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# ITERATED TILTED ALGEBRAS INDUCED FROM COVERINGS OF TRIVIAL EXTENSIONS OF HEREDITARY ALGEBRAS

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## Introduction.

Recently the relations between tilting theory and trivial extension algebras are deeply studied. Let $A$ and $B$ be basic connected artin algebras over a commutative artin ring $C$. In [6] Tachikawa and Wakamatsu showed that the existence of stably equivalence between categories over the trivial extension algebras $T(A)=A\ltimes DA$ and $T(B)=B\ltimes DB$ under the assumption that there is a tilting module $T_A$ with $B=\mathrm{End}(T_A)$. In case $C$ is a field, Hughes and Waschbüsch proved that if $T(B)$ is representation-finite of Cartan class $\Delta$, then there exists a tilted algebra $A$ of Dynkin type $\Delta$ such that $T(B)\cong T(A)$ [4]. Assem, Happel and Roldan showed that, for an algebra $B$ over an algebraically closed field, $T(B)$ is representation-finite iff $B$ is an iterated tilted algebra of Dynkin type [1]. However in case $T(B)$ is not of finite representation type the condition $T(B)\cong T(A)$ with $A$ hereditary does not forces $B$ to be an iterated tilted algebra.

Let's consider the covering $\hat{A}$ of $T(A)$ [4]. The author proved that the condition $\hat{A}\cong\hat{B}$ implies $T(A)\cong T(B)$ and that the converse holds if $T(A)$ is representation-finite [5]. In this paper, we prove that the condition $\hat{B}\cong\hat{A}$ with $A$ hereditary implies that $B$ is an iterated algebra obtained from $A$. It is to be noted that in case $A$ is not necessary representation-finite. Moreover, the proof of our theorem shows that such an algebra $B$ is obtained by at most 3m times processes tilting from $A$, where $m$ is the number of non-isomorphic primitive idempotents of $A$.

## 1. Preliminaries.

In this section, we recall some definitions and important results. Let $A$ be an artin algebra. An $A$-module $T_A$ is said to be a tilting module provided the following three conditions are satisfied,

(1) proj. dim $T_A\leq 1$

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(2) $\mathrm{Ext}_A^1(T_A, T_A)=0$.

(3) There is an exact sequence $0\to A\to T_1\to T_2\to 0$ with $T_1$, $T_2$ direct sums of direct summands of $T_A$.

Bongartz [2] showed that $T_A$ is a tilting module if and only if $T_A$ satisfied the three conditions (1), (2) and (4) instead of (3).

(4) $T_A$ has $m$ non-isomorphic indecomposable direct summands where $m$ is the number of non-isomorphic simple modules of mod $A$.

Moreover let $B=\mathrm{End}\, T_A$, $\mathcal{T}(T_A)=\{X\in \mathrm{mod}\, A \mid \mathrm{Ext}_A^1(T, X)=0\}$=the full subcategory of all modules generated by $T_A$ and $\mathcal{F}(T_A)=\{X\in \mathrm{mod}\, A \mid \mathrm{Hom}_A(T, X)=0\}$ =the full subcategory of all modules cogenerated by $\tau_A T_A$. Then $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ forms a torsion theory for mod $A$, and there are two corresponding full subcategories of mod $B$ defined by $\mathcal{X}(_BT)=\{Y\in \mathrm{mod}\, B \mid Y\otimes_B T=0\}$ and $\mathcal{Y}(_BT)=\{Y\in \mathrm{mod}\, B \mid \mathrm{Tor}_1^B(Y, T)=0\}$. Then we have the following;

THEOREM OF BRENNER-BUTLER.

$_BT$ is also a tilting module with $\mathrm{End}\, _BT\cong A$. $\mathcal{T}(T_A)$, $\mathcal{Y}(_BT)$ are equivalent under the restrictions of $\mathrm{Hom}_A(T_A, -)$, $-\otimes_B T$ which are mutually inverse each other, and similarly, $\mathcal{F}(T_A)$, $\mathcal{X}(_BT)$ are equivalent under the restrictions of $\mathrm{Ext}_A^1(T_A, -)$, $\mathrm{Tor}_1^B(-, _BT)$ which are mutually inverse to each other.

