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ON DEDEKIND SUMS AND ANALOGS

By

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Abstract The purpose of this paper is to point out a connection between an identity due to Subrahmanyam and the Peterson-Knopp identity for the classical Dedekind sum. We then consider the same connection with regard to the Apostol-Vu generalization of the Dedekind sum. We also consider some sums related to the classical Dedekind sum.

1. Introduction.

We define the function $((x))$ by

$$((x))=\begin{cases} x-[x]-1/2 & \text{if } x \text{ is not an integer} \\ 0 & \text{otherwise.}\end{cases}$$

Let h and k be positive integers. The Dedekind sum $S(h, k)$ is defined as

$$S(h, k)=\sum_{n(\mathrm{mod}\ k)}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{nh}{k}\right)\right). \tag{1.1}$$

In [9] H. Rademacher and E. Grosswald have given a survey of the properties of $S(h, k)$. P. Subrahmanyam in [10] has shown that

$$\sum_{b(\mathrm{mod}\ n)} S(h+bk, nk)=\sum_{d\mid n}\mu(d)S(hd, k)\sigma(n/d), \tag{1.2}$$

where $\mu(n)$ is the Möbius function and $\sigma(n)$ is the sum of divisors function. In [5] M. I. Knopp proved the following identity for $S(a, h)$:

$$\sum_{\substack{ad=n \\ d>0}}\ \sum_{b(\mathrm{mod}\ d)} S(ah+bk, dk)\sigma(n)S(h, k). \tag{1.3}$$

This generalized an older identity of Petersson. Proofs of (1.3) have been given by Goldberg in [4] using the identity (1.2) and Parson in [7] using Hecke operators. In [8] Parson and Rosen have extended Knopp's identity to generalized Dedekind sums.

Suppose $A(x)$ and $B(x)$ are given functions which are defined on the rationals and satisfy a relation of the form

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$$\sum_{b(\mathrm{mod}\ q)} F(x+b/q)=q^{\nu(F)}F(qx),$$

for every positive integer q, every rational number x and some constant $\nu(F)$ depending only on F. In [1] Apostol and Vu define a class of functions

$$f(h, k)=\sum_{n(\mathrm{mod}\ k)} A(n/k)B(hn/k),$$

which are called Dedekind sums of type $(\nu(A), \nu(B))$. It may be observed that the classical Dedekind sum $S(h, k)$, defined by (1.1), is of type (0, 0) and is obtained by taking $A(x)=B(x)=((x))$.

In [1] Apostol and Vu generalize (1.2) and (1.3). Let $f(h, k)$ be a Dedekind sum of type $(\nu(A), \nu(B))$. Let $\lambda=1-\nu(A)-\nu(B)$ and

$$\sigma_\lambda(n)=\sum_{d\mid n} d^\lambda.$$

They prove the following results. If k is a given integer, then

$$\sum_{b(\mathrm{mod}\ n)} f(h+bk, nk)=n^{1-\lambda}\sum_{d\mid n}\mu(d)d^{-\nu(A)}f(hd, k)\sigma_\lambda(n/d) \tag{1.5}$$

and if n is a positive integer, then

$$\sum_{\substack{ad=n\\ d>0}} d^{-\nu(B)}\sum_{b(\mathrm{mod}\ d)} f(ah+bk, dk)=n^{\nu(A)}\sigma_\lambda(n)f(h, k). \tag{1.6}$$

The purpose of the paper is to point out the intrinsic connection between Subrahmanyan's identity and Knopp's identity as well as qetween Apostol and Vu's generalizations of these identities via a basic inversion principle. Incidentally, we derive a few analogues of Knopp's identity. In this connection, we also introduced two sums $T(h, k)$ and $S'(h, k)$ which are related to $S(h, k)$.

2. An Inversion Principle.

Let $f(m, n)$ and $g(m, n)$ be complex valued functions defined for all positive in tegers m and n. Define the two arithmetic functions $e_0(n)$ and $e(n)$ by $e_0(n) =[1/n]$ and $e(n)=1$ for all positive integers n. Let $\varepsilon(n)$ and $\eta(n)$ be two arithmetic functions related by the identity

$$\sum_{d\mid n}\varepsilon(n/d)\eta(d)=e_0(n). \tag{2.1}$$

We say that ε and η are Dirichlet inverses of each other.

THEOREM 1. *With ε and η as above we have*

$$f(m, n)=\sum_{d\mid n} g(md, n/d)\varepsilon(d)$$

if and only if

$$g(m, n)=\sum_{d\mid n} f(md, n/d)\eta(d).$$

PROOF. We have

$$\begin{aligned}\sum_{d\mid n} \eta(d)f(md, n/d) &= \sum_{d\mid n} \eta(d) \sum_{ts=n/d} \varepsilon(t)g(mdt, s)\\ &= \sum_{dts=n} \eta(d)\varepsilon(t)g(mdt, s)\\ &= \sum_{s\mid n} g(mn/s, s) \sum_{dt=n/s} \eta(d)\varepsilon(t)\\ &= \sum_{s\mid n} g(mn/s, s)e_0(n/s)\\ &= g(m, n),\end{aligned}$$

by (2.1) and the definition of e_0.

This proves that

$$\sum_{d\mid n} g(md, n/d)\varepsilon(d)=f(m, n) \quad \text{implies} \quad \sum_{d\mid n} f(md, n/d)\eta(d)=g(m, n)$$

The proof of the reverse implication is similar and is omitted. This completes the proof of Theorem 1.

COROLLARY 1.1. *We have*

$$f(m, n)=\sum_{d\mid n} g(md, n/d)$$

if and only if

$$g(m, n)=\sum_{d\mid n} \mu(d)f(md, n/d).$$

PROOF. The result follows immediately from Theorem 1 if we take $\varepsilon(n)=e(n)=1$ and $\eta(u)=\mu(n)$, since it is known (see [6, Theorem 4.6]) that (2.1) is valid for this choice of ε and η.

