

ON SPAN AND INVERSE LIMITS

By

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1. Introduction.

A compact metric space is called a *compactum* and a connected compactum is called a *continuum*. All maps in this paper are continuous. Let $f: X \rightarrow Y$ be a map between continua. Ingram [2] and Lelek [11] defined the *span*, *semispan*, *surjective span*, and *surjective semispan* of f by the following formulas (the map $p_i: X \times X \rightarrow X$ denotes the projection to the i -th factor, $i=1, 2$).

$$\tau = \sigma, \sigma_0, \sigma^*, \sigma_0^*.$$

$$\tau(f) = \sup \left\{ c \geq 0 \left| \begin{array}{l} \text{there exists a continuum } Z \subset X \times X \text{ such} \\ \text{that } Z \text{ satisfies the condition } \tau \text{ and} \\ d(f(x), f(y)) \geq c \text{ for each } (x, y) \in Z \end{array} \right. \right\},$$

where the condition τ is:

$$\begin{aligned} p_1(Z) = p_2(Z) & \text{ if } \tau = \sigma, & p_1(Z) \supset p_2(Z) & \text{ if } \tau = \sigma_0, \\ p_1(Z) = p_2(Z) = X & \text{ if } \tau = \sigma^*, & p_1(Z) = X & \text{ if } \tau = \sigma_0^*. \end{aligned}$$

The span of a continuum X is defined by $\sigma(id_X)$. The other cases are similar. In the same way, we can define the *symmetric span* of f by the formula

$$s(f) = \sup \left\{ c \geq 0 \left| \begin{array}{l} \text{there exists a continuum } Z \subset X \times X \text{ such that} \\ Z \text{ is symmetric (i. e. } (x, y) \in Z \text{ iff } (y, x) \in Z) \\ \text{and } d(f(x), f(y)) \geq c \text{ for each } (x, y) \in Z \end{array} \right. \right\}.$$

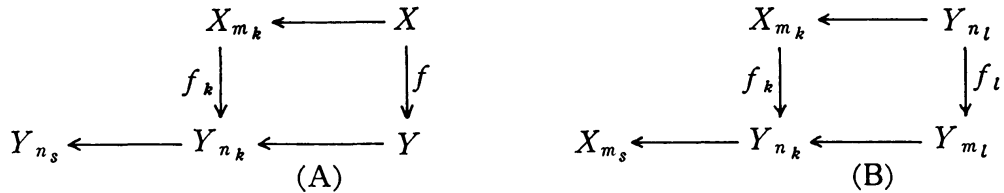
It is a mapping version of symmetric span of a continuum due to J. F. Davis [1].

Let $X = \varprojlim (X_n, p_{n, n+1})$ be a continuum, where $p_{n, n+1}: X_{n+1} \rightarrow X_n$. Ingram [2] and [4] showed that $\sigma(X) = 0$ if and only if there exists a cofinal subsequence $(n_i)_{i \geq 1}$ such that $\lim_j \sigma(p_{n_i, n_j}) = 0$ for each $i \geq 1$. In section 2 of this paper, we will prove a mapping version of this theorem. H. Cook proved essentially that the symmetric span of the dyadic solenoid is zero ([1], p. 134), while its span is positive. The author wishes to thank to the referee for pointing out this fact. In section 3, we generalize this to the poly-adic solenoid. Let f and $g: X \rightarrow Y$ be maps. $d(f, g)$ denotes $\sup\{d(f(x), g(x)) \mid x \in X\}$.

2. Span and a limit of maps.

Let $X = \varprojlim (X_n, p_{n, n+1})$ and $Y = \varprojlim (Y_n, q_{n, n+1})$ be compacta, where all X_n and Y_n are polyhedra and both of $p_{n, n+1}: X_{n+1} \rightarrow X_n$ and $q_{n, n+1}: Y_{n+1} \rightarrow Y_n$ are surjective for each $n \geq 0$. The maps $p_n: X \rightarrow X_n$ and $q_n: Y \rightarrow Y_n$ denote the projection maps. Under these notations, Mioduszewski showed the following [15].

THEOREM 1. 1) For every sequence (ϵ_n) of positive numbers with $\lim \epsilon_n = 0$, there exist cofinal increasing subsequences (m_k) and (n_k) and maps $f_k: X_{m_k} \rightarrow Y_{n_k}$ such that diagrams (A) and (B) are ϵ_k -commutative for each $s \leq k \leq l$.



2) Conversely, if we are given diagram (B), then we can find a map $f: X \rightarrow Y$ which satisfies diagram (A) for each k . If all f_k 's are surjective, f can be constructed so as to be surjective.