A series $(A_i, T_i)_{0\leq i\leq s}$ will be called a splitting tilting series if it satisfies following three conditions;

(1) $A_i$ is an artin algebra for $0\leqq i\leqq s$ and $T_i$ is an $A_i$-tilting module for $0\leqq i\leqq s-1$.

(2) $A_{i+1}=\mathrm{End}\, T_i$ for $0\leqq i\leqq s-1$.

(3) The induced tortion theories $(\mathcal{X}(T_i), \mathcal{Y}(T_i))$ are all splitting.

An artin algebra $B$ will be called an iterated tilted algebra if there exists a splitting tilting series $(A_i, T_i)_{0\leq i\leq s}$ such that $A_0$ is hereditary and $A_s\cong B$. On the other hand Hoshino [3] proved that $(\mathcal{X}(_BT), \mathcal{Y}(_BT))$ is splitting if and only if inj. dim $X\leqq 1$ for all $X\in \mathcal{F}(T_A)$.

Again let $T_A$ be a tilting module with $\mathrm{End}\, T_A=B$. Tachikawa and Wakamatsu [7] showed the existence of stable equivalence $S$ between $T(A)$ and $T(B)$, and it satisfies that $S(X)\cong \mathrm{Hom}_A(T, X)$ for $X\in \mathcal{T}(T_A)$ and $S(Y)\cong \Omega_{T(B)}\, \mathrm{Ext}_A^1(T, Y)$ for $Y\in \mathcal{F}(T_A)$ where $\Omega_{T(B)}$ is the loop functor of Heller.

Hughes and Waschbüsch [4] introduced the following doubly infinite matrix algebra;

$$\begin{pmatrix} * & * & & & & & & \\ & * & * & & & & & \\ & & A_{n-1} & M_{n-1} & & & & \\ & & & A_n & M_n & & & \\ & & & & A_{n+1} & M_{n+1} & & \\ & & & & & * & * & \\ & & & & & & * & * \end{pmatrix}$$

in which matrices are assumed to have only finitely many entries different from zero, $A_n \cong A$ and $M_n \cong DA$ for all integers $n$, all the remaining entries are zero, and the multiplication is induced from the canonical maps $A \otimes_A DA \rightarrow DA$, $DA \otimes_A A \rightarrow DA$ and a zero map $DA \otimes_A DA \rightarrow 0$. The author [5] proved that $\hat{A} \cong \hat{B}$ if and only if $A$ and $B$ has the following triangular matrix decompositions (*);

$$A = \begin{pmatrix} S_1 & M_1 & & & & \\ & S_2 & M_2 & & & \\ & & * & * & & \\ & & & * & * & \\ & & & & S_{n-1} & M_{n-1} \\ & & & & & S_n \end{pmatrix}$$

$$B = \begin{pmatrix} S_1 & & & & & \\ D(M_1) & S_2 & & & & \\ & * & * & & & \\ & & * & * & & \\ & & & D(M_{n-2}) & S_{n-1} & \\ & & & & D(M_{n-1}) & S_n \end{pmatrix}$$

where $S_i$ is an algebra for all $i$, $M_j$ is an $S_j - S_{j+1}$-bimodule for all $j$ and all the remaining entries are zero.

## 2. Construction of tilting modules.

First we will state the main result of this paper.

THEOREM. *Let $A$ be a hereditary algebra. If $\hat{B} \cong \hat{A}$, then $B$ is an iterated tilted algebra obtained from $A$.*

This theorem can be proved by using the following proposition repeatedly.