THEOREM 2. *Suppose $f(m, n)$ and $g(m, n)$ are related by*

$$f(m, n)=\sum_{d\mid n} g(md, n/d). \tag{2.2}$$

If G and H are arithmetic functions which are related by

$$G(n)=\sum_{d\mid n} H(d), \tag{2.3}$$

then

$$\sum_{ad=n} G(a)g(ah, d)=\sum_{d\mid n} H(d)f(hd, n/d).$$

PROOF. By Theorem 4.7 of [6], we have, from (2.3),

$$H(n)=\sum_{d\mid n} \mu(d)G(n/d). \tag{2.4}$$

By (2.2) and Corollary 1.1, we have

$$\begin{aligned}\sum_{ad=n} G(a)g(ah, d) &= \sum_{ad=n} G(a) \sum_{st=d} \mu(s)f(ahs, t) \\ &= \sum_{ast=n} G(a)\mu(s)f(ahs, t) \\ &= \sum_{t\mid n} f(ht, n/t) \sum_{as=n} G(a)\mu(s) \\ &= \sum_{t\mid n} H(t)f(ht, n/t),\end{aligned}$$

by (2.4). This completes the proof of Theorem 2.

THEOREM 3. *The two identities* (1.2) *and* (1.3) *are equivalent.*

PROOF. We have only to appeal to Corollary 1.1 with appropriate choices of $f(m, n)$ and $g(m, n)$. We take

$$f(m, n) = \sigma(n)S(m, n)$$

and

$$g(m, n) = \sum_{b (\mathrm{mod}\ n)} S(m+bn, mn).$$

Then (1.2) is equivalent to

$$g(m, n) = \sum_{d\mid n} \mu(d)f(md, n/d),$$

which, by Corollary 1.1, is equivalent to

$$f(m, n) = \sum_{d\mid n} g(md, n/d).$$

This establishes the equivalence and completes the proof.

THEOREM 4. *Let G and H be arithmetic functions which are related by* (2.3). *Then*

$$\sum_{ad=n} G(a) \sum_{b (\mathrm{mod}\ d)} S(ah+bk, dk) = \sum_{d\mid n} H(d)S(hd, k)\sigma(n/d). \tag{2.5}$$

PROOF. Denote the left hand side of (2.5) by L. Then, by (1.2), we have

$$L = \sum_{ad=n} G(a) \sum_{c\mid d} \mu(c)S(ahc, k)\sigma(d/c).$$

If we let $m = ac$, we have

$$\begin{aligned}L &= \sum_{\substack{m\mid n \\ md=cn}} \sum_{c\mid m} G(m/c)\mu(c)S(mh, k)\sigma(n/m) \\ &= \sum_{\substack{m\mid n \\ md=cn}} S(mh, k)\sigma(n/m) \sum_{c\mid m} G(m/c)\mu(c) \\ &= \sum_{m\mid n} H(m)S(mh, k)\sigma(n/m),\end{aligned}$$

by (2.4). This completes the proof of the theorem.

COROLLARY 4.1. *We have*

(1) $$\sum_{\substack{ad=n\\(a,m)=1}}\sum_{b(\bmod d)} S(ah+bk, dk)=\sum_{d\mid(m,n)}\mu(d)S(hd, k)\sigma(n/d),$$

for any positive integer m,

(2) $$\sum_{a^2d=n}\sum_{b(\bmod d)} S(a^2h+bk, dk)=\sum_{d\mid n}\lambda(d)S(hd, k)\sigma(n/d),$$

where λ is Liouville's function,

(3) $$\sum_{\substack{ad=n\\a\mid m}} a\sum_{b(\bmod d)} S(ah+bk, dk)=\sum_{d\mid n} c(m, d)S(hd, k)\sigma(n/d),$$

where $c(m, n)$ is Ramanujan's trigonometric sum defined by

(2.6) $$c(m, n)=\sum_{\substack{K(\bmod n)\\(k,n)=1}}\exp(2\pi ikm/n),$$

(4) $$\sum_{ad=n} a\sum_{b(\bmod d)} S(ah+bk, dkk)=\sum_{d\mid n}\varphi(d)S(hd, k)\sigma(n/d),$$

where φ denotes Euler's quotent function, and

(5) $$\sum_{ad=n}\log a\sum_{b(\bmod d)} S(ah+bk, dk)=\sum_{d\mid n}\Lambda(d)S(hd, k)\sigma(u/d),$$

wчere Λ is the von-Mangoldt function defined by

$$\Lambda(n)=\begin{cases}\log p & \textit{if } n \textit{ is a power of the prime } p\\ 0 & \textit{otherwise.}\end{cases}$$

PRRF OF (1). If we take $G(n)=e_0((m, n))$, then, by (2.4), $H(n)=\mu(n)$. The result follows from Theorem 4.

PROOF OF (2). Recall that $\lambda(n)$ is defined by $\lambda(n)=(-1)^{\Omega(n)}$, where $\Omega(n)$ counts the total number of prime factors of n. Then, we have

$$\sum_{d\mid n}\lambda(d)=\begin{cases}1 & \text{if } n \text{ is a perfect square}\\ 0 & \text{otherwise.}\end{cases}$$

(see [4, p. 111]). The result follows from Theorem 4 if we take $H(n)=\lambda(n)$ and

$$G(n)=\begin{cases}1 & \text{if } n \text{ is a perfect square}\\ 0 & \text{otherwise.}\end{cases}$$

PROOF OF (3). From (2.6) we have

$$\sum_{d\mid n} c(m, d)=\begin{cases}n & \text{if } n\mid m\\ 0 & \text{otherwise.}\end{cases}$$

Thus, if we take $H(n)=c(m, n)$ and

$$G(n)=\begin{cases} n & \text{if } n \mid m \\ 0 & \text{otherwise,} \end{cases}$$

then the result follows from Theorem 4.

PROOF OF (4). Here we take $H(n)=\varphi(n)$ and $G(n)=n$. Then for this choice of G and H we see that (2.3) holds by Theorem 2.17 of [4]. The result then follows from Theorem 4.

PROOF OF (5). By the definition of $\Lambda(n)$ and unique factorization we see that

$$\sum_{d \mid n} \Lambda(d)=\log n .$$

Thus, if we take $H(n)=\Lambda(n)$ and $G(n)=\log n$ in Theorem 4, then the result follows from Theorem 4.

This completes the proof of the corollary.