Notice that the map f is defined by $q_{n_s} f = \lim_k q_{n_s n_k} f_k p_{m_k}$.

We say that f is weakly induced by the sequence (f_k) . This terminology is due to Oversteegen and Tymchatyn [13].

THEOREM 2. Let $f: X \rightarrow Y$ be a map between continua which is weakly induced by a sequence $(f_k: X_{m_k} \rightarrow Y_{n_k})$. Then,

$\tau(f) = 0$ if and only if there exists a cofinal subsequence (n_{k_j}) of (n_k) such that $\lim_j \tau(q_{n_{k_i} n_{k_j}} f_{k_j}) = 0$ for each i . Where, $\tau = \sigma, \sigma^*, \sigma_0, \sigma_0^*$, and s .

The basic idea of the proof is in [2] and [3]. But we need some preparations. Throughout this section, τ denotes $\sigma, \sigma_0, \sigma^*, \sigma_0^*$, and s unless otherwise stated.

PROPOSITION 3. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be maps.

1) $\tau(gf) \leq \tau(g)$. 2) If $\tau(f) = 0$, then $\tau(gf) = 0$.

PROPOSITION 4. Let $(f_n: X \rightarrow Y)$ be a sequence of maps which converges uniformly to a map $f: X \rightarrow Y$. Then $\tau(f) = \lim \tau(f_n)$.

The proof of the above two propositions are easy and will be omitted.

PROPOSITION 5. 1) Let X_n 's and X be continua in a metric space M and let Y_n 's and Y be continua in a metric space N . Suppose that $f: X \rightarrow Y$, $f_n: X_n \rightarrow Y_n$, $p_n: X \rightarrow X_n$, and $q_n: Y \rightarrow Y_n$ satisfy the following conditions.

- a) $\text{Lim } X_n = X$, $\text{Lim } Y_n = Y$. Both of $X \cup \bigcup_{n=1}^{\infty} X_n$ and $Y \cup \bigcup_{n=1}^{\infty} Y_n$ are compact.
- b) Both of the maps p_n and q_n are $1/2^n$ -translation (that is, $d(x, p_n(x)) < 1/2^n$ for each $x \in X$ etc.).
- c) There exists a decreasing sequence of positive numbers ϵ_n 's with $\lim \epsilon_n = 0$, such that $d(q_n f, f_n p_n) < \epsilon_n$.
- d) Define $F: X \cup \bigcup X_n \rightarrow Y \cup \bigcup Y_n$ by $F|X = f$, $F|X_n = f_n$. Then F is well defined and continuous.

Then $\tau(f) = \lim \tau(f_n)$.

2) We can replace condition d) by

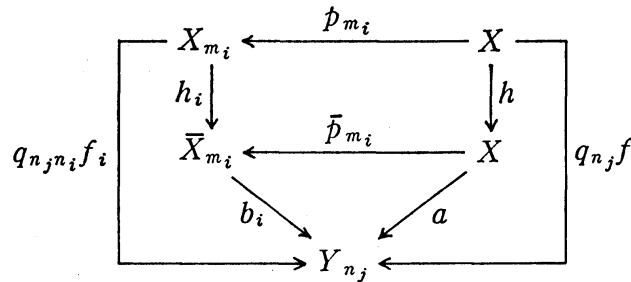
e) Each p_n is surjective.

Reasoning the same way as in [10, 3.1] and [5, 2.1], we can show two inequalities; $\limsup \tau(f_n) \leq \tau(f) \leq \liminf \tau(f_n)$, which imply the conclusion.

PROOF OF THEOREM 2. To simplify the notations, a cofinal subsequence of (n_i) is also denoted by (n_i) . First we assume that $\tau(f) = 0$. Take any subsequence (n_i) and an integer $j > 0$. It suffices to prove that $\lim_i \tau(q_{n_j n_i} f_i) = 0$. Let A be a compactum satisfying the following conditions.

- 1) $A = \bar{X} \cup \bigcup \bar{X}_{m_k}$, where \bar{X} and \bar{X}_{m_k} are homeomorphic to X and X_{m_k} respectively. $\bar{X} \cap \bar{X}_{m_k} = \emptyset = \bar{X}_{m_k} \cap \bar{X}_{m_l}$ for each $k \neq l$.
- 2) Let $h: X \rightarrow \bar{X}$ and $h_k: X_{m_k} \rightarrow \bar{X}_{m_k}$ be homeomorphisms. There exists an ϵ_k -translation $\bar{p}_{m_k}: \bar{X} \rightarrow \bar{X}_{m_k}$ satisfying $h_k p_{m_k} = \bar{p}_{m_k} h$.
- 3) $\text{Lim } \bar{X}_{m_k} = \bar{X}$.

