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LARGE DEVIATION PRINCIPLE FOR DIFFUSION PROCESSES

By

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§1. Introduction.

Let $(\Omega, \boldsymbol{F}, P)$ be a probability space with an increasing family $\{\boldsymbol{F}_t;\ t\geqq 0\}$ of sub-σ-algebras of $\boldsymbol{F}$ and let $W(t)$ be a d-dimensional Brownian motion process adapted to $\boldsymbol{F}_t$. Then, we consider, on the Euclidean d-space R^d, the system of the stochastic differential equations;

$$(1.1)\qquad dX^{\varepsilon}(t)=b(X^{\varepsilon}(t))dt+\varepsilon^{1/2}\sigma(X^{\varepsilon}(t))dW(t)\,,\qquad X^{\varepsilon}(0)=x\in R^d\,,$$

where $\varepsilon>0$ is a small parameter, $b(x)=(b_i(x))_{i=1,\dots,d}$ is a d-vector function and $\sigma(x)=(\sigma_{ij}(x))_{i,j=1,\dots,d}$ is a $d\times d$-matrix function. Throughout the paper we assume that $b(x)$ and $\sigma(x)$ satisfy a local Lipschitz condition with respect to $x\in R^d$.

We shall study the behavior of $X^{\varepsilon}(t)$ as $\varepsilon\to 0$. This behavior will depend on the behavior of solutions of the dynamical system;

$$(1.2)\qquad dX^0(t)=b(X^0(t))dt\,,\qquad X^0(0)=x\in R^d\,,$$

The system (1.1) can be considered as a small random perturbation of (1.2), with randomness expressed by a diffusion term $\varepsilon^{1/2}\sigma dW$. Set $a(x)=\sigma(x)\sigma^*(x)$, where the * means transpose. When $a(x)$ is uniformly elliptic and bounded, Freidlin and Wentzell [2] and also Friedman [3] obtain the large deviation principle for $X^{\varepsilon}(t)$. The former assumes the boundedness condition on $b(x)$ and $\sigma(x)$ together with a global Lipschitz condition in R^d. The latter assumes the boundedness condition on $a(x)$ and $b(x)$ together with a global Hölder condition with exponent $0<\alpha\leqq 1$ in the whole space R^d. Recently, under the positive definiteness condition on $a(x)$, Stroock [6] shows the large deviation principle, only assuming that $b(x)$ and $\sigma(x)$ satisfy a global Lipschitz condition in R^d.

The first purpose of this paper is to obtain a large deviation principle for $X^{\varepsilon}(t)$ under a satisfaction of some growth restriction on $b(x)$ and $\sigma(x)$. The most illustrative application is Theorem 3.1 with Example 2.1. It asserts that, in the problem of large deviations, the classical condition of linear growth in the phase variable of the coefficients can be weakened by allowing a logarithmic

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factor.

Next, let $f(u)$ and $g(u)$ be scalar functions which satisfy a local Lipschitz condition with respect to $u\in R^1$, and introduce the function

$$F(u)=\int_0^u f(s)ds .$$

Then, we consider the following systems of the two-dimensional stochastic differential equations;

$$X^\varepsilon(t)=(X_1^\varepsilon(t),\ X_2^\varepsilon(t)),$$

$$\text{(1.3)}\qquad \begin{cases} dX_1^\varepsilon(t)=[X_2^\varepsilon(t)-F(X_1^\varepsilon(t))]dt ,\\ dX_2^\varepsilon(t)=-g(X_1^\varepsilon(t))dt+\varepsilon^{1/2}dw(t) ,\end{cases}$$

$$X^\varepsilon(0)=x\in R^2 ,$$

$$\tilde{X}^\varepsilon(t)=(\tilde{X}_1^\varepsilon(t),\ \tilde{X}_2^\varepsilon(t)),$$

$$\text{(1.4)}\qquad \begin{cases} d\tilde{X}_1^\varepsilon(t)=[\tilde{X}_2^\varepsilon(t)-\varepsilon F(\tilde{X}_1^\varepsilon(t))]dt ,\\ d\tilde{X}_2^\varepsilon(t)=-g(\tilde{X}_1^\varepsilon(t))dt+\varepsilon^{1/2}dw(t) ,\end{cases}$$

$$\tilde{X}^\varepsilon(0)=x\in R^2 ,$$

where $w(t)$ is a one-dimensional Brownian motion process adapted to $\boldsymbol{F}_t$. The solutions of (1.3) and (1.4) can be regarded respectively as the responses of the harmonic oscillators of the *Liénard type*

$$\text{(1.3)}'\qquad \ddot{u}+f(u)\dot{u}+g(u)=\varepsilon^{1/2}\dot{w}$$

and

$$\text{(1.4)}'\qquad \ddot{u}+\varepsilon f(u)\dot{u}+g(u)=\varepsilon^{1/2}\dot{w}$$

with damping f and restoring force g to the (formal) white noise $\dot{w}$, where the dotted notation stands for the symbolic derivative d/dt. The second purpose of this paper is to get large deviation results for solutions of (1.3) and (1.4). The results of Theorem 4.1 and Theorem 5.1 generalize the estimates of Dubrovskii [1] which treats a special case of (1.3)′ with $f\equiv 0$ and g satisfying a global Lipschitz condition in R^1. We note that (1.3)′ and (1.4)′ contain the oscillator of the *Van Der Pol type*;

$$f(u)=u^2-1 ,\qquad g(u)=u ,$$

where the deterministic system has a stable limit cycle in the Liénard plane (u, v) with

$$v=\dot{u}+F(u)\quad \text{or}\quad v=\dot{u}+\varepsilon F(u) .$$

In accordance with Stroock [6, p. 23] and Varadhan [7, p. 3] we give the following definitions.

DEFINITION 1.1. Let X be a complete separable metric space with Borel field $\mathcal{B}$. We say that the function $I: X\to[0, \infty]$ is a *rate function* if

(i) $I\not\equiv\infty$,

(ii) I is lower semi-continuous,

(iii) for any $l\geqq 0$, $\{x; I(x)\leqq l\}$ is compact.

DEFINITION 1.2. A family $\{\mu^{\varepsilon}; \varepsilon>0\}$ of probability measures on $(X, \mathcal{B})$ is said to satisfy the *large deviation principle* with a rate function I if

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu^{\varepsilon}(C)\leqq -\inf_{x\in C} I(x) \tag{1.5}$$

for all closed sets C in X, and

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu^{\varepsilon}(\mathcal{G})\geqq -\inf_{x\in \mathcal{G}} I(x) \tag{1.6}$$

for all non-empty open sets $\mathcal{G}$ in X. When X is a metric space with metric ρ, we shall use the equivalent estimates later on:

$(\mathrm{I})_{\mathrm{eq}}$ for any $\delta>0$, $\gamma>0$ and $s_0>0$ there exists an $\varepsilon_0>0$ such that

$$\mu^{\varepsilon}\{y; \rho(x, y)<\delta\}\geqq\exp\left\{-\frac{1}{\varepsilon}(I(x)+\gamma)\right\}$$

for all $\varepsilon\leqq\varepsilon_0$ and all $x\in\Phi(s_0)$, where $\Phi(s)=\{x; I(x)\leqq s\}$;

(II) for any $\delta>0$, $\gamma>0$ and $s>0$ there exists an $\varepsilon_0>0$ such that

$$\mu^{\varepsilon}\{y; \rho(y, \Phi(s))\geqq\delta\}\leqq\exp\left\{-\frac{1}{\varepsilon}(s-\gamma)\right\}$$

for all $\varepsilon\leqq\varepsilon_0$, where $\rho(y, \Phi(s))=\inf_{x\in\Phi(s)}\rho(y, x)$.