Note that (1) of Corollary 4.1 is a generalization of the Petersson-Knopp identity (1.3), which is the case $m=1$.

THEOREM 5. *The two identities* (1.5) *and* (1.6) *are equivalent.*

PROOF. Let $f(h, k)$ be a Dedekind sum of type $(\nu(A), \nu(B))$. Let

$$F(h, n)=n^{\nu(A)}\sigma_{\lambda}(n)f(h, k)$$

and

$$G(h, n)=n^{-\nu(B)} \sum_{b \pmod n} f(h+bk, nk),$$

where $\lambda=1-\nu(A)-\nu(B)$.

The identity (1.5) then states that

$$\begin{aligned}\sum_{b \pmod n} f(h+bk, nk)&=n^{1-\lambda} \sum_{d \mid n} \mu(d)d^{-\nu(A)}f(hd, k)\sigma_{\lambda}(n/d)\\ &=n^{\nu(A)+\nu(B)} \sum_{d \mid n} \mu(d)d^{-\nu(A)}f(hd, k)\sigma_{\lambda}(n/d)\\ &=n^{\mu(B)} \sum_{d \mid n} \mu(d)(n/d)^{\nu(A)}f(hd, k)\sigma_{\lambda}(n/d)\\ &=n^{\nu(B)} \sum_{d \mid n} \mu(d)F(hd, n/d)\end{aligned}$$

or

$$G(h, n)=\sum_{d \mid n} \mu(d)F(hd, n/d). \tag{2.7}$$

By Theorem 2, (2.7) is equivalent to

$$F(h, n)=\sum_{d \mid n} G(hd, n/d)$$

or

$$n^{\nu(A)}\sigma_\lambda(n)f(h, k)=\sum_{d\mid n} d^{-\nu(B)} \sum_{b(\mathrm{mod}\ d)} f(hd+bk, dk),$$

which is (1.6).

Thus (1.5) implies (1.6). The reverse implication is obtained by taking the above steps in reverse order. This completes the proof of the theorem.

THEOREM 6. *Let G and H be arithmetical functions related by*

$$G(n)=n^{\nu(B)}\sum_{d\mid n} H(d).$$

Then

$$\sum_{ad=n} G(a) \sum_{b(\mathrm{mod}\ d)} f(ah+bk, dk)=n^{1-\lambda}\sum_{d\mid n} H(d)d^{-\nu(A)}f(hd, k)\sigma_\lambda(n/d),$$

where f is a Dedekind sum of type $(\nu(A), \nu(B))$ *and* $\lambda=1-\nu(A)-\nu(B)$.

This result generalizes (1.6). The proof is similar to that of Theorem 4 and so we omit it.

3. Analogues of Knopp's Identity.

THEOREM 7. *If* $(h, k)=1$, *then*

$$\sum_{ad=n} \sum_{b(\mathrm{mod}\ d)} \mu((b, d))S(ah(b, d), k)=\varphi(n)S(h, k).$$

PROOF. We have

$$\begin{aligned}\sum_{ad=n} \sum_{b(\mathrm{mod}\ d)} \mu((b, d))S(ah(b, d), k) &= \sum_{ad=n} \sum_{c\mid d}\mu(c)S(ahc, k)\varphi(d/c)\\ &= \sum_{\substack{m\mid n\\ md=cn}} S(mh, k)\varphi(n/m)\sum_{c\mid m}\mu(c)\\ &=\varphi(n)S(h, k),\end{aligned}$$

by Theorem 4.6 of [4]. This completes the proof of Theorem 7.

THEOREM 8. *If f is an arithmetic function, then*

$$\sum_{a(\mathrm{mod}\ n)} f((a, n)) \sum_{b(\mathrm{mod}\ n/(a, n))} S((a, n)h+bk, nk/(a, n)) \tag{3.1}$$
$$=\sum_{rs=n} S(rh, k)\sigma(s)\sum_{c\mid r}\mu(c)\varphi(cs)f(r/c).$$

PROOF. If we denote by L the left hand side of (3.1), we have, by (1.2),

$$L=\sum_{dt=n}\varphi(t)f(d)\sum_{c\mid t}\mu(c)S(hdc,\ k)\sigma(t/c).$$

$$=\sum_{\substack{m\mid n\\ md=nc}}\sum_{c\mid m}\mu(c)S(mh,\ k)\sigma(n/m)f(m/c)\varphi(cn/m)$$

$$=\sum_{rs=n}S(rh,\ k)\sigma(s)\sum_{c\mid r}\mu(c)f(r/c)\varphi(cs),$$

which gives the right hand side of (3.1) and completes the proof of Theorem 8.

COROLLARY 8.1. *If n is square-free, then*

$$\sum_{a(\mathrm{mod}\ n)}\mu((a,\ n))\sum_{b(\mathrm{mod}\ n/(a,n))}S((a,\ n)h+bk,\ nk/(a,\ n))$$

$$=\sum_{rs=n}r\mu(r)S(rh,\ k)\sigma(s)\varphi(s).$$

PROOF. Let $f(n)=\mu(n)$ in Theorem 8. Since n is square-free and $r\mid n$ we see that r is square-free. Thus

$$\mu(c)\mu(r/c)=\mu(r)$$

for all $c\mid r$. Thus, if $f(n)=\mu(n)$, then the right hand side of (3.1) is equal to

$$\sum_{rs=n}S(rh,\ k)\sigma(s)\sum_{c\mid r}\mu(c)\mu(r/c)\varphi(cs)=\sum_{rs=n}S(rh,\ k)\sigma(s)\mu(r)\sum_{c\mid r}\varphi(cs). \tag{3.2}$$

Again n square-free and $rs=n$ implies that $(r,\ s)=1$. Since $c\mid r$ in the inner sum we see that $(c,\ s)=1$ and since φ is a multiplicative function [4, Theorem 2.15], we see that the inner sum in (3.2) is equal to

$$\sum_{c\mid r}\varphi(cs)=\varphi(s)\sum_{c\mid r}\varphi(c)=r\varphi(s),$$

by Theorem 2.17 of [4]. If we combine these results we get the result of the corollary and complete the proof.

Before giving our next analogue of the Petersson-Knopp identity (1.3) we prove some lemmas.

