exact rows and columns of the diagram

$$\begin{array}{ccccccc}
\Delta_W^3(S) & \longrightarrow & \Delta_W{}^3(M) & \longrightarrow & 0 & & \\
\cong\uparrow & & \uparrow & & & & \\
\Delta_W(S) & \longrightarrow & \Delta_W(M) & \longrightarrow & \mathrm{Coker}\,\Delta_W(\alpha) & \longrightarrow & 0 \\
 & & \uparrow & & & & \\
 & & 0 & & & &
\end{array}$$

show that $\mathrm{Coker}\,\Delta_W(\alpha)=0$ so $\Delta_W(\alpha)$ is an epimorphism. Thus 1) holds.

We remark that if R and S are finite dimensional algebras, and $_RW$ and W_S are finitely generated, then Theorem 1, Lemma 2, and Proposition 3 remain true if we replace Mod-S and R-Mod by mod-S and R-mod, respectively.

Recall that $U\in R$-Mod has *dominant dimension* at least n (dom. dim. $_RU\geqq n$) if there is an exact sequence

$$0 \longrightarrow {}_RU \longrightarrow {}_RE_1 \longrightarrow \cdots \longrightarrow {}_RE_n$$

where each E_i is both projective and injective. If R is a finite dimensional algebra the dominant dimensions of ${}_RR$ and R_R are equal (see [8], [9], [10]) and this number is called the dominant dimension of the algebra R. Such algebras of dominant dimension greater than or equal to one are also known as QF-3 algebras. A ring R is a left QF-3 ring if it has a *minimal faithful* left module, i. e. a module which is isomorphic to a direct summand of every faithful module (see [10], for example). Of course, a minimal faithful module is both projective and injective and is isomorphic to a left ideal Re for some idempotent $e \in R$.

PROPOSITION 4. *Suppose R is a finite dimensional algebra over a field and that dom. dim. $R \geqq 2$. Let ${}_RE$ be a minimal faithful left R-module with $S = \mathrm{End}_R(E)^{op}$. The following are equivalent:*

1) *R is QF.*
2) *E_S is injective.*
3) *Δ_E^2 is right exact on R-Mod.*

PROOF. Recall that ${}_RE_S$ is a balanced bimodule [10, Proposition 7.1]. Condition 2) implies condition 1) since E_S is a generator in Mod-S, hence if E_S is injective, S is QF and E_S is a progenerator, so R is Morita equivalent of S. Clearly condition 1) implies condition 3) since in this case ${}_RE$ is a progenerator so both ${}_RE$ and E_S are injective. Assume condition 3). By Proposition 3(3) and the remark following, to prove 2) it suffices to show that Δ_E^2 preserves monomorphisms in mod-S. If M is a finitely generated (hence finitely presented) module in mod-S, then since S is E-reflexive and Δ_E^2 is right exact on Mod-S (${}_RE$ is injective) it follows that M is E-reflexive. Thus Δ_E^2 is exact (hence preserves monomorphisms) on mod-S.

Recall that $U \in R$-Mod is a *tilting module* in case U has projective dimension at most one ($\mathrm{pd}_RU \leqq 1$), $\mathrm{Ext}_R^1(U, U) = 0$, and there is an exact sequence $0 \to {}_RR \to {}_RU_1 \to {}_RU_2 \to 0$ where $U_1, U_2 \in$ add-U. We refer to [2] and the references given there for basic results concerning tilting modules. We next note that rings of positive dominant dimension have a canonical tilting module.

PROPOSITION 5. *Suppose R is a finite dimensional algebra over a field with dom. dim. $R \geqq n$ where $n \geqq 1$, and let $U = E \oplus E({}_RR)/R$ where ${}_RE$ is a minimal faithful left R-module. Then ${}_RU$ is a tilting module and dom. dim. ${}_RU \geqq n-1$.*

PROOF. Let ${}_RQ = E({}_RR)$. Since ${}_RE$ is both projective and injective $\mathrm{Ext}_R^1(U, U) = 0$ will follow from $\mathrm{Ext}_R^1(Q/R, Q/R) = 0$. Since ${}_RQ$ is injective and $\mathrm{pd}_R(Q/R) \leqq 1$, this is guaranteed by the exactness of the sequence

$$0=\mathrm{Ext}^1(Q/R,\ Q) \longrightarrow \mathrm{Ext}^1(Q/R,\ Q/R) \longrightarrow \mathrm{Ext}^2(Q/R,\ R)=0$$

which is induced by the exact sequence $0\to{}_RR\to{}_RQ\to{}_R(Q/R)\to 0$. This latter exact sequence has both Q and Q/R in add-U and since Q is projective it is clear that $\mathrm{pd}_R U\leqq 1$. Finally, since dom. dim. $R\geqq n$ and Q is projective and injective, it is clear that dom. dim. ${}_R(Q/R)\geqq n-1$ so dom. dim. ${}_RU\geqq n-1$ as well.

