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ON THE NEUMANN PROBLEM FOR SOME LINEAR HYPERBOLIC SYSTEMS OF SECOND ORDER

By

Yoshihiro SHIBATA

Abstract. It is obtained the unique existence theorem of solutions to the mixed problem for linear hyperbolic systems of second order with inhomogeneous Neumann boundary condition. And, it is investigated that the constants appearing in the first energy inequality depend essentially only on $\mathscr{B}^{1+\mu}$-norms $(0<\mu\ll 1)$ of coefficients of operators.

§1. Introduction.

Let Ω be a domain in n-dimensional Euclidean space $\boldsymbol{R}^n$ having a boundary Γ which is a C^∞ and compact hypersurface. Throughout this paper except §8, we assume that $n\geqq 2$. Let $x=(x_1, \cdots, x_n)$ represent points of $\boldsymbol{R}^n$ and t a time variable. For differentiations we use the symbols $\partial_t=\partial/\partial t$ and $\partial_j=\partial/\partial x_j$. In this paper, we consider the following problem:

$$
\text{(N)}\qquad \begin{cases} P(t)[\vec{u}]=\partial_t^2\vec{u}+H(t, \bar{\partial}^1)\partial_t\vec{u}+A(t, \bar{\partial}^2)\vec{u}=\vec{f} & \text{in } [0, T]\times\Omega\,, \\ Q(t)[\vec{u}]=B(t, \bar{\partial}^1)\vec{u}+H^0(t, x)\partial_t\vec{u}=\vec{g} & \text{on } [0, T]\times\Gamma\,, \\ \vec{u}(0, x)=\vec{u}_0(x), \quad \partial_t\vec{u}(0, x)=\vec{u}_1(x) & \text{in } \Omega\,, \end{cases}
$$

where $H(t, \bar{\partial}^1)$, $A(t, \bar{\partial}^2)$ and $B(t, \bar{\partial}^1)$ are $m\times m$ matrices of differential operators of the forms:

(1.1.a) $H(t, \bar{\partial}^1)\vec{v}=2H^j(t, x)\partial_j\vec{v}+H^{n+1}(t, x)\vec{v}\,,$

(1.1.b) $A(t, \bar{\partial}^2)\vec{v}=-\partial_i(A^{ij}(t, x)\partial_j\vec{v})+A^j(t, x)\partial_j\vec{v}+A^{n+1}(t, x)\vec{v}\,,$

(1.1.c) $B(t, \bar{\partial}^1)\vec{v}=\nu_i(x)A^{ij}(t, x)\partial_j\vec{v}+B^i(t, x)\partial_i\vec{v}+B^{n+1}(t, x)\vec{v}\,,$

$\vec{v}={}^t(v_1, \cdots, v_m)$ (${}^tM$ means the transposed vector or matrix of M).

Here and hereafter, the summation convention is understood and in the summation the subscripts and superscripts i, i', j, j' (resp. p, p', q, q'; resp. a, a', b) take all values 1 to n (resp. 1 to $n-1$; resp. 1 to m). The $\nu_i(x)$ are real

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valued functions in $C_0^\infty(\boldsymbol{R}^n)$ such that $\nu(x)=(\nu_1(x), \cdots, \nu(x))$ represents the unit outer normal to Γ at $x\in\Gamma$. Put

$$H^k(t, x)=(H^k{}_a{}^b(t, x)), \quad A^l(t, x)=(A^l{}_a{}^b(t, x)), \quad A^{ij}(t, x)=(A^i{}_a{}^{jb}(t, x)),$$
$$B^l(t, x)=(B^l{}_a{}^b(t, x))$$

for $k=0, 1, \cdots, n+1$, $l=1, \cdots, n+1$, $i, j=1, \cdots, n$. The subscript a and superscript b denote the row and column, respectively. If $\vec{u}$ satisfies (N), then we say that $\vec{u}$ is a solution to (N) with initial data $\vec{u}_0$, $\vec{u}_1$, right member $\vec{f}$ and boundary data $\vec{g}$.