PROPOSITION. *Let A and B be the following matrix algebras;*

$$A=\begin{bmatrix} e_1Ae_1 & 0 & 0 \\ e_2Ae_1 & e_2Ae_2 & e_2Ae_3 \\ 0 & 0 & e_3Ae_3 \end{bmatrix} \qquad B=\begin{bmatrix} e_1Ae_1 & 0 & 0 \\ e_2Ae_1 & e_2Ae_2 & 0 \\ 0 & e_3D(A)e_2 & e_3Ae_3 \end{bmatrix}$$

where $e_1$, $e_2$ and $e_3$ are orthogonal idempotents of $A$ and $e_2\neq 0\neq e_3$. Assume that $(e_2+e_3)A(e_2+e_3)$ is hereditary. Then there exists a splitting tilting series $(A_i, T_i)_{0\leq i\leq 3}$ such that $A_0\cong A$ and $A_3\cong B$.

REMARK. The assumptions of this proposition immediately imply the following;

(1) $A$ submodule of $e_3A$ is an $A$-projective module and $e_3A$ has no non-zero injective direct summands.

(2) $A$ quotient module of $D(A(e_2+e_3))$ is an $A$-injective module.

PROOF OF THE PROPOSITION.

Let $F_i=\mathrm{Hom}_{A_i}(T_i, -)$ and $F_i'=\mathrm{Ext}^1_{A_i}(T_i, -)$ for $0\leqq i\leqq 2$.

(1) First tilting.

Let

$$T_0=(e_1+e_2)A\oplus\tau_A^{-1}(e_3A).$$

(i) proj. dim $T_0\leqq 1$.

It is sufficient to show that $\mathrm{Hom}_A(DA, \tau_A(T_0))\cong\mathrm{Hom}_A(DA, e_3A)=0$. If $f$ is a morphism from $DA$ to $e_3A$, then the image of $f$ is projective and injective, and then it is zero.

(ii) $\mathrm{Ext}^1_A(T_0, T_0)=0$.

We have

$$\begin{aligned}\mathrm{Ext}^1_A(T_0, T_0)&\cong D\overline{\mathrm{Hom}}_A(T_0, \tau_A(T_0))\\ &\cong D\overline{\mathrm{Hom}}_A(T_0, e_3A)\\ &\cong D\overline{\mathrm{Hom}}_A(\tau_A^{-1}(e_3A), e_3A)\end{aligned}$$

and $\mathrm{Hom}_A(\tau_A^{-1}(e_3A), e_3A)=0$ because $\tau_A^{-1}(e_3A)$ has no non-zero projective direct summands.

(i), (ii) and the number of indecomposable summands of $T_0$ show that $T_0$ is a tilting module.

(iii) $(\mathcal{X}(T_0), \mathcal{Y}(T_0))$ is splitting.

By definition $\mathcal{I}(T_0)=\text{add}\, e_3A$ where add $e_3A$ is the full subcategory of all direct sums of direct summands of $T_0$. From the assumption of $A$, the injective envelope of $e_3A$ is included in add $D(Ae_3)$. Then inj. dim $e_3A \leqq 1$.

(2) Second tilting.

Let

$$T_1=F_0(e_1A)\oplus F_0(e_2A/e_2Ae_3)\oplus F_0'(e_3A).$$

(i) proj. dim $T_1 \leqq 1$.

(a) $F_0(e_1A)$ is projective.

(b) proj. dim $F_0'(e_3A) \leqq 1$,

Since $T_0$ is an $A$-tilting module, there is an exact sequence

$$0 \longrightarrow e_3A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow 0$$

where $X_1$ and $X_2$ are contained in add $T_0$. Then we have the following resolution;

$$0 \longrightarrow F_0(X_0) \longrightarrow F_0(X_1) \longrightarrow F_0'(e_3A) \longrightarrow 0$$

(c) proj. dim $F_0(e_2A/e_2Ae_3) \leqq 1$.

We consider the exact sequence

$$0 \longrightarrow e_2Ae_3 \longrightarrow e_2A \longrightarrow e_2A/e_2Ae_3 \longrightarrow 0$$

By the assumption $e_2Ae_3$ is contained in add $e_3A=\mathcal{I}(T_0)$. Then we have an exact sequence

$$0 \longrightarrow F_0(e_2A) \longrightarrow F_0(e_2A/e_2Ae_3) \longrightarrow F_0'(e_2Ae_3) \longrightarrow 0.$$

Projectivity of $F_0(e_2A)$ and (b) provide that proj. dim $F_0(e_2A/e_2Ae_3) \leqq 1$.