LEMMA 6. *If ${}_RU$ is a tilting module then Ker* $\mathrm{Tor}_1^R(\_,U)$ *is closed under taking submodules.*

PROOF. Since $\mathrm{pd}_R U\leqq 1$, there is an exact sequence $0\to P_2\to P_1\to U\to 0$ in R-Mod with P_i projective. Suppose $0\to M\to N$ is exact in Mod-R and $\mathrm{Tor}_1^R(N,U)=0$. These two sequences innduce the commutative diagram

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & N\otimes P_2 & \longrightarrow & N\otimes P_1 \\ & & & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Tor}_1^R(M,U) & \longrightarrow & M\otimes P_2 & \longrightarrow & M\otimes P_1 \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

which, since P_1 and P_2 are projective, has exact rows and columns and from which $\mathrm{Tor}_1^R(M,U)=0$ follows.

LEMMA 7. *Suppose ${}_RU$ and V_R are tilting left and right modules, respectively. Let $S=\mathrm{End}_R(U)^{op}$ and $T=\mathrm{End}_R(V)$. If V_R is a submodule of a flat module, $\mathrm{Ext}_T^1(V,V)=0$, and ${}_TV_R$ is a balanced bimodule, then there are canonical isomorphisms $\mathrm{Hom}_T(V,V\otimes_R U)\cong U_S$ and $\mathrm{Hom}_T(V\otimes_R U,V\otimes_R U)\cong S$.*

PROOF. It suffices to establish the first isomorphism since, then, we have canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_T(V\otimes_R U,\ V\otimes_R U)&\cong\mathrm{Hom}_R(U,\ \mathrm{Hom}_T(V,\ V\otimes_R U))\\ &\cong\mathrm{Hom}_R(U,\ U)\\ &\cong S.\end{aligned}$$

Using the hypothesis on ${}_RV$ and Lemma 6, we have that $\mathrm{Tor}_1^R(V,U)=0$ so by our hypothesis that $\mathrm{Ext}_T^1(V,V)=0$ (hence $\mathrm{Ext}_T^1(V,V\otimes P_2)=0$) an exact sequence

$$0\longrightarrow P_2\longrightarrow P_1\longrightarrow U\longrightarrow 0$$

in R-Mod with P_i projective induces an exact sequence

$$0\longrightarrow \mathrm{Hom}_T(V,\ V\otimes P_2)\longrightarrow \mathrm{Hom}_T(V,\ V\otimes P_1)\longrightarrow \mathrm{Hom}_T(V,\ V\otimes U)\longrightarrow 0$$

so since ${}_RP_1$ and ${}_RP_2$ are finitely generated and projective and ${}_TV_R$ is balanced, the natural isomorphisms $\mathrm{Hom}_T(V, V\otimes P_i)\cong P_i$ induce the required isomorphism $\mathrm{Hom}_T(V, V\otimes U)\cong U$.

LEMMA 8. *Suppose that $V_R\in$ mod-R is a submodule of a flat module and that ${}_RU$ is a tilting module with dom. dim. ${}_RU\geqq 2$. Let $T=\mathrm{End}_R(V)$. Then ${}_TV$ and ${}_T(V\otimes_R U)$ finitely cogenerate each other.*

PROOF. By Lemma 6 $\mathrm{Tor}_1^R(V, U)=0$ so an exact sequence $0\to R\to U_1\to U_2\to 0$ in R-Mod with $U_i\in$ add-U induces an exact sequence $0\to{}_T(V\otimes R)\to{}_T(V\otimes U_1)$. Thus, since $U_1\in$ add-U there is an injection of ${}_TV$ into ${}_T(V\otimes U)^n$ for some n. Since dom. dim. ${}_RU\geqq 2$ there is an exact sequence $0\to U\to E_1\to E_2$ in R-Mod with E_i projective and injective. Hence, since E_2 is projective, by Lemma 6 we obtain an injection of ${}_T(V\otimes U)$ into ${}_T(V\otimes E_1)$, and, since E_1 is projective, one of ${}_T(V\otimes E_1)$ into ${}_TV^m$ for some m.