The following remark follows from Freidlin and Wentzell [2, pp. 84-85].

REMARK 1.1. When $I(x)$ is a rate function on X, (1.5) is equivalent to (II) and also (1.6) is equivalent to $(\mathrm{I})_{\mathrm{eq}}$.

§ 2. Explosion and upper bound of solutions.

Here we evaluate the asymptotic probability with which the solutions of (1.1), (1.3) and (1.4) leave the bounded domain. For this purpose we treat the general form of the system;

$$dX^{\varepsilon}(t)=b^{\varepsilon}(X^{\varepsilon}(t))dt+\varepsilon^{1/2}\sigma(X^{\varepsilon}(t))dW(t), \qquad X^{\varepsilon}(0)=x\in R^{d}, \tag{2.1}$$

where $b^\varepsilon(x)=(b_i^\varepsilon(x))_{i=1,\dots,d}$ is a d-vector function depending on $\varepsilon>0$ and satisfying a local Lipschitz condition with respect to $x\in R^d$. Denote by L^ε the differential generator associated with (2.1), i.e.,

$$L^\varepsilon=\sum_{i=1}^d b_i^\varepsilon(x)\frac{\partial}{\partial x_i}+\frac{1}{2}\varepsilon\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}, \tag{2.2}$$

where $a(x)=(a_{ij}(x))_{i,j=1,\dots,d}$ is defined by $a(x)=\sigma(x)\sigma^*(x)$. We shall use the following notations. For $x\in R^d$ and $y\in R^d$, let $\langle x, y\rangle$ be the inner product of x and y and let $|x|$ be the Euclidean norm of x. For a $d\times d$-matrix $M=(m_{ij})_{i,j,\dots,d}$, define $|M|=(\sum_{i,j=1}^d m_{ij}^2)^{1/2}$. Denote by $C^2(R^d)$ the family of scalar functions which are twice continuously differentiable with respect to $x\in R^d$. For the future use we give the following theorem.

THEOREM 2.1. *Let $X^\varepsilon(t)$ be the solution of* (2.1) *with the initial state $X^\varepsilon(0)=x\in R^d$, and let $e^\varepsilon(x)$ be the explosion time of $X^\varepsilon(t)$. Suppose that there exist positive constants c and r and that there exist a non-negative function $V\in C^2(R^d)$ and a non-decreasing differentiable function $\beta:[0,\infty)\to[0,\infty)$, satisfying the following conditions;*

$$\sup_{0<\varepsilon\leq 1} L^\varepsilon V(x)+\frac{1}{2}|\sigma^*(x)\operatorname{grad}V(x)|^2\frac{1}{1+\beta(V(x))} \tag{2.3}$$
$$\leqq c\beta(V(x)) \qquad \textit{for all } |x|\geqq r,$$

where L^ε is defined by (2.2),

$$V_R\equiv\inf_{|x|\geq R}V(x)\longrightarrow\infty \qquad \textit{as } R\to\infty, \tag{2.4}$$

$$\int_0^\infty\frac{du}{1+\beta(u)}=\infty. \tag{2.5}$$

Then, for each $0<\varepsilon\leqq 1$ we have

$$P(e^\varepsilon(x)=\infty)=1 \qquad \textit{for all } x\in R^d.$$

Further, let us assume $v_r\equiv\sup_{\substack{|x|\leq r\\0<\varepsilon\leq 1}}L^\varepsilon V(x)<\infty$.

Then, for each $T>0$

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\leq t\leq T}|X^\varepsilon(t)|\geqq R)=-\infty. \tag{2.6}$$

PROOF. For any constant $\lambda>0$, set $H(x)=\exp\{\lambda k(V(x))\}$, where $k(v)=\int_0^v\frac{du}{1+\beta(u)}$. Then, a simple calculation yields

$$L^\varepsilon H(x)=L^\varepsilon V(x)\frac{\lambda}{1+\beta(V(x))}H(x)$$
$$+\frac{1}{2}\varepsilon|\sigma^*(x)\,\mathrm{grad}\,V(x)|^2\frac{\{-\lambda\beta'(V(x))+\lambda^2\}}{\{1+\beta(V(x))\}^2}H(x)$$
$$\leqq\frac{\lambda}{1+\beta(V(x))}\Big[L^\varepsilon V(x)+\frac{1}{2}\varepsilon|\sigma^*(x)\,\mathrm{grad}\,V(x)|^2\frac{\lambda}{1+\beta(V(x))}\Big]H(x)$$

for all $x\in R^d$, since $\beta'\geqq 0$ by the assumption. Now, take $\lambda=1/\varepsilon$ with $0<\varepsilon\leqq 1$, so that

$$(2.7)\qquad L^\varepsilon H(x)\leqq\frac{1}{\varepsilon}\frac{1}{1+\beta(V(x))}$$
$$\times\Big[L^\varepsilon V(x)+\frac{1}{2}|\sigma^*(x)\,\mathrm{grad}\,V(x)|^2\frac{1}{1+\beta(V(x))}\Big]H(x)$$

for all $x\in R^d$. Combine (2.7) with (2.3) and notice that $\beta/(1+\beta)\leqq 1$. Then, under our assumptions, we have

$$L^\varepsilon H(x)\leqq\frac{1}{\varepsilon}\tilde{c}(1+H(x))\qquad\text{for all }|x|\geqq r$$

with a constant $\tilde{c}>0$ being independent of ε. Define $\tilde{H}(x)=1+H(x)$. Then, (2.4) and (2.5) imply that $\inf_{|x|\geqq R}\tilde{H}(x)\to\infty$ as $R\to\infty$. Moreover, $\tilde{H}(x)$ satisfies

$$(2.8)\qquad L^\varepsilon\tilde{H}(x)\leqq\frac{1}{\varepsilon}\tilde{c}\,\tilde{H}(x)\qquad\text{for all }|x|\geqq r.$$

This inequality, as follows from Narita [5, pp. 397-398], leads us to non-occurrence of the explosion for any initial state in R^d. Put $\tau_R^\varepsilon=\inf\{t\,;\,|X^\varepsilon(t)|\geqq R\}$ if such a time exists, and let $\tau_R^\varepsilon=\infty$ otherwise. For each $t\geqq 0$, set $t_R^\varepsilon=t\wedge\tau_R^\varepsilon$, where $a\wedge b$ stands for the smaller of a and b. Apply Ito's formula concerning stochastic differentials to $X^\varepsilon(t)$ and $\tilde{H}(x)$. Then, since we can choose $\tilde{c}$ by the assumption $v_r<\infty$ so that (2.8) may hold for every $x\in R^d$, we have

$$E[\tilde{H}(X^\varepsilon(t_R^\varepsilon))]\leqq\tilde{H}(x)+\frac{1}{\varepsilon}\tilde{c}\int_0^t E[\tilde{H}(X^\varepsilon(s_R^\varepsilon))]ds\,,$$

where $s_R^\varepsilon=s\wedge\tau_R^\varepsilon$. So, the Gronwall-Bellman inequality yields

$$E[\tilde{H}(X^\varepsilon(t_R^\varepsilon))]\leqq\tilde{H}(x)\exp\Big(\frac{1}{\varepsilon}\tilde{c}t\Big).$$