(ii) $\text{Ext}^1_{A_1}(T_1, T_1)=0$.

(a) $\text{Ext}^1_{A_1}(T_1, F_0'(e_3A))=0$.

Because $F_0'(e_3A)$ is an injective module.

(b) $\text{Ext}^1_A(F_0(e_2A/e_2Ae_3\oplus e_0A), F_0(e_2A/e_2Ae_3\oplus e_1A))=0$.

We have the following isomorphisms;

$$\begin{aligned}&\text{Ext}^1_{A_1}(F_0(e_2A/e_2Ae_3\oplus e_1A), F_0(e_2A/e_2Ae_3\oplus e_1A))\\ &\quad\cong \text{Ext}^1_A(e_2A/e_2Ae_3\oplus e_1A, e_2A/e_2Ae_3\oplus e_1A)\\ &\quad\cong \text{Ext}^1_{(e_1+e_2)A(e_1+e_2)}(e_2A/e_2Ae_3\oplus e_1A, e_2A/e_2Ae_3\oplus e_1A)\\ &\quad=0,\end{aligned}$$

because $e_2A/e_2Ae_3$ and $e_1A$ are $(e_1+e_2)A(e_1+e_2)$-projective modules.

(c) $\text{Ext}^1_{A_1}(F_0'(e_3A), F_0(e_2A/e_2Ae_3\oplus e_1A))=0$.

Since $\tau_{A_1}F'_0(e_3A)\cong F_0(D(Ae_3))$, then

$$\mathrm{Ext}^1_{A_1}(F'_0(e_3A),\ F_0(e_2A/e_2Ae_3\oplus e_1A))$$
$$\cong D\overline{\mathrm{Hom}}_{A_1}(F_0(e_2A/e_2Ae_3\oplus e_1A),\ F_0(D(Ae_3)))\,.$$

and

$$\mathrm{Hom}_{A_1}(F_0(e_2A/e_2Ae_3\oplus e_1A),\ F_0(D(Ae_3)))$$
$$\cong \mathrm{Hom}_A(e_2A/e_2Ae_3\oplus e_1A,\ D(Ae_3))=0\,.$$

(i) and (ii) shows that $T_1$ is an $A_1$-tilting module.

(iii) $(\mathcal{X}(T_1),\ \mathcal{Y}(T_1))$ is splitting.

Let $X$ be contained in $\mathcal{T}(T_1)$ and $I_0$ the injective hull of $X$. Since $T_1$ has $F_0(e_1A)$ as a direct summand, $I_0$ is contined in add $F'_0(e_3A)\oplus F_0(D(Ae_2))$. The construction of $T_0$ provides that a quotient module of $F'_0(e_3A)\oplus F_0(D(Ae_2))$ is again contained in add $F'_0(e_3A)\oplus F_0(D(Ae_2))$, then inj. dim $X\leqq 1$.

(3) Third tilting.

Let $D(Ae_3)e_3=P\oplus M$ where $P$ is a projective $A$-module and $M$ has no non-zero projective direct summands. Then $F_0(M)$ is contained in $\mathcal{T}(T_1)$ because

$$\mathrm{Hom}_{A_1}(T_1,\ F_0(M))\cong \mathrm{Hom}_{A_1}(F_0(e_1A\oplus e_2A/e_2Ae_3),\ F_0(M))$$
$$\cong \mathrm{Hom}_A(e_1A\oplus e_2A/e_2Ae_3,\ M)=0\,.$$

Let

$$T_2=F_1F_0(e_1A\oplus e_2A/e_2Ae_3)\oplus F_1F'_0(P)\oplus F'_1F_0(M)\,.$$

(i) proj. $\dim_{A_2}T_2\leqq 1$.