The purpose of this paper is to prove the unique existence theorem of solutions to (N) under suitable conditions and to investigate how constants appearing in energy inequalities depend on the coefficients of the operators $P(t)$ and $Q(t)$. The latter is quite important in proving the existence theorem of solutions to quasilinear operators (cf. Shibata [10] and Shibata-Nakamura [11]). If we consider 3-dimensional elasticities, in many cases the equations of motions are described by 3×3 quasilinear hyperbolic systems of 2nd order with Neumann boundary conditions (cf. [11]). And then, the linearized problems are described by (N). It is one of the reasons why we must consider the system of 2nd order.

In case that $m=1$ (i.e., scalar operator case), Ikawa [2, 3], Miyatake [6] and Yagi [13] proved the unique existence theorem of solutions. Especially, Miyatake [6] gave a necessary and sufficient condition for L^2-wellposedness. Ikawa [3] treated the case where the inequality:

$$\nu_i(x)H^i(t, x)+H^0(t, x)\geqq\delta \qquad \text{on } [0, T]\times\Gamma, \tag{1.2}$$

is valid for some constant $\varepsilon>0$. Roughly speaking, his method was as follows. First, using the Hille-Yoshida theorem, he proved the unique existence theorem when the coefficients of operators are independent of t. Secondly, he derived the energy inequalities by the usual energy method. Finally, by Cauchy's zigzag line method, he proved the unique existence theorem in time dependent case. To show the convergence of Cauch's zigzag line and the uniqueness of solutions, the energy inequalities play an essential role. In this manner, it is sufficient to assume that the coefficients are in $\mathscr{B}^2$.

In case where

$$\nu_i(x)H^i(t, x)+H^0(t, x)\geqq 0 \qquad \text{on } [0, T]\times\Gamma \tag{1.3}$$

and the boundary data $g\not\equiv 0$, Miyatake [6] first got the energy inequalities and the unique existence theorem of solutions to (N). Roughly speaking, his method of getting energy inequalities was as follows. Making Laplace transform in t and Fourier transform in tangential variables near the boundary and using simple localizations, he got some kind of *a priori* estimate with zero initial data by the method of finding so called the operator Q for P (this Q of course differs from our boundary operator Q). This method was first developped by Kreiss [4] and Sakamoto [8] to get *a priori* estimate for the general hyperbolic mixed problem under the uniform Lopatinski condition. And then, in the similar manner to Sakamoto [9], first by *a priori* estimate mentioned above and Riesz's representation theorem he got the unique existence theorem of solutions with zero initial data; secondly, using solutions with zero initial data, he obtained the usual energy inequalities. Furthermore, using the existence theorem with zero initial data and energy inequalities he proved the unique existence theorem. In his method, the theory on L^2-boundedness of pseudo-pifferential operators played an essential role. Thus, he must assume that the coefficients of the operators are sufficiently smooth, for example in $\mathscr{B}^\infty$. To remove the smoothness assumptions on the coefficients of the operators, to the auther it seems that more delicate discussions are needed.

The originalities of this paper are as follows.

1° The operators are systems of second order.

2° It is assumed that the coefficients of operators are in $\mathscr{B}^2$ while the boundary condition is inhomogeneous and (1.3) is assumed.

3° It is proved that the constants in first energy inequalities depend essentially only on the $\mathscr{B}^{1+\mu}$-norms of coefficients of the operators $(0<\mu<1)$.

Even in the case that $m=1$ (scalar operator case), under 2° the unique existence theorem of solutions to (N) and the assertion 3° do not follow immediately from results due to Miyatake [6]. We need other ideas.