Put $\tilde{H}=1+H$. Then

$$E[\tilde{H}(X^\varepsilon(t_R^\varepsilon))]\geqq E[H(X^\varepsilon(t_R^\varepsilon))\,;\,\tau_R^\varepsilon\leqq t]$$
$$\geqq\exp\Big\{\frac{1}{\varepsilon}k(V_R)\Big\}P(\tau_R^\varepsilon\leqq t)\,,$$

where $V_R=\inf_{|x|\geqq R} V(x)$. Accordingly, we get

$$P(\tau_R^\varepsilon\leqq t)\leqq\exp\left\{-\frac{1}{\varepsilon}k(V_R)+\frac{1}{\varepsilon}\tilde{c}t\right\}\left[1+\exp\left\{\frac{1}{\varepsilon}k(V(x))\right\}\right].$$

Since $\log(1+t)\leqq\log 2t$ for $t\geqq 1$, we see

$$\varepsilon\log P(\tau_R^\varepsilon\leqq t)\leqq -k(V_R)+\tilde{c}t+\varepsilon\log 2+k(V(x)),$$

and so

$$\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\leqq s\leqq t}|X^\varepsilon(s)|\geqq R)\leqq -k(V_R)+\tilde{c}t+k(V(x)).$$

Since $k(V_R)\to\infty$ as $R\to\infty$ by (2.4) and (2.5), we obtain (2.6) and complete the proof.

REMARK 2.1. When $b^\varepsilon(x)\equiv b(x)$ and $\sigma(x)\equiv 0$, (2.3) becomes

$$(\#)\qquad V'_{(1.2)}\equiv\sum_{i=1}^d b_i(x)\frac{\partial V}{\partial x_i}\leqq c\beta(V(x))\qquad\text{for all } |x|\geqq r.$$

Then, the condition (#) together with (2.4) and (2.5) gives the existence of the global solution of the dynamical system (1.2) (see LaSalle and Lefschetz [4; § 24, Chap. 4]).

REMARK 2.2. In particular, we notice that for the function $V(x)=|x|^2$, $L^\varepsilon V(x)$ has the form

$$L^\varepsilon V(x)=2\langle x, b^\varepsilon(x)\rangle+\varepsilon|\sigma(x)|^2.$$

Hence the condition (2.3) holds for $V(x)=|x|^2$, once the following inequality is satisfied:

$$(2.9)\qquad \sup_{0<\varepsilon\leqq 1}2\langle x, b^\varepsilon(x)\rangle+|\sigma(x)|^2+2|\sigma^*(x)x|^2\frac{1}{1+\beta(|x|^2)}$$
$$\leqq c\beta(|x|^2)\qquad\text{for all } |x|\geqq r.$$

Replacing $b^\varepsilon(x)$ of (2.9) by $b(x)$, we apply Theorem 2.1 to to the system (1.1) and get the following result.

COROLLARY 2.1. *Let $X^\varepsilon(t)$ be the solution of* (1.1) *with the initial state $X^\varepsilon(0)=x\in R^d$, and let $e^\varepsilon(x)$ be the explosion time of $X^\varepsilon(t)$. Suppose that there exist positive constants c and r and that there exists a non-decreasing differentiable function β: $[0,\infty)\to[0,\infty)$, satisfying the following condition;*

$$(2.10)\qquad 2\langle x, b(x)\rangle+|\sigma(x)|^2+2|\sigma^*(x)x|^2\frac{1}{1+\beta(|x|^2)}\leqq c\beta(|x|^2)\quad\textit{for all } |x|\geqq r,$$

where β satisfies (2.5). *Then, for each $0<\varepsilon\leqq 1$ we have*

$$P(e^\varepsilon(x)=\infty)=1\qquad\textit{for all } x\in R^d.$$

Moreover, for each $T>0$

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\leqq t\leqq T}|X^{\varepsilon}(t)|\geqq R)=-\infty.$$

EXAMPLE 2.1. Suppose that

$$|b(x)|^2+|\sigma(x)|^2\leqq K\lambda(|x|^2)\qquad\text{for all } |x|\geqq r \tag{2.11}$$

with some constants $K>0$ and $r>0$, where $\lambda(u)=(1+u)\log(1+u^{1/2})$. Then, (2.10) holds with the function $\beta(u)=u+\lambda(u)$. Therefore, the condition (2.10) corresponds to a generalization of (2.11) which is a restriction on growth of the coefficients in Yershov [8, Theorem 5.2].

In the following we consider the systems (1.3) and (1.4). Define the matrix σ and the Brownian motion process $W(t)$ by

$$\sigma=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\quad\text{and}\quad W(t)=\begin{pmatrix}w_0(t)\\ w(t)\end{pmatrix},$$

where $w_0(t)$ is a (dummy) Brownian motion process which is independent of $w(t)$. Then, (1.3) and (1.4) can be written respectively as the following forms;

$$dX^{\varepsilon}(t)=b(X^{\varepsilon}(t))dt+\varepsilon^{1/2}\sigma dW(t),\quad X^{\varepsilon}(0)=x\in R^2, \tag{2.12}$$

$$d\tilde{X}^{\varepsilon}(t)=b^{\varepsilon}(\tilde{X}^{\varepsilon}(t))dt+\varepsilon^{1/2}\sigma dW(t),\quad \tilde{X}^{\varepsilon}(0)=x\in R^2, \tag{2.13}$$

where for $x=(x_1, x_2)\in R^2$, $b(x)$ and $b^{\varepsilon}(x)$ are given by

$$b(x)=\begin{pmatrix}x_2-F(x_1)\\ -g(x_1)\end{pmatrix}\quad\text{and}\quad b^{\varepsilon}(x)=\begin{pmatrix}x_2-\varepsilon F(x_1)\\ -g(x_1)\end{pmatrix}.$$

We shall need the following assumption.

ASSUMPTION 2.1. (A_1) $\quad ug(u)>0\quad$ for all $u\neq 0$,

$$(A_2)\qquad -g(u)F(u)\leqq\alpha(1+G(u))\qquad\text{for all } u\in R^1$$

with a constant $\alpha>0$ and

$$(A_3)\qquad G(u)\longrightarrow\infty\qquad\text{as } |u|\to\infty,$$

where $G(u)=\int_0^u g(s)ds$.

We shall use the function

$$V(x)=G(x_1)+x_2^2/2\qquad\text{for } x=(x_1, x_2)\in R^2. \tag{2.14}$$

Denote by L^{ε} the differential operator associated with (2.12) or (2.13), and consider $L^{\varepsilon}V(x)$ for the function $V(x)$ defined by (2.14). Then, by Theorem 2.1 we get the following result.

COROLLARY 2.2. *Under Assumption* 2.1, *denote solutions* $X^{\varepsilon}(t)$ *of* (1.3), $\tilde{X}^{\varepsilon}(t)$ *of* (1.4), *by the same symbel* $Y^{\varepsilon}(t)$, *and let* $e^{\varepsilon}(x)$ *be the explosion time of* $Y^{\varepsilon}(t)$ *with the initial state* $Y^{\varepsilon}(0)=x\in R^2$. *Then, for each* $0<\varepsilon\leqq 1$ *we have*

$$P(e^{\varepsilon}(x)=\infty)=1 \qquad \text{for all } x\in R^2 .$$

Moreover, for each $T>0$

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\leq t\leq T}|Y^{\varepsilon}(t)|\geqq R)=-\infty .$$

§ 3. Large deviation principle for non-degenerate diffusions.