It is sufficient to show that proj. $\dim_{A_2}F'_1F_0(M)\leqq 1$. First we consider the projective resolution of $M$

$$0\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

where $P_0$ and $P_1$ is contained in add $e_3A$. And we have

$$0\longrightarrow F_0(M)\longrightarrow F'_0(P_1)\longrightarrow F'_0(P_0)\longrightarrow 0$$

and $F'_0(P_0)$ and $F'_0(P_1)$ is conteined in add $T_1$. Then

$$0\longrightarrow F_1F'_0(P_1)\longrightarrow F_1F'_0(P_0)\longrightarrow F'_1F_0(M)\longrightarrow 0$$

is the projective resolution of $F'_1F_0(M)$.

(ii) $\mathrm{Eet}^1_{A_2}(T_2,\ T_2)=0$.

It is sufficient to show that $\mathrm{Ext}^1_{A_2}(F'_1F_0(M),\ T_2)=0$.

(a) $\mathrm{Ext}^1_{A_2}(F'_1F_0(M),\ F'_1F_0(M))=0$.

We have the following isomorphisms

$$\mathrm{Ext}^1_{A_2}(F_1'F_0(M),\ F_1'F_0(M))\cong\mathrm{Ext}^1_A(M,\ M)$$
$$\cong\mathrm{Ext}^1_{e_3Ae_3}(M,\ M)=0$$

because $M$ is an injective $e_3Ae_3$-module.

(b) $\mathrm{Ext}^1_{A_2}(F_1'F_0(M),\ F_1F_0(e_1A\oplus e_2A/e_2Ae_3))=0$.

By the result of Tachikawa and Wakamatsu, we get

$$\begin{aligned}&\mathrm{Ext}^1_{T(A_2)}(F_1'F_0(M),\ F_1F_0(e_1A\oplus e_2A/e_2Ae_3))\\ &\quad\cong D\,\underline{\mathrm{Hom}}_{T(A_2)}(F_1F_0(e_1A\oplus e_2A/e_2Ae_3),\ \tau_{T(A_2)}F_1'F_0(M))\\ &\quad\cong D\,\underline{\mathrm{Hom}}_{T(A_1)}(F_0(e_1A\oplus e_2A/e_2Ae_3),\ \Omega_{T(A_1)}F_0(M))\\ &\quad\cong D\,\underline{\mathrm{Hom}}_{T(A)}(e_1A\oplus e_2A/e_2Ae_3,\ \Omega_{T(A)}M)\end{aligned}$$

And the socle of $\Omega_{T(A)}M$ is contained in add $e_3A/\mathrm{rad}\,e_3A$, then

$$\mathrm{Hom}_{T(A)}(e_1A\oplus e_2A/e_2Ae_3,\ \Omega_{T(A)}M)=0.$$

(c) $\mathrm{Ext}^1_{A_2}(F_1'F_0(M),\ F_1F_0(P))=0$.

Let $M'$ and $P'$ be indecomposable non-zero direct summands of $M$ and $P$ respectively. Then there exists a primitive idempotent $e'$ of $A$ such that $P'\cong D(Ae')e_3$ Let

$$0\longrightarrow F_1F_0'(P')\longrightarrow F_1(N)\oplus F_1'(N')\longrightarrow F_1'F_0(M')\longrightarrow 0$$

be a non-split exact sequence where $N$ and $N'$ is contained in $\mathcal{T}(T_1)$ and $\mathcal{F}(T_1)$ respectively. Then we have the exact sequence

$$0\longrightarrow N'\longrightarrow F_0(M')\longrightarrow F_0'(P')\longrightarrow N\longrightarrow 0$$

and $N$ and $N'$ are contained in $\mathcal{X}(T_0)$ and $\mathcal{Y}(T_0)$ respectively. So there exists a projective $A$-module $Q$ such that $N\cong F_0'(Q)$ and non-splitness of the first sequence shows that $Q$ has no direct summands isomorphic to $P'$. If $Q$ is non-zero, there is a monomorphism from $P'$ to $Q$, and then the inclusion map from $P'$ to $D(Ae')$ is extended to the map from $Q$ to $D(Ae')$. The existence of this extended map contradicts that $P'\cong D(Ae')e_3$. Then we assume that $N=0$. Applying $\otimes_{A_1}T_0$ to the second exact sequence, we get the non-split exact sequence

$$0\longrightarrow P'\longrightarrow N'\otimes_{A_1}T_0\longrightarrow M'\longrightarrow 0$$