The paper is organized as follows. In § 2, we explain the basic notations and introduce assumptions, and then the main results are stated. In §§ 3 and 4, we make preparations for proving the existence theorem of solutions to operators satisfying (1.2) rather than (1.3). This existence theorem plays an important role in proving the existence theorem and energy inequalities of the original problem. The discussions in §§ 3 and 4 are essentially the same as in Ikawa [3]. In § 5, we show some kind of *a priori* estimate with zero initial data in the spirits of Kreiss [4], Sakamoto [8] and Miyatake [6]. In § 6, the first energy inequality is obtained under the assumption that initial data are zero. In § 7, the usual energy inequalities and the existence theorem of (N) are obtained. In § 8, in the case where $n=1$ we prove the unique existence theorem of solutions to (N) and obtain the energy inequalities.

§2. Notations, assumptions and main results.

First, we explain basic notations used throughout the paper. We always assume that functions are real-valued, except for §5 and Appendix. Let $\delta^{\alpha\beta}$, $\delta_\alpha{}^\beta$, $\delta_{\alpha\beta}$ be Kronecker's delta symbols, i.e., $\delta^{\alpha\beta}=\delta_\alpha{}^\alpha=\delta_{\alpha\alpha}=1$ and $\delta^{\alpha\beta}=\delta_\alpha{}^\beta=\delta_{\alpha\beta}=0$ for $\alpha\neq\beta$. Let G be a domain in $\boldsymbol{R}^n$. By $\|\cdot\|_G$ we denote the usual L^2-norm for scalar functions defined on G. For any integer $L\geqq 0$, scalar functions u, v and vector-valued functions $\vec{u}$, $\vec{v}$, put

$$\vec{u}\cdot\vec{v}=\sum u_a v_a,\ |\vec{u}|^2=\vec{u}\cdot\vec{u},\ \|\vec{u}\|_G^2=\sum\|u_a\|_G^2,\ (u, v)_G=\int_G u(x)v(x)dx\ ,$$

$$(\vec{u}, \vec{v})_G=\sum(u_a, v_a)_G,\ \|u\|=\|u\|_\Omega,\ \|\vec{u}\|=\|\vec{u}\|_\Omega,\ (u, v)=(u, v)_\Omega\ ,$$

$$(\vec{u}, \vec{v})=(\vec{u}, \vec{v})_\Omega,\ \langle u, v\rangle=\int_\Gamma u(x)v(x)dS_x\ \ (dS_x\text{: the surface element of }\Gamma).$$

$$\langle\vec{u}, \vec{v}\rangle=\sum\langle u_a, v_a\rangle,\ \partial_t^k\partial_x^\alpha\vec{u}={}^t(\partial_t^k\partial_x^\alpha u_1, \cdots, \partial_t^k\partial_x^\alpha u_m),\ D^L u=(\partial_t^k\partial_x^\alpha u\ ;\ k+|\alpha|=L),$$

$$\bar{D}^L u=(\partial_t^k\partial_x^\alpha u\ ;\ k+|\alpha|\leqq L),\ \partial^L u=(\partial_x^\alpha u\ ;\ |\alpha|=L),\ \bar{\partial}^L u=(\partial_x^\alpha u\ ;\ |\alpha|\leqq L)\ ,$$

$$D^L\vec{u}=(D^L u_a\ ;\ a=1, \cdots, m),\ \bar{D}^L\vec{u}=(\bar{D}^L u_a\ ;\ a=1, \cdots, m)\ ,$$

$$\partial^L\vec{u}=(\partial^L u_a\ ;\ a=1, \cdots, m),\ \bar{\partial}^L\vec{u}=(\bar{\partial}^L u_a\ ;\ a=1, \cdots, m)\ .$$