That such space A exists is well known. As each bonding map is surjective, we can take each \bar{p}_{m_k} to be surjective. Consider the following diagram.



Where, $a = q_{n_j} f h^{-1}$ and $b_i = q_{n_i n_j} f_i h_i^{-1}$. Then,

$$4) \quad d(a, b_i \bar{p}_{m_i}) = d(q_{n_j} f h^{-1}, q_{n_j n_i} f_i h_i^{-1} h_i p_{m_i} h^{-1}) \\ = d(q_{n_j n_i} q_{n_i} f, q_{n_j n_i} f_i p_{m_i}) < \varepsilon_i$$

by the ε_i -commutativity of (A). It is easy to see that $\tau(a) = \tau(q_{n_j} f)$ and $\tau(b_i) = \tau(q_{n_j n_i} f_i)$. Applying Proposition 3.2), Proposition 5 and by condition 4), we have

$$\lim_i \tau(q_{n_j n_i} f_i) = \lim_i \tau(b_i) = \tau(a) = \tau(q_{n_j} f) = 0.$$

Next we assume that a cofinal subsequence satisfies the hypothesis. By Proposition 4 and Proposition 3.1),

$$\tau(q_{n_j} f) = \lim_i \tau(q_{n_j n_i} f_i p_i) \\ \leq \lim_i \tau(q_{n_j n_i} f_i) = 0.$$

To show that $\tau(f) = 0$, we take any continuum Z in $X \times X$ satisfying condition τ . There exists a point $(x^j, y^j) \in Z$ such that $q_{n_j} f(x^j) = q_{n_j} f(y^j)$, because $\tau(q_{n_j} f) = 0$ for each j . We can assume that $(x^j, y^j) \rightarrow (x, y)$ as $j \rightarrow \infty$. If $j < i$,

$$q_{n_j} f(x^i) = q_{n_j n_i} q_{n_i} f(x^i) \\ = q_{n_j n_i} q_{n_i} f(y^i) = q_{n_j} f(y^i).$$

Tending i to infinity, we have

$$q_{n_j} f(x) = q_{n_j} f(y) \quad \text{for each } j \text{ and hence } f(x) = f(y).$$

This completes the proof.

THEOREM 6. *Suppose that X, Y, f and f_n satisfy the hypothesis of Theorem 2. If there exists a cofinal subsequence (n_i) such that $\lim_j \tau(f_{n_i} p_{n_i n_j}) = 0$, then $\tau(f) = 0$.*

PROOF. For each $s < i$, $\tau(q_{n_s n_i} f_{n_i} p_{n_i}) = 0$, because by Proposition 3,

$$\tau(f_{n_i} p_{n_i}) = \lim_j \tau(f_{n_i} p_{n_i n_j} p_{n_j}) \\ \leq \lim_j \tau(f_{n_i} p_{n_i n_j}) = 0.$$

Using the ε_j -commutativity of the diagram (A) and (B), we have $\tau(q_{n_i} f) \leq \tau(q_{n_i n_j} f_j p_{n_j}) + 2\varepsilon_j = 2\varepsilon_j$ for each $j > i$. Therefore $\tau(q_{n_i} f) = 0$ for each i and $\tau(f) = 0$.

COROLLARY 7 [8 and 10]. *Let $X = \varprojlim (X_n, p_{n n+1})$ be a continuum represented as the inverse limit of continua and onto bonding maps. Then the followings are equivalent.*

- 1) $\tau(X) = 0$.
- 2) There exists a cofinal subsequence (n_i) such that $\lim_j \tau(p_{n_i n_j}) = 0$ for each i .

3) For each n , $\tau(p_n)=0$.

In Theorem 2 and 6, no conditions on p_n 's and q_n 's, on X_n 's and Y_n 's are required. If we add some conditions, the followings are obtained.

PROPOSITION 8. Suppose X, Y, f, f_n, p_n and q_n satisfy the hypothesis of Theorem 2. Moreover assume that:

1) All p_n 's are monotone. or

2) X is tree-like and each X_n is a finite tree. Each p_n is an open onto map.

$\tau=\sigma, \sigma_0$, and s .

If $\tau(f)=0$, then $\lim_n \tau(f_n)=0$.