But the last exact sequence is considered as an element of $\mathrm{Ext}^1{}_{e_3Ae_3}(M',\ P')$ and $P'$ is an injective $e_3Ae_3$-module.

(iii) $(\mathcal{X}(T_2),\ \mathcal{Y}(T_2))$ is splitting.

The algebra $A_2$ can be represented by

$$\mathrm{End}_{A_1}(T_1)\cong\begin{bmatrix}\mathrm{End}_{A_1}(F_0'(e_3A)) & \mathrm{Hom}_{A_1}(F_0(e_1A\oplus e_2A/e_2Ae_3,\ F_0'(e_3A)))\\ 0 & \mathrm{End}_{A_1}(F_0(e_1A\oplus e_2A/e_2Ae_3))\end{bmatrix}$$

and $\mathrm{End}_{A_1}(F'_0(e_3A))\cong e_3Ae_3$ is hereditary. On the other hand $T_2$ has $F_1F_0(e_1A\oplus e_2A/e_2Ae_3)$ as a direct summand, then a module contained in $\mathcal{I}(T_2)$ is considered as $\mathrm{End}_{A_1}(F'_0(e_3A))$-module and its injective resolution as $\mathrm{End}_{A_1}(F'_0(e_3A))$-module coincides with that as $A_2$-module.

(4) $\mathrm{End}_{A_2}(T_2)\cong B$.

We have the following isomorphisms;

$$\begin{aligned}\mathrm{End}_{A_2}(T_2)&\cong\underline{\mathrm{End}}_{T(A_2)}(T_2)\\&\cong\underline{\mathrm{End}}_{T(A_1)}(F_0(e_1A\oplus e_2A/e_2Ae_3)\oplus F'_0(P)\oplus\Omega^{-1}_{T(A_1)}F_0(M))\\&\cong\underline{\mathrm{End}}_{T(A)}(e_1A\oplus e_2A/e_2Ae_3\oplus\Omega^{-1}_{T(A)}(P)\oplus\Omega^{-1}_{T(A)}(M))\\&\cong\underline{\mathrm{End}}_{T(A)}(e_1A\oplus e_2A/e_2Ae_3\oplus\Omega^{-1}_{T(A)}(D(Ae_3)e_3)).\end{aligned}$$

Let $J$ denote $\Omega^{-1}_{T(A)}(D(Ae_3)e_3)$, and $e_3T(A)$ is the projective cover of $J$ in mod $T(A)$

$$0\longrightarrow D(Ae_3)e_3\longrightarrow e_3T(A)\longrightarrow J\longrightarrow 0.$$

Since the socle of $e_3T(A)$ and $J/\mathrm{rad}\,J$ are contained in add $e_3A/\mathrm{rad}\,e_3A$, we get

$$\mathrm{Hom}_{T(A)}(J,\ e_1A\oplus e_2A/e_2Ae_3)=0 \qquad \text{and}$$

$$\underline{\mathrm{Hom}}_{T(A)}(e_2A/e_2Ae_3\oplus e_1A,\ J)\cong\mathrm{Hom}_{T(A)}(e_2A/e_2Ae_3\oplus e_1A,\ J).$$

If $f$ is a $T(A)$-homomorphism from $J$ to $e_3T(A)$, the $A$-homomorphism, induced by $f$, from $J\cdot DA\cong D(Ae_3)/D(Ae_3)e_3$ to $e_3T(A)\cdot DA\cong e_3D(A)$ is zero, and then $f$ factors through $D(Ae_3)e_3$. We have

$$\underline{\mathrm{End}}_{T(A)}(J)\cong\mathrm{End}_{T(A)}(J).$$

Then

$$\mathrm{End}_{A_2}(T_2)\cong\mathrm{End}_{T(A)}(e_1A\oplus e_2A/e_2Ae_3\oplus J).$$

(i) $\mathrm{End}_{T(A)}(e_1A)\cong e_1Ae_1$.