The $H^L(G)$ denote the usual Sobolev spaces over G with norms $\|\bar{\partial}^L u\|_G$. The $H^L(G)\times\cdots\times H^L(G)$ denote Sobolev spaces for vector valued functions and for the sake of simplicity we denote them also by $H^L(G)$. Let G' be a set in $\boldsymbol{R}^k$ $(k\geqq 1)$ and X, Y represent points of $\boldsymbol{R}^k$. For any integer $l\geqq 0$ and $\sigma\in(0, 1)$, put

$$|u|_{\infty, l, G'}=\sum_{|\alpha|\leqq l}\sup_{X\in G'}|(\partial^\alpha u)(X)|\qquad(\text{here, }\partial^\alpha=(\partial/\partial X_1)^{\alpha_1}\cdots(\partial/\partial X_k)^{\alpha_k})\ ,$$

$$|u|_{\infty, l+\sigma, G'}=|u|_{\infty, l, G'}+\sum_{|\alpha|=l}\sup_{\substack{X, Y\in G'\\ X\neq Y}}\frac{|(\partial^\alpha u)(X)-(\partial^\alpha u)(Y)|}{|X-Y|^\sigma}\ .$$

For any matrix $U=(U_a{}^b)$, put $|U|_{\infty, l+\kappa, G'}=\sum|U_a{}^b|_{\infty, l+\kappa, G'}$ $(0\leqq\kappa<1)$. Since Γ is a C^∞ and compact hypersurface, we may assume that there exist finite numbers of open sets $\mathcal{O}_l$ $(l=1, \cdots, N)$ integers $d(l)$ $(1\leqq d(l)\leqq n)$, $\sigma_l>0$ and C^∞ functions ρ_l such that $\mathcal{O}_l\cap\Gamma=\{x\in\boldsymbol{R}^n\,|\,x_{d(l)}=\rho_l(x'),\ |x'|<\sigma_l\}$ where $x'=(x_1, \cdots, x_{d(l)-1}, x_{d(l)+1}, \cdots, x_n)$. We may also assume that

$$\mathcal{O}_l\cap\Omega\subset\{x\in\boldsymbol{R}^n\,|\,x_{d(l)}<\rho_l(x'),\ |x'|<\sigma_l\}. \tag{2.1}$$

Put $\Phi_k^l(x)=x_k$, $1\leqq k\leqq d(l)-1$; $\Phi_{k-1}^l(x)=x_k$, $d(l)+1\leqq k\leqq n$; $\Phi_n^l(x)=\rho_l(x')-x_{d(l)}$. Define the map $\Phi_l(x)$ by $\Phi_l(x)=(\Phi_1^l(x), \cdots, \Phi_n^l(x))$. We may assume that the Φ_l are C^∞-diffeomorphisms of $\mathcal{O}_l$ onto $Q(\sigma_l)$ and that $\Phi_l(\mathcal{O}_l\cap\Omega)=Q^+(\sigma_l)$, where

$Q(\sigma_l)=\{y=(y_1, \cdots, y_n)\in \boldsymbol{R}^n \mid |y'|=|(y_1, \cdots, y_{n-1})|<\sigma_l, |y_n|<\sigma_l\}$, $Q^+(\sigma_l)=\{y\in \boldsymbol{R}^n_+ \mid |y'|<\sigma_l, y_n<\sigma_l\}$ and $\boldsymbol{R}^n_+=\{y\in\boldsymbol{R}^n \mid y_n>0\}$. Let Ψ_l be the inverse maps of Φ_l. Let ϕ_k, $k=0, 1, \cdots, N$, be functions having the following properties:

(2.2.a) $$\phi_0\in C_0^\infty(\Omega),$$

(2.2.b) $$\phi_l\in C_0^\infty(\mathcal{O}_l), \qquad l=1, \cdots, N,$$

(2.2.c) $$\sum_{k=0}^{N}\phi_k^2(x)=1 \qquad \text{on } \bar{\Omega}.$$

Let ϕ_l', $l=1, \cdots, N$, be functions in $C_0^\infty(\mathcal{O}_l)$ such that