Suppose R is a finite dimensional algebra over a field. Then, if ${}_RU$ is a tilting module and $S=\mathrm{End}_R(U)^{op}$ then U_S is also a tilting module, ${}_RU_S$ is a balanced bimodule, and R and S have the same number of isomorphism classes of simple modules [2, Theorem 1.5] (our references to [2] do not require the standing hypothesis of algebraic closure made there).

THEOREM 9. *Suppose R is a finite dimensional algebra over a field and that dom. dim. $R\geqq 1$. Let F_R be a minimal faithful right module, let ${}_RU$ be a tilting left module, and $S=\mathrm{End}_R(U)^{op}$. Then $(F\otimes_R U)_S$ is an injective module. Consequently if dom. dim. ${}_RU\geqq 1$, then the functors Δ_U° preserve monomorphisms both in* Mod-S *and in* R-Mod. *Furthermore, if dom. dim. ${}_RU\geqq 2$ then Δ_U^2 is left exact on* Mod-S.

PROOF. Let $H_S=E((F\otimes_R U)_S)$. The evaluation $\mathrm{Hom}_S(U, H)\otimes_R U_S\to H_S$ is an isomorphism by [2, Proposition 1.5a] and the evaluation $\mathrm{Hom}_S(U, F\otimes_R U)\otimes_R U_S\to F\otimes_R U_S$ is an isomorphism since F_R is finitely generated and projective and R is the endomorphism ring of U_S. Since the injective module $\mathrm{Hom}_S(U, F\otimes_R U)_R=F_R$ is a direct summand of $\mathrm{Hom}_S(U, H)$, $(F\otimes_R U)_S$ is a direct summand of the injective module $\mathrm{Hom}_S(U, H)\otimes_R U_S=H_S$. Thus $(F\otimes_R U)_S$ is injective. In order to prove the remaining assertions, identify F_R with a right ideal fR and let Re be a minimal faithful left R-module where f and e are idempotents of R. Considering $R=\mathrm{End}_S(U)$, f is the canonical projection of U_S onto $F\otimes_R U_S$ so we have $fR\subseteqq I$ where I is the ideal of Theorem 1. Suppose dom. dim. ${}_RU\geqq 1$. Since fR_R is faithful and ${}_RU$ is R-torsionless, $\mathrm{Ann}_U(fR)=0$ so $\mathrm{Ann}_U(I)=0$ also. Thus

Δ_U^2 preserves monomorphisms in Mod-S by Theorem 1(4). Also, since dom. dom. $R\geqq 1$, $_RRe$ cogenerates $_RR$ hence also $_RU$ (since $_RU$ is a snbmodule of a projective). Thus Δ_U^2 preserves monomorphisms in R-Mod by Theorem 1(1) and [2, Corollary to Theorem 2.1]. The final assertion follows from Lemma 2 since the minimal faithful left R-module cogenerates any projective and is a direct summand of $_RU$.

PROPOSITION 10. *Suppose R is a finite dimensional algebra over a field. Assume that $_RU$ and V_R are tilting left and right modules respectively, that V_R is torsionless and that dom. dim. $_RU\geqq 2$ The following are equivalent:*

1) *$_TV$ is injective.*
2) *$_TV$ is a cogenerator.*
3) *$_T(V\otimes U)$ is a cogenerator.*
4) *R is QF.*

PROOF. Suppose T has n simple modules. By [2, Theorem 2.1], $_TV$ has n isomorphism classes of indecomposable direct summands. Hence, if $_TV$ is injective, every indecomposable injective is a direct summand of $_TV$ so $_TV$ is a cogenerator. Similarly, if $_TV$ is a cogenerator, $_TV$ is injective since it has n isomorphism classes of indecomposable injective direct summands. Thus conditions 1) and 2) are equivalent and their equivalence with 3) follows from Lemma 8. Now $_TV_R$ is a balanced bimodule so if 1) and 2) hold then $_TV_R$ defines a Morita Duality. Hence V_R is injective. Again by [2, Theorem 2.1], V_R has exactly n isomorphism classes of indecomposable direct summands and this is the number of simple R-modules. Thus R is QF. Finally, condition 4) implies conditions 1) and 2) by [2, Corollary to Theorem 2.1].