LEMMA 9.1. *If x is any real number, we have*

$$\sum_{\substack{n\ (\mathrm{mod}\ k)\\ (n,k)=1}}\left(\left(\frac{n}{k}+x\right)\right)=\mu(d)((kx/d)).$$

PROOF. We have

$$\sum_{\substack{n\ (\mathrm{mod}\ k)\\ (n,k)=1}}\left(\left(\frac{n}{k}+x\right)\right)=\sum_{n(\mathrm{mod}\ k)}\left(\left(\frac{n}{k}+x\right)\right)\sum_{\substack{d\mid n\\ d\mid k}}\mu(d)$$

$$=\sum_{d\mid k}\mu(d)\sum_{\substack{n\ (\mathrm{mod}\ k)\\ d\mid n}}\left(\left(\frac{n}{k}+x\right)\right)$$

$$=\sum_{d\mid k}\mu(d)\sum_{m(\mathrm{mod}\ k/d)}\left(\left(\frac{m}{kd}+x\right)\right)$$
$$=\sum_{d\mid k}\mu(d)((kx/d)),$$

by Lemma 1 of [9]. This completes the proof.

LEMMA 9.2. *If x is any real number, then*

$$\sum_{m(\mathrm{mod}\ k)}\left(\left(x+\frac{am}{k}\right)\right)=(a,\ k)\left(\left(\frac{kx}{(k,\ a)}\right)\right).$$

PROOF. Let $g=(a,\ k)$ and define a' and k' by $k'=k/g$ and $a'=a/g$. It follows from the definition of $((x))$ that it is periodic of period 1. Thus, for any integer n, we have $((x+n))=((x))$. Thus

$$\sum_{m(\mathrm{mod}\ k)}\left(\left(x+\frac{am}{k}\right)\right)=\sum_{m(\mathrm{mod}\ gk')}\left(\left(x+\frac{a'm}{k'}\right)\right)$$
$$=\sum_{n=0}^{k'-1}\sum_{m=0}^{g-1}\left(\left(x+\frac{a'(k'm+n)}{k'}\right)\right)$$
$$=\sum_{n=0}^{k'-1}\sum_{m=0}^{g-1}\left(\left(x+a'm+\frac{a'n}{k'}\right)\right)$$
$$=g\sum_{n(\mathrm{mod}\ k')}\left(\left(x+a'm+\frac{a'n}{k'}\right)\right)$$
$$=g\sum_{n(\mathrm{mod}\ k')}\left(\left(x+\frac{a'n}{k'}\right)\right)$$
$$=g\sum_{m(\mathrm{mod}\ k')}\left(\left(x+\frac{m}{k'}\right)\right)$$
$$=g((k'x)),$$

by Lemma 1 of [9] and the fact that since $(a',\ k')=1$ as n runs through a complete residue system modulo k' so does $a'n$. This completes the proof.

LEMMA 9.3. *For any real number x and integers a and k we have*

$$\sum_{\substack{n\ (\mathrm{mod}\ k)\\ (n,k)=1}}\left(\left(x+\frac{an}{k}\right)\right)=\sum_{d\mid k}\frac{\mu(d)}{d}(k,\ ad)\left(\left(\frac{kx}{(k,\ ad)}\right)\right).$$

PROOF. By Lemma 9.2, we have

$$\sum_{\substack{n\ (\mathrm{mod}\ k)\\ (n,k)=1}}\left(\left(x+\frac{an}{k}\right)\right)=\sum_{n(\mathrm{mod}\ k)}\left(\left(x+\frac{an}{k}\right)\right)\sum_{\substack{d\mid n\\ d\mid k}}\mu(d)$$
$$=\sum_{d\mid k}\mu(d)\sum_{n(\mathrm{mod}\ k/d)}\left(\left(x+\frac{am}{k/d}\right)\right)$$

$$=\sum_{d\mid k}\mu(d)(a,\ k/d)\left(\left(\frac{kx/d}{(a,\ k/d)}\right)\right)$$

$$=\sum_{d\mid k}\frac{\mu(d)}{d}(k,\ ad)\left(\left(\frac{kx}{(k,\ ad)}\right)\right),$$

which proves the result.

THEOREM 9. *We have*

$$\sum_{\substack{b(\mathrm{mod}\ n)\\(b,n)=1}} S(h+bk,\ nk)=\sum_{d\mid n}\frac{\mu(d)}{d}\sum_{rs\mid n}\mu(s)rS(hs,\ k(rs,\ d)),$$

PROOF. By Lemma 9.3, we have

$$\sum_{\substack{b(\mathrm{mod}\ n)\\(b,n)=1}} S(h+bk,\ nk)=\sum_{\substack{b(\mathrm{mod}\ n)\\(b,n)=1}}\ \sum_{m(\mathrm{mod}\ nk)}\left(\left(\frac{m}{nk}\right)\right)\left(\left(\frac{(h+bk)m}{nk}\right)\right)$$

$$=\sum_{m(\mathrm{mod}\ nk)}\left(\left(\frac{m}{nk}\right)\right)\sum_{\substack{b(\mathrm{mod}\ n)\\(b,n)=1}}\left(\left(\frac{mh}{nk}+\frac{bm}{n}\right)\right)$$

$$=\sum_{m(\mathrm{mod}\ nk)}\left(\left(\frac{m}{nk}\right)\right)\sum_{d\mid n}\frac{\mu(d)}{d}(n,\ md)\left(\left(\frac{mh}{k(n,\ md)}\right)\right)$$

$$=\sum_{d\mid n}\frac{\mu(d)}{d}\sum_{m(\mathrm{mod}\ nk)}\left(\left(\frac{m}{nk}\right)\right)(n,\ md)\left(\left(\frac{mh}{k(n,\ md)}\right)\right)$$

$$=\sum_{d\mid n}\frac{\mu(d)}{d}\sum_{r\mid n}r\sum_{\substack{m(\mathrm{mod}\ nk)\\(n,\ md)=r}}\left(\left(\frac{m}{nk}\right)\right)\left(\left(\frac{mh}{kr}\right)\right)$$

$$=\sum_{d\mid n}\frac{\mu(d)}{d}\sum_{rst=n}\mu(s)rS(hst,\ k(rs,\ d)t)$$

$$=\sum_{d\mid n}\frac{\mu(d)}{d}\sum_{rs\mid n}\mu(s)rS(hs,\ k(rs,\ d)),$$

since by (5) of [1], we have

$$S(qh,\ qk)=S(h,\ k) \tag{3.3}$$

for any positive integer q, since the classical Dedekind sum is a Dedekind sum of type of (0, 0). This completes the proof.