Let us consider the system (1.1), and denote by $X^{\varepsilon}(t, x)$ the solution $X^{\varepsilon}(t)$ of (1.1) with the initial state $X^{\varepsilon}(0)=x\in R^d$, i.e.,

$$X^{\varepsilon}(t, x)=x+\int_0^t b(X^{\varepsilon}(s, x))ds+\varepsilon^{1/2}\int_0^t \sigma(X^{\varepsilon}(s, x))dW(s) .$$

Also, consider the solution $X^0(t)$ of the system (1.2), i.e.,

$$X^0(t)=x+\int_0^t b(X^0(s))ds .$$

REMARK 3.1. Assume the same conditions as in Theorem 2.1 except that $b^{\varepsilon}(x)$ is replaced by $b(x)$. Then, Theorem 2.1 and Remark 2.1 imply that both $X^{\varepsilon}(t, x)$ and $X^0(t)$ become global solutions.

Hereafter, by $C([0, \infty); R^d)$ (resp. $C([0, T]; R^d)$) we denote the space of all continuous functions $\phi(t)$, $0\leqq t<\infty$ (resp. $0\leqq t\leqq T$), with range in R^d. By $C_K^{\infty}(R^d)$ we shall denote the family of scalar functions which are infinitely differentiable with respect to $x\in R^d$, having compact support. The main result of this section is the following theorem.

THEOREM 3.1. *Under the same assumption as in Corollary* 2.1, *set* $a(x)=\sigma(x)\sigma^*(x)=(a_{ij}(x))_{i,j=1,\cdots,d}$, *and suppose that* $a(x)$ *is positive definite for all* $x\in R^d$ *and that* $a_{ij}\in C^2(R^d)$ *for all* $i, j=1, \cdots, d$. *Let* $X^{\varepsilon}(t, x)$ *be the solution* $X^{\varepsilon}(t)$ *of* (1.1) *with the initial state* $X^{\varepsilon}(0)=x\in R^d$, *and let* P_x^{ε} *be the probability measure induced by* $X^{\varepsilon}(\cdot, x)$ *on* $C([0, \infty); R^d)$. *Define* $I_{x;T}^{a;b}$ *on* $C([0, T]; R^d)$ *by*

$$I_{x;T}^{a;b}(\phi)=\frac{1}{2}\int_0^T \langle\dot{\phi}(t)-b(\phi(t)),\ a^{-1}(\phi(t))(\dot{\phi}(t)-b(\phi(t)))\rangle dt \tag{3.1}$$

if $\phi(0)=x$ *and* $\phi|_{[0,T]}$ *is absolutely continuous,*

$$I_{x;T}^{a;b}(\phi)=\infty \qquad \text{otherwise.}$$

Then, $I_{x;T}^{a;b}$ *is a rate function. Moreover, for each* $T>0$ *and* $x\in R^d$, $\{P_x^{\varepsilon};\ \varepsilon>0\}$ *on* $C([0, T]; R^d)$ *satisfies the large deviation principle with respect to* $I_{x;T}^{a;b}$.

PROOF. Take a non-negative function $\eta\in C_K^\infty(R^d)$ such that

$$\eta(y)=1 \quad \text{for } |y|\leqq 1, \qquad \eta(y)=0 \quad \text{for } |y|\geqq 2,$$

satisfying $0\leqq\eta(y)\leqq 1$ for all $y\in R^d$. For each $R>0$, set $\eta_R(y)=\eta(y/R)$, and define $a_R(y)$ and $b_R(y)$ by

$$a_R(y)=\eta_R(y)a(y)+(1-\eta_R(y))I \quad \text{and} \quad b_R(y)=\eta_R(y)b(y),$$

where I is the identity matrix. Then, we see that $a_R(y)$ is positive definite for all $y\in R^d$. The (i, j)-element $a_{R,ij}$ of a_R belongs to $C^2(R^d)$ for all $i, j=1, \cdots, d$ and

$$\sup_{y\in R^d}\sup_{i,j,k,l}\left|\frac{\partial^2 a_{R,ij}}{\partial y_k\partial y_l}\right|\leqq M_R$$

with a constant $M_R>0$ depending on R. Thus, a_R has a unique non-negative square root σ_R which satisfies a global Lipschitz condition with respect to $y\in R^d$ (see Friedman [3, p. 129]). Combining this with the fact that $b_R(y)$ satisfies a global Lipschitz condition with respect to $y\in R^d$, we see that there exists a pathwise unique solution $X^{\varepsilon,R}(t, x)$ of the system (1.1) with $b=b_R$ and $\sigma=\sigma_R$, starting from $x\in R^d$. Denote by $P_x^{\varepsilon,R}$ the probability measure induced by $X^{\varepsilon,R}(\cdot, x)$ on $C([0, \infty); R^d)$. Then, as follows from Stroock [6, p. 85], $\{P_x^{\varepsilon,R};\ \varepsilon>0\}$ on $C([0, T]; R^d)$ satisfies the large deviation principle with a rate function $I_{x,T}^{a_R,b_R}$, where $I_{x,T}^{a_R,b_R}$ is defined by (3.1) with $a=a_R$ and $b=b_R$. In the following let $x\in R^d$ and $T>0$ be arbitrary and be fixed. For $\omega\in C([0, \infty); R^d)$, set

$$\zeta_R=\inf\{t;\ |\omega(t)|\geqq R\}$$

and denote by $\boldsymbol{M}_T$ the σ-algebra generated by $\{\omega(t);\ 0\leqq t\leqq T\}$. Then, we first notice that

$$P_x^{\varepsilon,R}(A\cap\{\zeta_R\geqq T\})=P_x^\varepsilon(A\cap\{\zeta_R\geqq T\}) \quad \text{for any } A\in\boldsymbol{M}_T \tag{3.2}$$

and that

$$I_{x,T}^{a_R,b_R}(\phi)=I_{x,T}^{a,b}(\phi) \quad \text{for any } \phi\in C([0, T]; R^d) \text{ satisfying } \|\phi\|_T\leqq R, \tag{3.3}$$

where $\|\phi\|_T=\sup_{0\leqq t\leqq T}|\phi(t)|$. For the proof of Theorem, we have only to show that $I_{x,T}^{a,b}$ satisfies (iii) of Definition 1.1 and that $\{P_x^\varepsilon;\ \varepsilon>0\}$ satisfies (1.5) and (1.6) of Definition 1.2. Using the estimate of Corollary 2.1, we can proceed the proof as in the proof of Stroock [6, p. 87].

