TSUKUBA J. MATH.
Vol. 12 No. 1, (1988) 231—233

ISOMETRIC IMMERSION OF RIEMANNIAN HOMOGENEOUS MANIFOLDS

By

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1. Introduction.

Bang-Yen Chen has introduced the notion of isometric immersion of finite type and proved that an equivariant isometric immersion of a compact Riemannian homogeneous manifold into a Euclidean space is of finite type [1].

In this paper we will prove the following theorem.

THEOREM. *Let M be a compact connected Riemannian homogeneous manifold with irreducible isotropy action. For an equivariant isametric immersion f of M into a Euclidean space E^N (considered as a Euclidean vector space) there exist a finite number of vector subspaces $E_0, E_1, \cdots, E_r$ of E^N, isometric immersions f_i of 1-type of M into E_i $(i=1, \cdots, r)$, constant vector v_0 in E_0 and positive constant $a_1, \cdots, a_r$ so that*

(1) $E^N=E_0+E_1+\cdots+E_r$ *(Euclidean direct sum)*

(2) $f=v_0+a_1f_1+\cdots+a_rf_r$.

REMARK. $a_1, \cdots, a_r$ satisfy $\sum_{i=1}^{r} a_i^2=1$.

2. Proof of Theorem.

Let M be a compact connected Riemannian homogeneous manifold with irreducible isotropy action. Let $G=I_0(M)$ be the identity component of the group of all isometries of M. G is a compact Lie group and acts on M transitively.

Let f be an equivariant isometric immersion of M into a Euclidean space E^N. Then there exists a Lie homomorphism ϕ of G into the isometry group $I(E^N)$ of E^N such that

$$f(g(p))=\phi(g)(f(p))$$

for any $g\in G$ and $p\in M$.

Since an isometric transformation of E^N is decomposed into a product of an orthogonal transformation and a parallel translation, we have a Lie homomorphism

Received April 21, 1987.

ρ of G into $SO(E^N)$ and an E^N-valned function α on M such that

$$f(g(p))=\rho(g)(f(p))+\alpha(g)$$

for any $g\in G$ and $p\in M$, where $SO(E^N)$ is the special orthogonal group of E^N.

Since (ρ, E^N) is a representation of a compact Lie group G, (ρ, E^N) is decomposed into the sum of irreducible subrepresentations $(\rho_1, E_1), \cdots, (\rho_m, E_m)$ such that

$$E^N=E_1+\cdots+E_m \qquad \text{(Euclidean direct sum).}$$

Let f'_i and α_i be the E_i-components of f and α respectively. Then we have

$$f'_i(g(p))=\rho_i(g)(f'_i(p))+\alpha_i(g) \qquad (i=1, \cdots, m)$$

for any $g\in G$ and $p\in M$.

The function α_i satisfies

$$\alpha_i(g_1g_2)=\rho_i(g_1)(\alpha_i(g_2))+\alpha_i(g_1)$$

for $g_1, g_2\in G$. Define a vector $v_i\in E_i$ by

$$v_i=\int_G \alpha_i(g)dg$$

where dg is the normalized Haar measure on G. Then we have

$$v_i=\rho_i(g)(v_i)+\alpha_i(g)$$

for $g\in G$. Put $h_i(p)=f'_i(p)-v_i$. h_i is an E_i-valued function on M and satisfies

$$h_i(g(p))=\rho_i(g)(h_i(p))$$

for $g\in G$ and $p\in M$.

Take a point $o\in M$ fixed and let K be the isotropy subgroup of G at the point o. In the following of this paper we identify M with the homogeneous space G/K in a natural way. In order to calculate the Laplacian Δh_i of the function h_i, we introduce a biinvariant Riemannian metric on G so that the canonical projection of G onto $M=G/K$ to be a Riemannian submersion. Let $X_1, \cdots, X_n$ ($n=\dim G$) be orthonormal basis of the Lie algebra of G which is the tangent space $T_e(G)$ of G at the unit element e as a vector space. Then the Laplacian Δh_i is calculated in the following way (See [2]):

$$\begin{aligned}\Delta h_i(p)&=-\sum_{\alpha=1}^n \frac{d^2}{dt^2}\Big|_{t=0} h_i(\exp tX_\alpha(p))\\ &=-\sum_{\alpha=1}^n \rho_i(X_\alpha)^2(h_i(p))\end{aligned}$$

where we denote the induced homomorphism of the Lie algebra $T_e(G)$ into the Lie algebra $\mathfrak{so}(E_i)$ of $SO(E_i)$ by the same ρ_i. Then $\rho_i(X_\alpha)$ is a skew-symmetric linear transformation of E_i and $\sum_\alpha \rho_i(X_\alpha)^2$ is a symmetric linear transformation.

For $g \in G$ we write $(a_{\alpha\beta})$ the matrix representation of $\mathrm{Ad}(g)$ with respect to the basis $X_1, \cdots, X_n$, that is,

$$\mathrm{Ad}(g)X_\beta = \sum_\alpha a_{\alpha\beta} X_\alpha .$$

Then the matrix $(a_{\alpha\beta})$ is an orthogonal matrix and we have

$$\begin{aligned}\rho_i(g)(\textstyle\sum_\alpha \rho_i(X_\alpha)^2)\rho_i(g^{-1}) &= \textstyle\sum_\alpha \rho_i(\mathrm{Ad}(g)X_\alpha)^2 \\ &= \textstyle\sum_{\alpha,\beta,\gamma} a_{\beta\alpha} a_{\gamma\alpha} \rho_i(X_\beta)\rho_i(X_\gamma) \\ &= \textstyle\sum_\alpha \rho_i(X_\alpha)^2 .\end{aligned}$$

Therefore, by Schur's lemma, there exists a constant λ_i such that

$$\sum_\alpha \rho_i(X_\alpha)^2 = -\lambda_i I_i$$

where I_i is the identity of E_i. Since $\rho_i(X_\alpha)$ is skew-symmetric, λ_i is non-negative. Then we obtain

$$\Delta h_i = \lambda_i h_i .$$

If $\lambda_i = 0$, h_i is constant and thus f_i' is also constant. We denote by E_0 the sum of these E_i and v_0 the sum of these constant f_i' for which $\lambda_i = 0$. If λ_i is positive, the induced metric $|df_i'|^2$ on M is invariant under the action of G. Since the linear isotropy representation is irreducible, $|df_i'|^2$ is a constant multiple of the original Riemannian metric on M, that is, there exists a positive constant a_i such that $|df_i'|^2 = a_i^2 |df|^2$. Put $f_i = a_i^{-1} f_i'$. Then f_i is an isometric immersion of 1-type of M into E_i. Reordering those E_i and f_i for which $\lambda_i > 0$, we complete the proof of the theorem.

References

[1] Bang-Yen Chen, Total Mean Curvature and Submanifolds of Finite Type, World Scientific (1984).

[2] Wallach, N., Minimal immersions of symmetric spaces into spheres, Symmetric Spaces, Pure and Applied Math. Series B, Dekker (1972).

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