THEOREM 11. *Suppose R is a finite dimensional algebra over a field and that dom. dim. $_RR\geqq 1$. Suppose $_RU$ and V_R are tilting left and right R-modules, respectively, each having dominant dimension at least* 1. *Let* $S=\mathrm{End}_R(U)^{op}$, $T=\mathrm{End}_R(V)$, *and* $_TW_S={_T(V\otimes_R U)_S}$. *Then $_TW_S$ is a balanced bimodule,* $\Delta_W(_TV)=U_S$, *and* $\Delta_W(U_S)={_TV}$. *Furthermore,*

1) *If dom. dom. $_RU\geqq 2$ and dom. dim. $V_R\geqq 2$, then the functors Δ_W^2 preserve monomorphisms both in* Mod-S *and in* T-Mod,
2) *If dom. dom. $_RU\geqq 3$ and dom. dim. $V_R\geqq 3$, then the functors Δ_W^2 are left exact both in* Mod-S *and* T-Mod, *and*
3) *If dom. dom. $_RU\geqq 2$ and dom. dim. $V_R\geqq 2$ then R is QF if and only if* Δ_W: T-Mod$\leftrightarrow$Mod-S: Δ_W *defines a Morita Duality.*

PROOF. The first assertion follows from Lemma 7. Suppose dom. dom. ${}_RU \geqq 2$ and dom. dim. $V_R \geqq 2$. By Theorem 9 Δ_V^2 is left exact on T-Mod and Δ_U^2 is left exact on Mod-S. Hence by Theorem 1, U_S cogenerates $E(U_S)$ and ${}_TV$ cogenerates $E({}_TV)$. By Lemma 8 $E({}_TV)$ cogenerates $E({}_TW)$ so ${}_TV$ cogenerates $E({}_TW)$, but then, since ${}_TW$ cogenerates ${}_TV$, ${}_TW$ cogenerates $E({}_TW)$. Thus Δ_W^2 preserves monomorphisms in Mod-S by Theorem 1. Similarly, Δ_W^2 preserves monomorphisms in T-Mod. Next assume that dom. dim. ${}_RU \geqq 3$ and dom. dim. $V_R \geqq 3$. Let F_R be a minimal faithful right R-module. There is an exact sequence

$$0 \longrightarrow V_R \longrightarrow F_R^1 \longrightarrow F_R^2 \longrightarrow F_R^3$$

where $F_R^i \in \text{add-}F_R$. Applying Lemma 6 twice, we conclude that the induced sequence

$$0 \longrightarrow W_S \longrightarrow (F^1 \otimes_R U)_S \longrightarrow (F^2 \otimes_R U)_S$$

is exact. Since $(F \otimes_R U)_S$ is injective by Theorem 9 and since F_R is isomorphic to a direct summand of V_R by [2, corollary to Theorem 2.1] we conclude that Δ_W^2 is left exact on T-Mod by Lemma 2. Similarly, Δ_W^2 is left exact on Mod-S. Statement 3) follows from Proposition 10.

EXAMPLE. Let A be the algebra of 3×3 lower triangular matrices over an field and let $R = A/J^2$ where J is the radical of A. Then R has dominant dimension 2 (and is not QF). Computation shows that, with notation as in Theorem 11 and ${}_RU$, V_R chosen as in Proposition 5 (and having dominant dimension 1), $\Delta_W : T\text{-Mod} \leftrightarrow \text{Mod-}S : \Delta_W$ does define a Morita Duality.

References

[1] Anderson, F. W. and Fuller, K. R., "Rings and Categories of Modules," Springer-Verlag, Inc., New York, 1973.

[2] Bongartz, K., Tilted algebras, Springer LNM **903** (1981), 16-32.

[3] Cohn, P. M., "Morita equivalence and duality," Mathematical Notes, Queen Mary College, University of London, 1966.

[4] Colby, R. R. and Fuller, K. R., Exactness of the double dual, Proc. Amer. Math. Soc. **82** (1981), 521-526.

[5] Colby, R. R. and Fuller, K. R., Exactness of the double dual and Morita duality for Grothendieck categories, J. Algebra **82** (1983), 546-558.

[6] Colby, R. R. and Fuller, K. R., On rings whose double dual functors preserve monomorphism—a generalization of QF-3 rings, J. Pure Appl. Algebra **28** (1983), 223-234.

[7] Colby, R. R. and Fuller, K. R., QF-3' rings and Morita duality, Tsukuba J. Math. **8** (1984), 183-188.

[8] Morita, K., Duality in QF-3 rings, Math. Z. **108** (1969), 237-252.

[9] Mueller, B. J., The classification of algebras by dominant dimension, Canad. J. Math. Soc. **20** (1968), 398-409.

[10] Tachikawa, H., "Quasi Frobenius rings and generalizations," Springer LNM 351, Springer-Verlag, Inc., Berlin and New York, 1973.

Department of Mathematics
University of Hawaii
Honolulu, HI 96734
U. S. A.