Let $\mathcal{G}$ be any open set in $C([0, T]; R^d)$. Then, we can see

$$\liminf_{\varepsilon\to 0}\varepsilon\log P_x^\varepsilon(\mathcal{G})\geqq-\inf_{\phi\in\mathcal{G}}I_{x,T}^{a,b}(\phi) \tag{3.4}$$

if we can obtain the following estimate:

$$\liminf_{\varepsilon\to 0} \varepsilon \log P_x^\varepsilon(\mathcal{G}) \geqq -I_{x,T}^{a,b}(\tilde{\phi}) \tag{3.4$'$}$$

for all $\tilde{\phi}\in C([0, \infty); R^d)\cap\mathcal{G}$ satisfying $\tilde{\phi}(0)=x$. So, we show (3.4)$'$. In fact, for such a $\tilde{\phi}$ we can choose $R>0$ so that $\|\tilde{\phi}\|_T<R$ and then set $B_R=\{\phi; \|\phi\|_T<R\}$. Apply the large deviation principle with respect to the family $\{P_x^{\varepsilon,R}; \varepsilon>0\}$. Then, by (3.2) and (3.3) we have

$$\begin{aligned}\liminf \varepsilon \log P_x^\varepsilon(\mathcal{G}) &\geqq \liminf_{\varepsilon\to 0} \varepsilon \log P_x^\varepsilon(\mathcal{G}\cap B_R)\\ &=\liminf_{\varepsilon\to 0} \varepsilon \log P_x^{\varepsilon,R}(\mathcal{G}\cap B_R)\\ &\geqq -\inf_{\phi\in\mathcal{G}\cap B_R} I_{x,T}^{a_R,b_R}(\phi)\\ &=-\inf_{\phi\in\mathcal{G}\cap B_R} I_{x,T}^{a,b}(\phi)\\ &\geqq -I_{x,T}^{a,b}(\tilde{\phi}),\end{aligned}$$

and hence we get (3.4)$'$. We next show that $I_{x,T}^{a,b}$ satisfies (iii) of Definition 1.1. Apply the estimate of Corollary 2.1 to $X^\varepsilon(t, x)$. Then, for any $l<\infty$ we can find an $R>0$ so that

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_x^\varepsilon((\bar{B}_R)^c)<-l. \tag{3.5}$$

Combining this with (3.4), we have

$$\inf_{\phi\in(\bar{B}_R)^c} I_{x,T}^{a,b}(\phi)>l,$$

and so $\{\phi; I_{x,T}^{a,b}(\phi)\leqq l\}\subseteqq \bar{B}_R$. This relation and (3.3) yield

$$\{\phi; I_{x,T}^{a,b}(\phi)\leqq l\}=\{\phi; I_{x,T}^{a_R,b_R}(\phi)\leqq l\}\cap\bar{B}_R.$$

Since $I_{x,T}^{a_R,b_R}$ is a rate function, the latter set of the above equation is compact in $C([0, T]; R^d)$, which implies the compactness of the former set.

Finally, let C be any closed set in $C([0, T]; R^d)$ and let $l<\infty$ be arbitrary, and choose R so as (3.5) holds. Then, by (3.2) and (3.3) we see

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon \log P_x^\varepsilon(C) &\leqq \limsup_{\varepsilon\to 0} \varepsilon \log\{P_x^{\varepsilon,R}(C\cap\bar{B}_R)+P_x^\varepsilon((\bar{B}_R)^c)\}\\ &\leqq\{-\inf_{\phi\in C\cap\bar{B}_R} I_{x,T}^{a_R,b_R}(\phi)\}\vee(-l)\\ &\leqq -[\{\inf_{\phi\in C} I_{x,T}^{a,b}(\phi)\}\wedge l],\end{aligned}$$

where $s\vee t=\max\{s, t\}$ and $s\wedge t=\min\{s, t\}$. Passing to the limit as $l\to\infty$ in the above equation, we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_x^\varepsilon(C) \leqq -\inf_{\phi\in C} I_{x;T}^{a;b}(\phi)\,.$$

Hence the proof is completed.

The following theorem deals with a slight generalization of Theorem 3.1.

THEOREM 3.2. *Under the same assumption as in Theorem* 3.1, *suppose that* $b^\varepsilon(x)=(b_i^\varepsilon(x))_{i=1,\dots,d}$ *satisfies* (2.9) *and that* $b_i^\varepsilon(x)\to b_i(x)$ *uniformly in all* $x\in R^d$ *as* $\varepsilon\to 0$. *Denote by* $\tilde{X}^\varepsilon(t, x)$ *the solution* $X^\varepsilon(t)$ *of* (2.1) *with the initial state* $X^\varepsilon(0)=x\in R^d$ *and let* $\tilde{P}_x^\varepsilon$ *be the probability measure induced by* $\tilde{X}^\varepsilon(\cdot, x)$ *on* $C([0, \infty); R^d)$. *Then, for each* $T>0$ *and* $x\in R^d$, $\{\tilde{P}_x^\varepsilon; \varepsilon>0\}$ *on* $C([0, T]; R^d)$ *satisfies the large deviatian principle with a rate function* $I_{x;T}^{a;b}$ *of* (3.1).

PROOF. Freidlin and Wentzell's theorem [2, p. 154] states that the above Theorem 3.2 holds for bounded smooth a_{ij} and b_i^ε. Noting that (2.6) holds under (2.9) we can prove Theorem 3.2 in the same way as the proof of Theorem 3.1.

§ 4. Oscillators of the Liénard type.

Before proceeding to the systems (1.3) and (1.4) we shall need some preparations. Let $C([0, T]; R^1)$ be the space of all continuous functions $\psi: [0, T]\to R^1$, and set

$$C_0^T=\{\psi\in C([0, T]; R^1);\ \psi(0)=0\}.$$

Let $f(u)$ and $g(u)$ be functions which satisfy a local Lipschitz condition with respect to $u\in R^1$. For any $x=(x_1, x_2)\in R^2$ and $T>0$, we consider a solution of the following system;

$$(4.1)\qquad \begin{cases} \phi_1(t)=x_1+\int_0^t [\phi_2(s)-F(\phi_1(s))]ds\,, \\ \phi_2(t)=x_2-\int_0^t g(\phi_1(s))ds+\psi(t)\,, \qquad 0\leqq t\leqq T\,, \end{cases}$$

associating with each function $\psi\in C_0^T$, where $F(u)=\int_0^u f(s)ds$. When (4.1) has a unique solution, we shall use the following notations and definition.

NOTATION 4.1. For $x=(x_1, x_2)\in R^2$ and $T>0$, define an operator $A=A(x, T)$ on C_0^T by $A\psi=\phi$, where $\phi=(\phi_1, \phi_2)$ satisfies (4.1). Define $\mathcal{W}=\mathcal{W}(x, T)=A[C_0^T]$, which denotes the set of all admissible paths initiating from x.

Set $A=A^{F,g}$ and $\mathcal{W}=\mathcal{W}^{F,g}$, in case we need to emphasize that the system (4.1) depends on the functions F and g.

DEFINITION 4.1. Define the functional on $\mathcal{W}(x, T)$ as follows;

$$S_{x;T}^{f;g}(\phi)=\frac{1}{2}\int_0^T\left[\frac{d^2\phi_1(t)}{dt^2}+f(\phi_1(t))\frac{d\phi_1(t)}{dt}+g(\phi_1(t))\right]^2 dt$$

if $\phi=(\phi_1, \phi_2)\in\mathcal{W}(x, T)$, ϕ is absolutely continuous and the integral exists,

$S_{x;T}^{f;g}(\phi)=\infty$ otherwise.

NOTATION 4.2. Define

$$\|x\|=|x_1|+|x_2| \qquad \text{for } x=(x_1, x_2)\in R^2,$$

$$\|\phi\|_t=\sup_{0\leq s\leq t}\|\phi(s)\| \qquad \text{for } \phi\in C([0, T]; R^2)$$

and

$$\|\psi\|_t=\sup_{0\leq s\leq t}|\psi(s)| \qquad \text{for } \psi\in C_0^T,$$

where $t\leqq T$. Further, define

$$|\phi(t)|=(\phi_1(t)^2+\phi_2(t)^2)^{1/2} \qquad \text{for } \phi=(\phi_1, \phi_2)\in C([0, T]; R^2),$$

where $t\leqq T$.