4. Carlitz's sum $b_r(h,\ k)$.

Let $B_r(x)$ denote the rth Bernoulli polynomial and let $\bar{B}_r(x)=B_r(x-[x])$. In [2] Carlitz defined, for $(h,\ k)=1$,

$$C_r(h,\ k)=\sum_{n(\mathrm{mod}\ k)}\bar{B}_{p+1-r}(n/k)\bar{B}_r(hn/k). \tag{4.1}$$

In [1] it is stated that $C_r(h,\ k)$ is a Dedekind sum of type $(r-p,\ 1-r)$. Thus, by Theorem 1 of [1], we have

(4.2) $$\sum_{b(\mathrm{mod}\ d)} C_r(h+bk,\ dk)=d^{1-p}\sum_{t\mid d}\mu(t)t^{p-r}C_r(ht,\ k)\sigma_p(d/t).$$

Further, in [2], Carlitz defines the sums

(4.3) $$b_r(h,\ k)=\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}h^{r-s}C_s(h,\ k).$$

In [2] Carlitz proves (4.2) in the case when d is a prime q and then remarks that it does not seem possible a similar result for the sums in (4.3). We now state such a result, which is an analog of Subrahmanyam's identity.

THEOREM 10. *We have*

$$\sum_{m(\mathrm{mod}\ n)} b_r(h+mk,\ nk)=n^{1-p}\sum_{d\mid n} d^{p-r}\mu(d)\sigma_p(u/d)b_r(hd,\ k).$$

PROOF. This follows immediately from (4.2) and (4.3) and so we omit the details.

We can also give an analog of the Petersson-Knopp identity.

THEOREM 11. *We have*

$$\sum_{\substack{=ad\\ d>0}} d^{r-1}\sum_{m(\mathrm{mod}\ d)} b_r(ah+mk,\ dk)=n^{r-p}\sigma_p(n)b_r(n,\ k).$$

PROOF. If in Corollary 1.1 we take

$$f(m,\ n)=n^{r-p}\sigma_p(n)b_r(m,\ n) \quad\text{and}\quad g(m,\ n)=n^{r-1}\sum_{b(\mathrm{mod}\ n)} b_r(m+bk,\ nk),$$

then the result follows immediately.

5. The Sum $T(h,\ k)$.

For $x\geqq 0$ we define the fractional part of x by

$$\{x\}=x-[x].$$

It is known [3] that

$$\sum_{b(\mathrm{mod}\ d)}\left\{x+\frac{b}{q}\right\}=\frac{1}{2}(q-1)+\{qx\}.$$

Thus, if $q>1$, then

$$\sum_{b(\mathrm{mod}\ d)}\left\{x+\frac{n}{q}\right\}\neq\{qx\}$$

for all x. Therefore, the sum defined for relatively prime integers h and k by

(5.1) $$T(h,\ k)=\sum_{n(\mathrm{mod}\ k)}\left\{\frac{n}{k}\right\}\left\{\frac{nh}{k}\right\}$$

is not a sum of Dedekind type. However, $T(h,\ k)$ and $S(h,\ k)$ are closely related to one another.

THEOREM 12. *We have*

$$T(h, k)=S(h, k)+k/4. \tag{5.2}$$

PROOF. From [9] we have

$$\sum_{n(\mathrm{mod}\ k)}\left(\left(\frac{n}{k}\right)\right)=0$$

and if $(h, k)=1$, then

$$\sum_{n(\mathrm{mod}\ k)}\left(\left(\frac{nh}{k}\right)\right)=0.$$

Thus

$$\begin{aligned}T(h, k)&=\sum_{n(\mathrm{mod}\ k)}\left(\left(\left(\frac{n}{k}\right)\right)+\frac{1}{2}\right)\left(\left(\left(\frac{nh}{k}\right)\right)+\frac{1}{2}\right)\\&=\sum_{n(\mathrm{mod}\ k)}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{nh}{k}\right)\right)+\frac{1}{2}\sum_{n(\mathrm{mod}\ k)}\left(\left(\frac{n}{k}\right)\right)\\&\quad+\frac{1}{2}\sum_{n(\mathrm{mod}\ k)}\left(\left(\frac{n}{k}\right)\right)+\frac{k}{4}\\&=\sum_{n(\mathrm{mod}\ k)}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{nh}{k}\right)\right)+\frac{k}{4},\end{aligned}$$

which completes the proof.

THEOREM 13. *If $(h, k)=1$, then*

$$T(h, k)+T(k, h)=\frac{1}{4}(h+k-1)+\frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right). \tag{5.3}$$

PROOF. The reciprocity law for $S(h, k)$ is given by

$$S(h, k)+S(k, h)=-\frac{1}{4}(1/2)\left(h/k+k/h+\frac{1}{hk}\right) \tag{5.4}$$

(see [9]). Thus

$$T(h, k)+T(k, h)=S(h, k)+S(k, h)+(h+k)/4$$

and the result follows from (5.4) and completes the proof.

6. The Sum $S'(n, k)$.

We define the sum

$$S'(h, k)=\sum_{\substack{n(\mathrm{mod}\ k)\\(n, k)=1}}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{nh}{k}\right)\right). \tag{6.1}$$

Our first result is a relatiinship between the two sums $S(h, k)$ and $S'(h, k)$, which, unfortunately, is not symmetric in h and k.