PROOF. 1) For each $n \geq 0$ and for each continuum $Z \subset X_n \times X_n$ satisfying τ , $(p_n \times p_n)^{-1}(Z)$ is a continuum in $X \times X$ satisfying τ . There exists a $(x, y) \in (p_n \times p_n)^{-1}(Z)$ such that $f(x)=f(y)$. Then

$$\begin{aligned} d(f_n p_n(x), f_n p_n(y)) &\leq d(f_n p_n(x), q_n f(x)) + d(q_n f(y), f_n p_n(y)) \\ &\leq 2\varepsilon_n. \end{aligned}$$

Hence $\tau(f_n) \leq 2\varepsilon_n$ and this completes the proof.

2) We need the following theorem for the proof.

THEOREM 9 [14, p. 189]. Let X and Y be compacta and $f: X \rightarrow Y$ be a light open map from X onto Y . For each dendrite D in Y , there exists a dendrite D_1 in X such that $f(D_1)=D$ and $f|_{D_1}$ is a homeomorphism on D .

Using this Theorem, 2) is shown as follows.

Let n be a positive integer. There exists a continuum W_n and maps $r_n: X \rightarrow W_n, s_n: W_n \rightarrow X_n$ such that r_n is monotone and s_n is light open and $s_n r_n = p_n$. As X_n is a tree, there exists a dendrite T_n in W_n such that $s_n(T_n)=X_n$ and $s_n|_{T_n}$ is a homeomorphism by Theorem 9. For each continuum $Z \subset X_n \times X_n$ satisfying the condition τ ($\tau=\sigma, \tau_0$, and s), the set $(s_n \cdot (r_n|_{r_n^{-1}(T_n)}) \times s_n \cdot (r_n|_{r_n^{-1}(T_n)}))^{-1}(Z)$ is a continuum in $X \times X$ which also satisfies the condition τ . Arguing the same way as in 1), we obtain the conclusion.

An easy example shows that the converse of Proposition 8 does not hold. But by Theorem 6 and Proposition 3, we can prove:

If $\tau(f_n)=0$ for each n , then $\tau(f)=0$.

Monotone maps preserve span zero ([3], theorem 2). The author recently proved that open maps also preserve span zero [7]. Hence,

COROLLARY 10. Let $X = \varprojlim (X_n, p_{n, n+1})$ be a continuum as the inverse limit of continua and onto bonding maps. Suppose that all $p_{n, n+1}: X_{n+1} \rightarrow X_n$'s are monotone or all $p_{n, n+1}$'s are open. Then $\sigma(X) = 0$ if and only if $\sigma(X_n) = 0$ for each n .

3. Some examples.

In this section, we are concerned with circle-like continua.

PROPOSITION 11. Let $X = \varprojlim (X_n, p_{n, n+1})$, $Y = \varprojlim (Y_n, q_{n, n+1})$ be circle-like continua and $f: X \rightarrow Y$ be a map which is weakly-induced by a sequence of maps $(f_n: X_n \rightarrow Y_n)$. If all X_n 's and Y_n 's are simple closed curves and all $q_{n, n+1}$ are essential, then the followings are equivalent.

- a) $\sigma(f) = 0$.
- b) There exists a subsequence (n_j) such that $f_{n_j} \cong 0$ for each j .

As was shown in [5, 2.2], a map $f: X \rightarrow S^1$ from a continuum X to the unit circle S^1 is essential if and only if $\sigma(f) = \text{diam } S^1 > 0$. Using this result, this proposition is easily proved. (See also [16]).

H. Cook has essentially proved that the symmetric span of the dyadic solenoid is zero ([1], p. 134). Here we consider general p -adic solenoid. Let $p = (p_1, p_2, \dots)$ be a sequence of positive integers. The p -adic solenoid S_p is defined by the inverse limit of the unit circles $X_n = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, whose bonding maps $f_n: X_{n+1} \rightarrow X_n$ are defined by the formulas; $f_n(z) = z^{p_n}$. We show the following result.

PROPOSITION 12. Let S_p be the p -adic solenoid, $p = (p_1, p_2, \dots)$. Then $s(S_p) > 0$ if and only if there exists a positive integer N such that for each $n > N$, p_n is odd.

First we calculate the symmetric span of maps between the unit circles.

LEMMA 13. Let $f: S^1 \rightarrow S^1$ be the map between the unit circles defined by $f(z) = z^n$, where n is a positive integer. Then $s(f) = 0$ or $\text{diam } S^1 (= 2)$. Also, $s(f) = 0$ if and only if n is even.