REMARK 4.1. Suppose that $F(u)$ and $g(u)$ satisfy a global Lipschitz condition with respect to $u\in R^1$. Then, both (1.3) and (4.1) have unique solutions denoted by $X^\varepsilon(t)$ and $\phi(t)$ respectively, and $X^\varepsilon(t)$ has the form

$$X^\varepsilon=A(\varepsilon^{1/2}w).$$

Further, the operator A is continuous from $(C_0^T, \|\cdot\|_t)$ to $(\mathcal{W}, \|\cdot\|_t)$, having its inverse A^{-1}. Therefore, according to Freidlin and Wentzell [2, p. 81] and Varadhan [7, p. 5], Schilder's theorem of the large deviation for the Wiener measure with covariance $\varepsilon \min\{s, t\}$ can be transferred to the probability measure corresponding to X^ε.

Dubrovskii [1] treats the special case of the system (1.3) when $f\equiv 0$ and g satisfies a global Lipschitz condition in R^1 by using the method referred in Remark 4.1. Unfortunately, we cannot proceed as in Remark 4.1 because the operator A may not be continuous in general. So we take a truncation procedure in order to get around this difficulty.

For each $R>0$, let $\eta_R(u)$ be a smooth function on R^1 such that

$$\eta_R(u)=1 \quad \text{for } |u|\leqq R, \qquad \eta_R(u)=0 \quad \text{for } |u|\geqq 2R$$

satisfying $0\leqq\eta_R(u)\leqq 1$ for all $u\in R^1$. Now, set

$$f_R(u)=\eta_R(u)f(u), \quad g_R(u)=\eta_R(u)g(u) \quad \text{and} \quad F_R(u)=\int_0^u f_R(s)ds.$$

Then, since $F_R(u)$ and $g_R(u)$ satisfy a global Lipschitz condition with respect to

$u\in R^1$, there exist pathwise unique solutions of the systems (4.1) and (1.3) with $\{F, g\}$ replaced by $\{F_R, g_R\}$, initiating from $x\in R^2$, which are denoted by $\phi^R(t)$ and $X^{\varepsilon,R}(t)$ respectively. First, define the function $\phi(t)$ by

$$\phi(t)=\phi^R(t) \qquad \text{for } t<t_R, \quad R=1, 2, \cdots,$$

where $t_R=T\wedge\inf\{t; |\phi^R(t)|\geqq R\}$. Hereafter we call $\phi(t)$ a solution of (4.1). In general, $\phi(t)$ may not be continued to $t=T$. The following lemma assures us that $\phi(t)$ can be continued to $t=T$ for each $T>0$ under the assumption below and hence the existence-and-uniqueness for the solution of (4.1) holds on every finite interval $[0, T]$.

We shall need the following assumption.

ASSUMPTION 4.1.

$$g(u)^2\leqq\beta(1+G(u)) \qquad \text{for all } u\in R^1 \text{ with a constant } \beta>0,$$

where $G(u)=\int_0^u g(s)ds$.

The folloing functions satisfy Assumption 4.1;

$$g(u)=u, \quad g(u)=|u|^n \operatorname{sgn}(u) \qquad \text{with } 0<n<1.$$

LEMMA 4.1. *Let Assumption* 2.1 *and Assumption* 4.1 *hold. Then, for any* $x=(x_1, x_2)\in R^2$ *and* $T>0$, *each solution of* (4.1) *can be continued to* $t=T$.

PROOF. To the contrary, we assume that (4.1) has a solution $\phi(t)=(\phi_1(t), \phi_2(t))$ defined on $[0, T_0)$ and satisfying

$$\lim_{t\uparrow T_0}|\phi(t)|=\infty \qquad \text{for some } T_0<T.$$

Here by $|\ |$ we mean the usual Euclidean norm. In the following we take such a solution $\phi(t)$ on $[0, T_0)$ and evaluate $V(\phi(t))$, where V is the function defined by (2.14).

Observe the first equation of (4.1), so that

$$dG(\phi_1(t))=g(\phi_1(t))(\phi_2(t)-F(\phi_1(t)))dt.$$

Notice that $2ab\leqq a^2+b^2$ for $a\in R^1$ and $b\in R^1$. Then, (A_2) of Assumption 2.1 yields

$$g(\phi_1(t))(\phi_2(t)-F(\phi_1(t)))\leqq\frac{1}{2}\{g(\phi_1(t))^2+\phi_2(t)^2\}+\alpha(1+G(\phi_1(t))).$$

Next, apply the Schwarz inequality to the second equation of (4.1), so that

$$\frac{1}{2}\phi_2(t)^2\leqq\frac{3}{2}\left[x_2^2+\left\{\int_0^t g(\phi_1(s))ds\right\}^2+\psi(t)^2\right]$$

$$\leqq \frac{3}{2}\left[x_2^2+t\int_0^t g(\phi_1(s))^2 ds+\phi(t)^2\right].$$

Then, Assumption 4.1 together with the fact that $G \leqq V$ yields the following inequalities;

$$G(\phi_1(t)) \leqq c_1(T_0)+c_2\int_0^t V(\phi(s))ds \tag{4.2}$$

for all $0 \leqq t < T_0$ with some constants $c_1(T_0)>0$ depending on T_0 and $c_2>0$,

$$\frac{1}{2}\phi_2(t)^2 \leqq d_1(T_0)+d_2(T_0)\int_0^t V(\phi(s))ds \tag{4.3}$$

for all $0 \leqq t < T_0$ with some constants $d_1(T_0)>0$ and $d_2(T_0)>0$ depending on T_0.

Taking the sum of (4.2) and (4.3), we get

$$V(\phi(t)) \leqq k_1(T_0)+k_2(T_0)\int_0^t V(\phi(s))ds$$

for all $0 \leqq t < T_0$ with some constants $k_1(T_0)>0$ and $k_2(T_0)>0$ depending on T_0. By the Gronwall-Bellman inequality we have

$$0 \leqq V(\phi(t)) \leqq k_1(T_0)\exp\{k_2(T_0)t\} < \infty \qquad \text{for all } 0 \leqq t < T_0,$$

which is a contradiction. Hence the proof is completed.

REMARK 4.2. The oscillator of the *Van Der Pol type* with $f(u)=u^2-1$ and $g(u)=u$ satisfies Assumption 2.1 and Assumption 4.1, and hence Lemma 4.1 is applicable for the oscillator.

NOTATION 4.3. Let $R>0$ be arbitrary and be fixed. For $x=(x_1, x_2) \in R^2$ and $T>0$, define an operator $A^R=A^R(x, T)$ on C_0^T by $A^R(x, T)=A^{F_R, g_R}(x, T)$ and set $\mathcal{W}^R=\mathcal{W}^R(x, T)=A^R[C_0^T]$.

REMARK 4.3. The solution $X^{\varepsilon, R}(t)$ of the system (1.3) with $F=F_R$ and $g=g_R$ can be expressed by $X^{\varepsilon, R}=A^R(\varepsilon^{1/2}w)$. The operator A^R is continuous from $(C_0^T, \|\cdot\|_t)$ to $(\mathcal{W}^R, \|\cdot\|_t)$, and also A^R is invertible such that

$$[(A^R)^{-1}\phi](t)=\phi_2(t)-x_2+\int_0^t g_R(\phi_1(s))ds \quad \text{for any } \phi=(\phi_1, \phi_2) \equiv \phi^R \in \mathcal{W}^R.$$

Now, we proceed as in Freidlin and Wentzell [2, Chap. 3].