THEOREM 14. *We have*

$$S'(h,k)=\sum_{d\mid k}\mu(d)S(h,k/d).$$

PROOF. We have

$$\begin{aligned} S'(h,k) &= \sum_{n(\text{mod }k)}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{nh}{k}\right)\right)\sum_{\substack{d\mid n\\ d\mid k}}\mu(d) \\ &= \sum_{d\mid k}\mu(d)\sum_{\substack{n(\text{mod }k)\\ d\mid n}}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{nh}{k}\right)\right) \\ &= \sum_{d\mid k}\mu(d)\sum_{n(\text{mod }k/d)}\left(\left(\frac{n}{k/d}\right)\right)\left(\left(\frac{hn}{k/d}\right)\right) \\ &= \sum_{d\mid k}\mu(d)S(h,k/d), \end{aligned}$$

which proves the result.

The next theorem gives three results that are the exact analogues of the corresponding results for the classical Dedekind sum.

THEOREM 15. (1) *If* $h'\equiv\pm h$ (mod k), *then* $S'(h',k)=\pm S'(h,k)$.

(2) $h\bar{h}\equiv\pm1$ (mod k), *then* $S'(\bar{h},k)=\pm S'(h,k)$.

(3) *If* $h^2+1\equiv0$ (mod k), *then* $S'(h,k)=0$.

PROOF OF (1). By Theorem 4.1 of [6] we see that $((x))$ satisfies $((\pm x))=\pm((x))$ and recall that $((x))$ is periodic of period 1 so that for all integers m, n and q we have

$$\left(\left(\frac{m+qn}{n}\right)\right)=\left(\left(\frac{m}{n}\right)\right).$$

Thus

$$\begin{aligned} S'(h',k) &= \sum_{\substack{n(\text{mod }k)\\ (n,k)=1}}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{h'n}{k}\right)\right) \\ &= \sum_{\substack{n(\text{mod }k)\\ (n,k)=1}}\left(\left(\frac{n}{k}\right)\right)\left(\left(\pm\frac{hn}{k}\right)\right) \\ &= \pm\sum_{\substack{n(\text{mod }k)\\ (n,k)=1}}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{hn}{k}\right)\right) \\ &= \pm S'(h,k). \end{aligned}$$

PROOF OF (2). By part (1), we need only prove the case $h\bar{h}\equiv1$ (mod k) since $h\bar{h}\equiv-1$ (mod k) implies $h(-\bar{h})\equiv1$ (mod k). Also $h\bar{h}\equiv1$ (mod k) implies that $(h,k)=1$. Thus hn covers a reduced residue system modulo k if n does.

Thus, by part (1),

$$\begin{aligned} S'(\bar{h}, k) &= \sum_{\substack{n \pmod k \\ (n,k)=1}} \left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{\bar{h}n}{k}\right)\right) \\ &= \sum_{\substack{n \pmod k \\ (n,k)=1}} \left(\left(\frac{hn}{k}\right)\right)\left(\left(\frac{\bar{h}hn}{k}\right)\right) \\ &= \sum_{\substack{n \pmod k \\ (n,k)=1}} \left(\left(\frac{nh}{k}\right)\right)\left(\left(\frac{n}{k}\right)\right) \\ &= S'(h, k). \end{aligned}$$

PROOF OF (3). If $h^2+1\equiv 0 \pmod k$, then $hh\equiv -1 \pmod k$. Thus, by part (2), $S'(h, k)=-S'(h, k)$ or $S'(h, k)=0$.

This completes the proof.

THEOREM 16. *If $\omega(k)$ counts the number of distinct prime factors of k and $\gamma(k)$ equals their product, then*

$$S'(1, k)=\frac{\varphi(k)}{12}+\frac{(-1)^{\omega(k)}}{6k^2}\varphi(k)\gamma(k).$$

PROOF. We have

$$\begin{aligned} S'(1, k) &= \sum_{\substack{n=1 \\ (n,k)=1}}^{k} \left(\left(\frac{n}{k}\right)\right)^2 \qquad (6.2) \\ &= \sum_{\substack{n=1 \\ (n,k)=1}}^{k-1} \left(\frac{n}{k}-\frac{1}{2}\right)^2 \\ &= \frac{1}{k^2}\sum_{\substack{n=1 \\ (n,k)=1}}^{k-1} n^2-\frac{1}{k}\sum_{\substack{n=1 \\ (u,k)=1}}^{k-1} n+\frac{1}{4}\sum_{\substack{n=1 \\ (n,k)=1}}^{k-1} 1 \\ &= \frac{1}{k^2}S_2(k)-\frac{1}{k}S_1(k)+\frac{1}{4}S_0(k), \end{aligned}$$

say. By definition we see that $S_0(k)=\varphi(k)$. The values of $S_1(k)$ and $S_2(k)$ are reasonably well-known (see [6, pp. 51 and 114]). For future reference we give them explicitly:

$$S_1(k)=\frac{k\varphi(k)}{2} \qquad (6.3)$$

and

$$S_2(k)=\frac{k^2\varphi(k)}{3}+(-1)^{\omega(k)}\frac{\varphi(k)\gamma(k)}{6}. \qquad (6.4)$$

If we combine (6.2), (6.3) and (6.4) we obtain the result of the theorem and complete the proof.

THEOREM 17. *If $\zeta=\exp(2\pi i/k)$, then*

$$S'(h,\ k)=\frac{1}{4k^2}\sum_{m=1}^{k-1}\frac{1+\zeta^m}{1-\zeta^m}\sum_{n=1}^{k-1}\frac{1+\zeta^n}{1-\zeta^n}c(hm+n,\ k),$$

where $c(m,\ n)$ is Ramanujan's sum (2.6).

PROOF. On p. 114 of [9] the following identity is given

$$\left(\left(\frac{n}{k}\right)\right)=\frac{1}{k}\sum_{m=1}^{k-1}\left(\frac{\zeta^m}{1-\zeta^m}+\frac{1}{2}\right)\zeta^{mn} \tag{6.5}$$

$$=\frac{1}{2k}\sum_{m=1}^{k-1}\frac{1+\zeta^m}{1-\zeta^m}\zeta^{mn}.$$

Thus, by (6.1) and (6.5), we have

$$S'(h,\ k)=\sum_{\substack{n \pmod k\\ (n,k)=1}}\frac{1}{2k}\sum_{r=1}^{k-1}\frac{1+\zeta^r}{1-\zeta^r}\frac{1}{2k}\sum_{s=1}^{k-1}\frac{1+\zeta^s}{1-\zeta^s}\zeta^{nr+hsn}$$

$$=\frac{1}{4k^2}\sum_{r=1}^{k-1}\frac{1+\zeta^r}{1-\zeta^r}\sum_{s=1}^{k-1}\frac{1+\zeta^s}{1-\zeta^s}\sum_{\substack{n \pmod k\\ (n,k)=1}}\zeta^{n(r+hs)}$$

$$=\frac{1}{4k^2}\sum_{r=1}^{k-1}\frac{1+\zeta^r}{1-\zeta^r}\sum_{s=1}^{k-1}\frac{1+\zeta^s}{1-\zeta^s}c(r+hs,\ k),$$

which proves the result.