PROOF. $S^1 \times S^1$ is obtained from the rectangle $[0, 2\pi] \times [0, 2\pi]$ by identifying $(x, 0)$ and $(x, 2\pi)$, $(0, y)$ and $(2\pi, y)$ ($0 \leq x, y \leq 2\pi$). Let $F = \{(x, y) \in S^1 \times S^1 \mid f(x) = f(y)\}$. Then F contains diagonal set. Let

$$\begin{aligned} A_i &= [2\pi \cdot (i-1)/n, 2\pi \cdot i/n] \times 0, \\ B_i &= 0 \times [2\pi \cdot (i-1)/n, 2\pi \cdot i/n], \end{aligned}$$

$$C_i = [2\pi \cdot (i-1)/n, 2\pi \cdot i/n] \times 2\pi,$$

$$D_i = 2\pi \times [2\pi \cdot (i-1)/n, 2\pi \cdot i/n], \quad i=1, \dots, n.$$

A_i and C_i, B_i and D_i are identified in $S^1 \times S^1$ respectively. Let X_i be the tetragon bounded by F and A_i and D_{n+1-i} in $[0, 2\pi] \times [0, 2\pi]$, and \tilde{X}_i be the set in $S^1 \times S^1$ obtained from X_i by the identification. Notice that $s(f) > 0$ if and only if there exists a continuum Z in $S^1 \times S^1$ such that Z is symmetric and $Z \cap F = \emptyset$.

First we assume that n is odd. Then $(\pi, 0) (= (\pi, 2\pi))$ in $S^1 \times S^1$ and $(0, \pi) (= (2\pi, \pi))$ in $S^1 \times S^1$ do not belong to F . So we can join $(\pi, 0)$ and $(0, \pi)$ by the symmetric arc $A = \{(x, y) \in S^1 \times S^1 \mid |\arg x - \arg y| = \pi\}$. It is easy to see that $d(f(x), f(y)) = \text{diam } S^1 = 2$ for each (x, y) of A . Hence $s(f) = 2$.

Next we assume that n is even. Suppose that $s(f) > 0$. Then by the above remark, there exists a continuum Z in $S^1 \times S^1$ such that Z is symmetric and $Z \cap F = \emptyset$. For each $i=1, \dots, n$, let $Z_i = Z \cap \tilde{X}_i$. Then $Z_i^{-1} = Z \cap \tilde{X}_i^{-1}$. Let j be the first integer such that $Z_j \neq \emptyset$.

We claim that $Z_j \cap Z_j^{-1} = \emptyset$. If $j=1$, $\tilde{X}_1 \cap \tilde{X}_1^{-1} \subset (\text{diagonal}) \subset F$. Since $Z \cap F = \emptyset$, $Z_1 \cap Z_1^{-1} = \emptyset$. Assume $j > 1$. As n is even, $i \neq n+1-i$ for each integer. Hence $B_i \cap D_{n+1-i} \subset F$, $A_i \cap C_{n+1-i} \subset F$, and we have $\tilde{X}_j \cap \tilde{X}_j^{-1} \subset F$. As $Z \cap F = \emptyset$, we have the claim.

As Z is connected, $Z_j \cup Z_j^{-1} \neq Z$. If Z does not intersect $\text{Int}_{S^1 \times S^1}(\tilde{X}_{n+1-j}^{-1})$, then $Z_j \cup Z_j^{-1}$ is a clopen set in Z , because \tilde{X}_{n+1-j}^{-1} is the only one of the \tilde{X}_i 's which meets \tilde{X}_j in $S^1 \times S^1 - F$. So $Z \cap \text{Int } \tilde{X}_{n+1-j}^{-1} \neq \emptyset$. By the similar argument, we see that $\tilde{X}_j \cup \tilde{X}_{n+1-j}$ does not intersect any other \tilde{X}_i 's and \tilde{X}_i^{-1} 's in $S^1 \times S^1 - F$ and $\tilde{X}_j \neq \tilde{X}_{n+1-j}$. Therefore $Z_j \cup Z_{n+1-j}^{-1}$ is a clopen proper subset of Z . This is a contradiction which completes the proof.

PROOF OF PROPOSITION 12.

First we assume $s(S_p) > 0$. If there exists a cofinal subsequence (n_i) such that p_{n_i} is even, $s(f_{n_i+1, n_i+1}) = 0$ by Lemma 13. By Corollary 7, $s(S_p) = 0$, a contradiction.

Next suppose that there exists a positive integer N satisfying the hypothesis. Then for each $m > n > N$, $s(f_{nm}) = 2$. Therefore $\lim_m s(f_{nm}) > 0$ and $s(S_p) > 0$, as desired.

This completes the proof.

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