Clearly.

(ii) $\mathrm{End}_{T(A)}(e_2A/e_2Ae_3)\cong e_2Ae_2$.

The exact sequence in mod $A$

$$0\longrightarrow e_2Ae_3\longrightarrow e_2A\longrightarrow e_2A/e_2Ae_3\longrightarrow 0$$

induces following two isomorphisms;

$$\mathrm{Hom}_A(e_2A,\ e_2A)\cong\mathrm{Hom}_A(e_2A,\ e_2A/e_2Ae_3) \qquad \text{and}$$

$$\mathrm{Hom}_A(e_2A/e_2Ae_3,\ e_2A/e_2Ae_3)\cong\mathrm{Hom}_A(e_2A,\ e_2A/e_2Ae_3).$$

Moreover we have

$$\mathrm{End}_{T(A)}(e_2A/e_2Ae_3)\cong\mathrm{End}_A(e_2A/e_2Ae_3) \qquad \text{and}$$

$$\mathrm{End}_A(e_2A)\cong e_2Ae_2,$$

(iii) $\mathrm{End}_{T(A)}(J) \cong e_3Ae_3$.

We have $\mathrm{Hom}_{T(A)}(e_3T(A), J) \cong Je_3 \cong e_3Ae_3$. If $f$ is a $T(A)$-homomorphism $e_3T(A)$ to $J$, the kernel of $f$ contains $e_3D(A)$. Then

$$\mathrm{Hom}_{T(A)}(J, J) \cong \mathrm{Hom}_{T(A)}(e_3T(A), J) \cong e_3Ae_3 .$$

(iv) $\mathrm{Hom}_{T(A)}(e_1A, e_2A/e_2Ae_3) \cong e_2Ae_1$.

Because $\mathrm{Hom}_{T(A)}(e_1A, e_2A/e_2Ae_3) \cong \mathrm{Hom}_A(e_1A, e_2A/e_2Ae_3) \cong e_2Ae_1$.

(v) $\mathrm{Hom}_{T(A)}(e_2A/e_2Ae_3, e_1A)=0$.

Because $e_1Ae_2=0$.

(vi) $\mathrm{Hom}_{T(A)}(e_1A, J)=0$.

Because $e_3Ae_1 \cong e_1Ae_3=0$.

(vii) $\mathrm{Hom}_{T(A)}(e_2A/e_2Ae_3, J) \cong e_3D(A)e_2$.

Because $\mathrm{Hom}_{T(A)}(e_2A, J) \cong \mathrm{Hom}_A(e_2A, e_3D(A)/e_3D(A)e_3) \cong e_3D(A)e_2$ and the kernel of an element of $\mathrm{Hom}_{T(A)}(e_2A, J)$ includes $e_2Ae_3$.

(viii) $\mathrm{Hom}_{T(A)}(J, e_2A/e_2Ae_3 \oplus e_1A)=0$.

Because $(e_2A/e_2Ae_3 \oplus e_1A)e_3=0$.

(ix) Multiplication

By the following commutative diagram we know the existence of an algebra isomorphism from $B$ to $\mathrm{End}_{T(A_2)}(T_2)$.

$$\begin{array}{ccccccccccc}
e_1A & \longrightarrow & e_1A & \longrightarrow & e_2A & \longrightarrow & e_2A & \longrightarrow & e_3T(A) & \longrightarrow & e_3T(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
e_1A & \longrightarrow & e_1A & \longrightarrow & e_2A/e_2Ae_3 & \longrightarrow & e_2A/e_2Ae_3 & \longrightarrow & J & \longrightarrow & J .
\end{array}$$

This completes the proof of the proposition.