COROLLARY 17.1. *We have*

$$S'(h,\ k)=\frac{1}{4k^2}\sum_{r=1}^{k-1}\sum_{s=1}^{k-1}\cot\left(\frac{\pi s}{k}\right)\cot\left(\frac{\pi s}{k}\right)c(r+hs,\ k).$$

PROOF. This follows immediately from Theorem 17 since

$$\cot\left(\frac{\pi r}{k}\right)=\frac{\zeta^r+1}{\zeta^r-1}\quad\text{and}\quad\cot\left(\frac{\pi s}{k}\right)=\frac{\zeta^s+1}{\zeta^s-1}.$$

The original aim in deriving the identities of Theorems 16 and 17 was to follow along the lines of various proofs of the reciprocity theorem for $S(h,\ k)$, (5.4), to prove a reciprocity theorem for $S'(h,\ k)$. Unfortunately, we have not succeeded in this goal. As seems to be indicated by the above results on $S'(h,\ k)$, as well as those that follow, the results for $S'(h,\ k)$ correspond closely to those for $S(h,\ k)$. Thus, a reciprocity theorem like (5.4) does not seem totally out of the question.

As another indication of how closely related $S(h,\ k)$ and $S'(h,\ k)$ we give the following congruence satisfied by the sum $S'(h,\ k)$.

THEOREM 18. *If* $k\geqq 3$ *and* $(h,\ k)=1$, *then*

$$6kS'(h,\ k)\equiv 2hk\varphi(k)+(-1)^{\omega(k)}h\varphi(\gamma(k))-3k\varphi(k)/2 \pmod 6. \tag{6.6}$$

PROOF. We have

$$S'(h,\ k)=\sum_{\substack{n \,(\mathrm{mod}\, k)\\ (n,k)=1}}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{nh}{k}\right)\right)$$

$$=\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\left(\left(\frac{n}{k}\right)\right)\left(\left(\frac{nh}{k}\right)\right)$$

$$=\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\left(\left(\frac{n}{k}-\frac{1}{2}\right)\right)\left(\left(\frac{nh}{k}\right)\right)$$

$$=\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\frac{n}{k}\left(\left(\frac{nh}{k}\right)\right)-\frac{1}{2}\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\left(\left(\frac{nh}{k}\right)\right).$$

Since $((x))$ is periodic of period 1 and also an odd function and since $(n,\ k)=1$ if and only if $(k-n,\ k)=1$ we see that

$$\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\left(\left(\frac{nh}{k}\right)\right)=0.$$

Thus

$$S'(h,\ k)=\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\frac{n}{k}\left(\frac{hn}{k}-\left(\frac{hn}{k}\right)-\frac{1}{2}\right)$$

$$=h\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\frac{n^2}{k^2}-\frac{1}{2}\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\frac{n}{k}\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}\frac{n}{k}\left(\frac{hn}{k}\right),$$

and so, using the notation above,

$$6k^2S'(h,\ k)=6hS_2(k)-3kS_1(k)-6k\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}n\left(\frac{nh}{k}\right).$$

Since

$$\sum_{\substack{n=1\\ (n,k)=1}}^{k-1}n\left(\frac{hk}{k}\right)$$

is an integer we see that

$$6k^2S'(h,\ k)\equiv 6hS_2(k)-3kS_1(k)\ (\mathrm{mod}\ 6k).$$

Thus, by (6.3) and (6.4), we have

$$6k^2S'(h,\ k)\equiv 2hk^2\varphi(k)+(-1)^{\omega(k)}h\varphi(k)\gamma(k)-3k^2\varphi(k)/2\ (\mathrm{mod}\ 6k). \tag{6.7}$$

Note that

$$\varphi(k)\gamma(k)=k\prod_{p\mid k}\left(1-\frac{1}{p}\right)\prod_{p\mid k}p=k\prod_{p\mid k}(p-1)=k\varphi(\gamma(k)). \tag{6.8}$$

Thus, by (6.7) and (6.8), we have

$$6k^2S'(h,\ k)\equiv 2hk^2\varphi(k)+(-1)^{\omega(k)}kh\varphi(\gamma(k))-3k^2\varphi(k)/2\ (\mathrm{mod}\ 6k),$$

or, dividing by k,

$$6kS'(h,\ k)\equiv 2hk\varphi(k)+(-1)^{\omega(k)}h\varphi(\gamma(k))-3k\varphi(k)/2\ (\mathrm{mod}\ 6),$$

which proves our result.

COROLLARY 18.1. *If* $k \geqq 3$ *and* $(h, k)=1$, *then*

1) $6kS'(h, k)$ *is an integer, and*

2) *if* $3 \mid h$, *then* $6kS'(h, k) \equiv 0 (\mathrm{mod}\ 3)$, *and so* $2kS'(h, k)$ *is an integer.*

PROOF. 1) Since $k \geqq 3$ implies that $\varphi(k)$ is an even integer we see that the right hand side of (6.6) is an integer and then so is the left hand side, which is $6kS'(h, k)$.

2) If $3 \mid h$, then 3 divides the right hand side of (6.6) and so 3 divides the left hand side of (6.6). Thus, since the left hand side of (6.6) is 3 times some integer we see that we must have that $2kS'(h, k)$ is an integer.

This completes the proof of the corollary.

As a final indication of the close correspondence between the two sums $S(h, k)$ and $S'(h, k)$ we give the S'-analogues for the identities (1.2) and (1.3).

We begin with the analogue of Subrahmanyam's identity. First we prove a lemma about the classical Dedekind sum.