DEFINITION 4.2. Define the action functional on C_0^T for the Wiener process as follows;

$$I_T^w(\phi)=\frac{1}{2}\int_0^T \left\{\frac{d\phi(t)}{dt}\right\}^2 dt \quad \text{if } \phi \text{ is absolutely continuous and the integral exists,}$$

$$I_T^w(\psi)=\infty \qquad \text{otherwise.}$$

Further, define the action along $\phi\equiv\phi^R\in\mathcal{W}^R$ by

$$S_{x,T}^R(\phi)=I_T^w(A^R(x, T)^{-1}\phi) \qquad \text{for } \phi\equiv\phi^R\in\mathcal{W}^R(x, T).$$

In terms of Definition 1.1, I_T^w is a rate function on C_0^T. Substituting the expression for $(A^R)^{-1}$ and noting that

$$d\phi_1(t)/dt=\phi_2(t)-F_R(\phi_1(t)) \qquad \text{for } \phi=(\phi_1, \phi_2)\in\mathcal{W}^R(x, T),$$

we find that

(4.4) $$S_{x,T}^R(\phi)=S_{x;T}^{f;g}(\phi) \qquad \text{with } f=f_R \text{ and } g=g_R$$

whenever $\phi=(\phi_1, \phi_2)\in\mathcal{W}^R$, ϕ is absolutely continuous and the integral exists.

THEOREM 4.1. *Under Assumption* 2.1. *and Assumption* 4.1, *let* $X^\varepsilon(t, x)$ *be the solution* $X^\varepsilon(t)$ *of* (1.3) *with the initial state* $X^\varepsilon(0)=x\in R^2$, *and let* P_x^ε *be the probability measure induced by* $X^\varepsilon(\cdot, x)$ *on* $C([0, \infty); R^2)$. *Then, the function* $S_{x;T}^{f;g}$ *given by Definition* 4.1 *is a rate function. Moreover, for each* $T>0$ *and* $x\in R^2$, $\{P_x^\varepsilon|_{\mathcal{W}}\ ; \varepsilon>0\}$ *satisfies the large deviation principle with respect to* $S_{x;T}^{f;g}$.

PROOF. For any $a\geqq 0$, put

$$\Phi_a(x, T)=\{\phi\in\mathcal{W}(x, T)\,;\, S_{x;T}^{f;g}(\phi)\leqq a\}$$

and consider $\phi=(\phi_1, \phi_2)\in\Phi_a(x, T)$. Then, ϕ is absolutely continuous and

$$\int_0^T|\ddot{\phi}_1(t)|^2dt\leqq B,$$

where B is a constant independent of ϕ. If $0\leqq t<t+h\leqq T$, then $\phi=(\phi_1, \phi_2)\in\Phi_a(x, T)$ satisfies

$$|\phi_i(t+h)-\phi_i(t)|=\left|\int_t^{t+h}\dot{\phi}_i(s)ds\right|\leqq\sqrt{h}\left\{\int_t^{t+h}|\dot{\phi}_i(s)|^2ds\right\}^{1/2}$$

for $i=1, 2$. Notice that $\phi=(\phi_1, \phi_2)$ is absolutely continuous and that $\dot{\phi}_1(s)=\phi_2(s)-F(\phi_1(s))$, so that

$$\begin{aligned}|\dot{\phi}_1(s)|^2&\leqq 2\{|\dot{\phi}_1(s)-\dot{\phi}_1(0)|^2+|\dot{\phi}_1(0)|^2\}\\&=2\left\{\left|\int_0^s\ddot{\phi}_1(r)dr\right|^2+|\dot{\phi}_1(0)|^2\right\}\\&\leqq 2\left\{s\int_0^s|\ddot{\phi}_1(r)|^2dr+|\dot{\phi}_1(0)|^2\right\}\end{aligned}$$

and also $|\dot{\phi}_2(s)|^2=|\ddot{\phi}_1(s)+f(\phi_1(s))\dot{\phi}_1(s)|^2$. Thus, we get

$$|\phi_i(t+h)-\phi_i(t)|\leqq\sqrt{h}\,B' \qquad \text{for } i=1, 2 \text{ if } \phi=(\phi_1, \phi_2)\in\Phi_a(x, T),$$

where B' is a constant independent of ϕ. Namely, $\Phi_a(x, T)$ is a class of equicontinuous functions. These functions are also uniformly bounded since $\phi(0)=x$ belongs to some bounded set in R^2. Therefore, the compactness of $\Phi_a(x, T)$ follows from the lemma of Áscoli-Arzela, and hence $S^{f;\,g}_{x;\,T}$ is a rate function.

In order to prove (1.6) for $P^{\varepsilon}_{x}|_{\mathcal{W}}$ we notice that $(\mathcal{W}, \|\cdot\|_t)$ is a metric space, and then by Remark 1.1 we have only to prove the following estimate; for any $\delta>0$, $\gamma>0$ and $a>0$ there exists an $\varepsilon_0>0$ such that

$$P^{\varepsilon}_{x}(\|X^{\varepsilon}-\phi\|_T<\delta)\geqq\exp\left\{-\frac{1}{\varepsilon}(S^{f;\,g}_{x;\,T}(\phi)+\gamma)\right\} \tag{4.5}$$

$$\text{for all } \varepsilon\leqq\varepsilon_0 \text{ and all } \phi\in\Phi_a(x, T).$$

Choose $R_0>0$ so that $\|\phi\|_T<R_0$. By the estimate of Corollary 2.2, for any $l>0$ we can find an $R_1>0$ so that for all $R>R_1$

$$\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\leq t\leq T}|X^{\varepsilon}(t)|\geqq R/\sqrt{2})<-l.$$

Thus, for every $\gamma>0$ and ε sufficiently small, we have

$$\varepsilon\log P(\sup_{0\leq t\leq T}|X^{\varepsilon}(t)|\geqq R/\sqrt{2})<-l+\gamma,$$

and so

$$P(\sup_{0\leq t\leq T}|X^{\varepsilon}(t)|\geqq R/\sqrt{2})<\exp\left\{-\frac{1}{\varepsilon}(l-\gamma)\right\}. \tag{4.6}$$

Put $R'=\max\{R_0, R_1\}$ and then consider any R such that $R>R'$. Let $X^{\varepsilon,R}(t)$ be the solution of (1.3) with $F=F_R$ and $g=g_R$, and let $P^{\varepsilon,R}_{x}$ be the probability measure induced by $X^{\varepsilon,R}(t)$ initiating from $x\in R^2$. Let $\delta>0$ be arbitrary and observe that

$$\{\|X^{\varepsilon}-\phi\|_T<\delta\}\supseteqq\left\{\|X^{\varepsilon}-X^{\varepsilon,R}\|_T<\frac{\delta}{2},\ \|X^{\varepsilon,R}-\phi\|_T<\frac{\delta}{2}\right\}.$$

Then, we obtain

$$P(\|X^{\varepsilon}-\phi\|_T<\delta)\geqq P\Big(\|X^{\varepsilon,R}-\phi\|_T<\frac{\delta}{2}\Big)-P\Big(\|X^{\varepsilon}-X^{\varepsilon,R}\|_T\geqq\frac{\delta}{2}\Big). \tag{4.7}$$