PROOF OF THE THEOREM.

Let $A$ and $B$ be the martix algebras $(*)$, and $R_i$ be the following matrix algebra;

$$\begin{bmatrix} S_1 & & & & & & & & & \\ D(M_1) & S_2 & & & & & & & & \\ & * & * & & & & & & & \\ & & * & * & & & & & & \\ & & & D(M_i) & S_{i+1} & M_{i+1} & & & & \\ & & & & & * & * & & & \\ & & & & & & * & * & & \\ & & & & & & & S_{n-1} & M_{n-1} & \\ & & & & & & & & S_n & \end{bmatrix}$$

then $R_0=A$, $R_{n-1}=B$ and we can apply the proposition to the pair $(R_i, R_{i+1})$ for $0\leqq i\leqq n-2$.

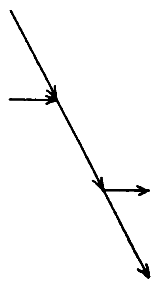
COROLLARY. *Let $A$ be a hereditary algebra. If $\hat{A}\cong\hat{B}$, then $B$ is given by at most $3m$ times tilting from $A$ where $m$ is the number of non-isomorphic primitive idempotents of $A$.*

### 3. Example.

Let $Q_0$ and $Q_6$ be the following quivers:

$Q_0$:

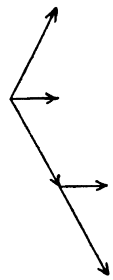
$Q_6$:



Let $A$ be the path algebra $kQ_0$ and $B$ the path algebra $kQ_6$ with relation $\mathrm{rad}^2 B=0$, where $k$ is a field. By the theorem we get the following splitting tilting series $(A_i, T_i)_{0\leq i\leq 6}$.

$$A_0=A \qquad T_0=\begin{array}{c}0\\00\\11\\0\end{array}\oplus\begin{array}{c}0\\00\\01\\1\end{array}\oplus\begin{array}{c}0\\11\\01\\0\end{array}\oplus\begin{array}{c}1\\21\\13\\1\end{array}\oplus\begin{array}{c}1\\10\\01\\0\end{array}\oplus\begin{array}{c}1\\11\\12\\1\end{array}$$

$A_1 = kQ_1$ $\quad Q_1:$

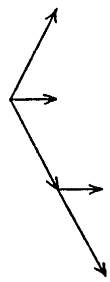


$T_1 = DA_1$

$A_2 = kQ_1$

| | | | | | | |
|---|---|---|---|---|---|---|
| $T_2=$ | 0 | 0 | 1 | 0 | 0 | 0 |
| | 00 | ⊕00 | ⊕10 | ⊕10 | ⊕11 | ⊕10 |
| | 01 | 00 | 00 | 00 | 00 | 11 |
| | 0 | 1 | 0 | 0 | 0 | 1 |

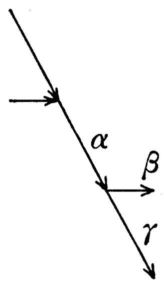
$A_3 = kQ_3$ $\quad Q_3:$



with relation $\mathrm{rad}^2\, A_3 = 0$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $T_3=$ | 0 | 0 | 1 | 0 | 0 | 1 |
| | 00 | ⊕00 | ⊕11 | ⊕00 | ⊕11 | ⊕10 |
| | 01 | 00 | 10 | 11 | 10 | 10 |
| | 0 | 1 | 0 | 1 | 0 | 0 |

$A_4 = kQ_4$ $\quad Q_4:$




with relation $\alpha\beta = \alpha\gamma = 0$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $T_4=$ | 0 | 0 | 1 | 0 | 1 | 0 |
| | 00 | ⊕00 | ⊕11 | ⊕00 | ⊕00 | ⊗10 |
| | 01 | 00 | 10 | 11 | 00 | 00 |
| | 0 | 1 | 0 | 1 | 0 | 0 |

$A_5 = B$ $\qquad T_5 = A_5$

$A_6 = B$.

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