LEMMA 19.1. *If* l *is a positive integer, then*

$$\sum_{b (\mathrm{mod}\ d)} S(h+blk, dk) = \sum_{rs \mid d} r\mu(s) S(hs, k(l, sr)),$$

PROOF. We have, by Lemma 9.2,

$$\begin{aligned}
\sum_{b (\mathrm{mod}\ d)} S(h+blk, dkk) &= \sum_{b (\mathrm{mod}\ d)} \sum_{n (\mathrm{mod}\ dk)} \left(\left(\frac{n}{dk}\right)\right)\left(\left(\frac{(h+blk)n}{dk}\right)\right) \\
&= \sum_{n (\mathrm{mod}\ dk)} \left(\left(\frac{n}{dk}\right)\right) \sum_{b (\mathrm{mod}\ d)} \left(\left(\frac{hn}{dk} + \frac{(nl)b}{d}\right)\right) \\
&= \sum_{n (\mathrm{mod}\ dk)} \left(\left(\frac{n}{dk}\right)\right)(d, nl)\left(\left(\frac{d}{(d, nl)} \cdot \frac{nh}{dk}\right)\right) \\
&= \sum_{r \mid d} r \sum_{\substack{n (\mathrm{mod}\ dk) \\ (d, nl)=r}} \left(\left(\frac{n}{dk}\right)\right)\left(\left(\frac{nh}{zk}\right)\right) \\
&= \sum_{r \mid d} \sum_{n (\mathrm{mod}\ dk)} \left(\left(\frac{n}{dk}\right)\right)\left(\left(\frac{nh}{rk}\right)\right) \sum_{\substack{rst=d \\ sr \mid nl}} \mu(s) \\
&= \sum_{rst=d} r\mu(s) \sum_{\substack{n (\mathrm{mod}\ dk) \\ sr/(l, sr) \mid n}} \left(\left(\frac{n}{dk}\right)\right)\left(\left(\frac{nh}{rk}\right)\right) \\
&= \sum_{rst=d} r\mu(s) \sum_{vrs/(l, rs) (\mathrm{mod}\ rstk)} \left(\left(\frac{v}{(l, sr)tk}\right)\right)\left(\left(\frac{ush}{(l, rs)k}\right)\right) \\
&= \sum_{rst=d} r\mu(s) \sum_{v (\mathrm{mod}\ tk(l, rs))} \left(\left(\frac{v}{tk(l, sr)}\right)\right)\left(\left(\frac{sthv}{tk(l, sr)}\right)\right)
\end{aligned}$$

$$= \sum_{rst=d} r\mu(s)S(hts,\ tk(l,\ sr))$$

$$= \sum_{rs\mid d} r\mu(s)S(hs,\ k(l,\ sr)),$$

by (3.3). This completes the proof.

THEOREM 19. *If d is square-free and $(d, k)=1$, then*

$$\sum_{b(\bmod d)} S'(h+bk,\ dk)= \sum_{t\mid dk} \mu(t) \sum_{s\mid d} \mu(s)S(hst,\ k)\sigma(d/s),$$

PROOF. We have, by Theorem 14 and (3.3),

$$\sum_{b(\bmod d)} S'(h+bk,\ dk)= \sum_{b(\bmod d)} \sum_{m\mid dk} \mu(m)S(h+bk,\ dk/m)$$

$$= \sum_{m\mid dk} \mu(m) \sum_{b(\bmod d)} S(hm+bmk,\ dk)$$

$$= \sum_{m\mid dk} \mu(m) \sum_{sr\mid d} r\mu(s)S(hms,\ k(m,\ sr)),$$

by Lemma 19.1. If $m\mid dk$, we see that since $(d, k)=1$ we can write $m=m_1m_2$, where $m_1\mid d$ and $m_2\mid k$. Then $(m, sr)=(m_1m_2, sr)=(m_1, sr)$ since $sr\mid d$ and $(m_2, d)=1$. Since d is square, free we have $(m_1, sr)=1$. Thus

$$\sum_{b(\bmod d)} S'(h+bk,\ dk)= \sum_{m_1m_2\mid dk} \mu(m_1m_2) \sum_{rs\mid d} r\mu(s)S(hsm_1m_2,\ k)$$

$$= \sum_{m\mid dk} \mu(m) \sum_{s\mid d} \mu(s)S(hs,\ mk)\sigma(d/s),$$

which completes the proof.

We now give the S'-analogue of the Petersson-Knopp identity (1.3).

THEOREM 20. *If n is square-free and $(n, k)=1$, then*

$$\sum_{\substack{ad=n\\ d>0}} \sum_{b(\bmod d)} S'(ah+bk,\ dk)=\sigma(n)S'(nh,\ nk).$$

PROOF. We have, by Theorem 19,

$$\sum_{\substack{ad=n\\ d>0}} \sum_{b(\bmod q)} S'(ah+bk,\ dk)= \sum_{\substack{ad=n\\ d>0}} \sum_{t\mid dk} \mu(t) \sum_{s\mid d} \mu(s)S(ahst,\ k)\sigma(d/s).$$

If we let $m=as$, then we have

$$\sum_{\substack{ad=n\\ d>0}} \sum_{b(\bmod d)} S'(ah+bk,\ dk)= \sum_{\substack{m\mid n\\ md=ns}} \sum_{t\mid dk} \mu(t) \sum_{s\mid m} \mu(s)S(mht,\ k)\sigma(n/m) \tag{6.9}$$

$$= \sum_{\substack{m\mid n\\ md=ns\\ m>0}} \sum_{s\mid dk} \mu(t)S(mht,\ k)\sigma(n/m) \sum_{s\mid m} \mu(s)$$

$$= \sum_{t\mid nk} \mu(t)S(ht,\ k)\sigma(n),$$

by Theorem 4.6 of [6].

By (3.3) we have

$$\sum_{t\mid nk}\mu(t)S(ht,\ k)=\sum_{t\mid nk}\mu(t)S(nh,\ nk/t) \tag{6.10}$$

$$=S'(nh,\ nk),$$

by Theorem 14. If we combine (6.9) and (6.10) we obtain the result and complete the proof of the theorem.

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