Since $\|\phi\|_T<R'<R$, the definition of F_R and g_R implies that

$$F(\phi_1)=F_R(\phi_1) \quad\text{and}\quad g(\phi_1)=g_R(\phi_1) \qquad \text{for such a } \phi=(\phi_1, \phi_2),$$

and hence $\phi\in\mathcal{W}^R(x, T)$. Moreover, it follows from (4.4) that

$$S^{f;\,g}_{x;\,T}(\phi)=S^{R}_{x,T}(\phi) \qquad \text{for } \|\phi\|_T<R.$$

Since F_R and g_R satisfy a global Lipschitz condition in R^1, by Remark 1.1 we can get

(4.8)
$$P\left(\|X^{\varepsilon,R}-\phi\|_T<\frac{\delta}{2}\right)\geqq\exp\left\{-\frac{1}{\varepsilon}(S^R_{x,T}(\phi)+\gamma)\right\}$$
$$=\exp\left\{-\frac{1}{\varepsilon}(S^{f;g}_{x;T}(\phi)+\gamma)\right\}.$$

On the other hand, we have $X^\varepsilon=X^{\varepsilon,R}$ on $\|X^\varepsilon\|_T<R$ and

$$\{\|x\|\geqq R\}\subseteqq\{|x|\geqq R/\sqrt{2}\}\qquad\text{for } x=(x_1, x_2)\in R^2,$$

where $|x|=(x_1^2+x_2^2)^{1/2}$. Then, it holds

$$P\left(\|X^\varepsilon-X^{\varepsilon,R}\|_T\geqq\frac{\delta}{2}\right)$$
$$=P\left(\|X^\varepsilon-X^{\varepsilon,R}\|_T\geqq\frac{\delta}{2},\ \|X^\varepsilon\|_T<R\right)+P\left(\|X^\varepsilon-X^{\varepsilon,R}\|_T\geqq\frac{\delta}{2},\ \|X^\varepsilon\|_T\geqq R\right)$$
$$\leqq P(\|X^\varepsilon\|_T\geqq R)$$
$$\leqq P(\sup_{0\leqq t\leqq T}|X^\varepsilon(t)|\geqq R/\sqrt{2}).$$

Apply (4.6) to the last term of the above equation. Then we get

(4.9)
$$P\left(\|X^\varepsilon-X^{\varepsilon,R}\|_T\geqq\frac{\delta}{2}\right)<\exp\left\{-\frac{1}{\varepsilon}(l-\gamma)\right\}.$$

Combining (4.9) and (4.8) with (4.7), we obtain

$$P(\|X^\varepsilon-\phi\|_T<\delta)\geqq\exp\left\{-\frac{1}{\varepsilon}(S^{f;g}_{x;T}(\phi)+\gamma)\right\}-\exp\left\{-\frac{1}{\varepsilon}(l-\gamma)\right\}.$$

Let $l\uparrow\infty$ in the above equation. Then we get (4.5).

In order to prove (1.5) for $P^\varepsilon_x|_{\mathcal{W}}$ we consider the opposite inequality and follow the same argument as in the preceding procedure. Then, for $P^\varepsilon_x|_{\mathcal{W}}$, we can obtain the equivalent estimate (II) introduced by Remark 1.1, completing the proof.

§5. Oscillators with damping multiplied by ε.

Here we treat the system (1.4) and establish the same result as in Theorem 4.1. For this purpose, let us consider the systems (1.3) and (4.1) with $F\equiv 0$, which have unique solutions under Assumption 2.1 and Assumption 4.1 on g, since Corollary 2.2 and Lemma 4.1 hold. For each $x\in R^2$ and $T>0$, define $\tilde{\mathcal{W}}=\tilde{\mathcal{W}}(x, T)$ by $\tilde{\mathcal{W}}=\mathcal{W}^{F,g}(x, T)$ with $F\equiv 0$, and then define the functional $\tilde{S}^g_{x;T}$ on $\tilde{\mathcal{W}}$ as follows;

(5.1) $\tilde{S}^{g}_{x,T}(\phi)=\frac{1}{2}\int_0^T\left[\frac{d^2\phi_1(t)}{dt^2}+g(\phi_1(t))\right]^2 dt$

if $\phi=(\phi_1, \phi_2)\in\tilde{\mathcal{W}}(x, T)$, ϕ is absolutely continuous and the integral exists,

$\tilde{S}^{g}_{x,T}(\phi)=\infty$ otherwise.

Further, define $\tilde{\mathcal{W}}^R=\tilde{\mathcal{W}}^R(x, T)$ by $\tilde{\mathcal{W}}^R=\tilde{\mathcal{W}}^{F,g}(x, T)$ with $F\equiv 0$ and $g=g_R$, and put

(5.2) $$\tilde{S}^{R}_{x,T}(\phi)=\tilde{S}^{g}_{x,T}(\phi) \quad \text{with } g=g_R .$$

Then, we obtain the following theorem.

THEOREM 5.1. *Under Assumption* 2.1 *and Assumption* 4.1 *on* g, *let* $\tilde{X}^{\varepsilon}(t, x)$ *be the solution* $\tilde{X}^{\varepsilon}(t)$ *of* (1.4) *with the initial state* $\tilde{X}^{\varepsilon}(0)=x\in R^2$, *and let* $\tilde{P}^{\varepsilon}_x$ *be the probability measure induced by* $\tilde{X}^{\varepsilon}(\cdot, x)$ *on* $C([0, \infty); R^2)$. *Then, for each* $T>0$ *and* $x\in R^2$, $\{\tilde{P}^{\varepsilon}_x|_{\tilde{\mathcal{W}}}\ ;\ \varepsilon>0\}$ *satisfies the large deviation principle with a rate function* $\tilde{S}^{g}_{x,T}$ *defined by* (5.1).

PROOF. Let $X^{\varepsilon,R}(t)$ be the solution of (1.3) with $F\equiv 0$ and $g=g_R$, and let $\tilde{X}^{\varepsilon,R}(t)$ be the solution of (1.4) with $F=F_R$ and $g=g_R$, initiating from the same state $x\in R^2$. Denote by $P^{\varepsilon,R}_x$ and $\tilde{P}^{\varepsilon,R}_x$ the probability measures induced by $X^{\varepsilon,R}$ and $\tilde{X}^{\varepsilon,R}$ on $C([0, \infty); R^2)$, respectively. Then, we can proceed as follows:

Step 1. For each $T>0$ there exists a constant $K(T, R)>0$ depending on T and R such that

(5.3) $$P(\|\tilde{X}^{\varepsilon,R}-X^{\varepsilon,R}\|_T\leq \varepsilon K(T, R))=1 .$$

Step 2. Since Theorem 4.1 applies to $P^{\varepsilon,R}_x$ except that $\{S^{f,g}_{x,T}, \mathcal{W}\}$ is replaced by $\{\tilde{S}^{R}_{x,T}, \tilde{\mathcal{W}}^R\}$, (5.3) implies that $\{\tilde{P}^{\varepsilon,S}_x|_{\tilde{\mathcal{W}}^R}\ ;\ \varepsilon>0\}$ satisfies the large deviation principle with respect to $\tilde{S}^{R}_{x,T}$, where $\tilde{S}^{R}_{x,T}$ is given by (5.2).

Step 3. The proof of the theorem from Step 2 is quite analogous as the proof of Theorem 4.1, and we omit the details.

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