(2.2.d) $$\phi_l'(x)=1 \qquad \text{on supp}\,\phi_l.$$

For any real number $r\neq 0$, put

$$\langle u\rangle_r^2=\sum_{l=1}^{N}\int_{R^{n-1}}|\hat{v}_l(\xi')|^2(1+|\xi'|^2)^r d\xi', \ \xi'=(\xi_1, \cdots, \xi_{n-1}),$$

where $v_l(y')=u(\Psi_l(y', 0))\phi_l'(\Psi_l(y', 0))$ and the $\hat{v}_l(\xi')$ denotes the Fourier transform of v_l. Put

$$\langle u\rangle_0^2=\langle u, u\rangle, \quad \langle \vec{u}\rangle_r=\sum\langle u_a\rangle_r^2,$$

$$H^r(\Gamma)=\{u \text{ (resp. } \vec{u})\mid \langle u\rangle_r \text{ (resp. } \langle\vec{u}\rangle_r)<\infty\}.$$

For any non-negative integer s and an interval I of $\boldsymbol{R}$, put

$$|u|_{\infty, s, I\times\Gamma}=\sum_{l=1}^{N}|v_l'|_{\infty, s, I\times B(\sigma_l)},$$

where $v_l'(t, y')=\phi_l'(\Psi_l(y', 0))u(t, \Psi_l(y', 0))$ and $B(\sigma_l)'=\{y'\in\boldsymbol{R}^{n-1} \mid |y'|<\sigma_l\}$. For a matrix $(U_a{}^b)$, put $|U|_{\infty, s, I\times\Gamma}=\sum|U_a{}^b|_{\infty, s, I\times\Gamma}$. For any interval I of $\boldsymbol{R}$ and a Banach space X, $C^L(I; X)$ denotes the set of all X-valued continuous functions in I having all derivatives $\leqq L$ continuous in I. When I is bounded, $C^L(I; X)$ is equipped with uniform topology, i.e., what u_k converges to u in $C^L(I; X)$ as $k\to\infty$ means that $\sup\{\|u_k(s)-u(s)\|_X ; s\in I\}\to 0$ as $k\to\infty$ where $\|\cdot\|_X$ denotes the norm of X. Put $E^L(I, G)=\bigcap_{l=0}^{L}C^l(I; H^{L-l}(G))$, $E^L(I)=E^L(I; \Omega)$ and $E^{L,r}(I, \Gamma)=\bigcap_{l=0}^{L}C^l(I; H^{L+r-l}(\Gamma))$. μ always refers to a very small positive number $(\in(0, 1))$. For any intervals I and J of $\boldsymbol{R}$, put

$$\mathcal{M}_{I,J}(0)=\sum_{i,j=1}^{n}|A^{ij}|_{\infty,0,I\times\Omega}+\sum_{k=1}^{n+1}|A^k|_{\infty,0,I\times\Omega}+\sum_{i=1}^{n}|H^i|_{\infty,0,J\times\Omega}$$
$$+|H^{n+1}|_{\infty,0,I\times\Omega}+|H^0|_{\infty,0,J\times\Gamma}+\sum_{k=1}^{n+1}|B^k|_{\infty,0,I\times\Gamma},$$

$$\mathcal{M}_{I,J}(r)=\sum_{i,j=1}^{n}|A^{ij}|_{\infty,r,I\times\Omega}+\sum_{i=1}^{n}|H^{i}|_{\infty,r,J\times\Omega}+|H^{0}|_{\infty,r,J\times\Gamma}+B^{n+1}|_{\infty,1,I\times\Gamma}$$

$$+\sum_{i=1}^{n}|B^{i}|_{\infty,r,I\times\Gamma}+\sum_{k=1}^{n+1}|A^{k}|_{\infty,0,I\times\Omega}+|H^{n+1}|_{\infty,0,I\times\Omega},\ 1\leqq r<2,$$

$$\mathcal{M}_{I,J}(2)=\sum_{i,j=1}^{n}|A^{ij}|_{\infty,2,I\times\Omega}+\sum_{i=1}^{n}|H^{i}|_{\infty,1+\mu,J\times\Omega}+|H^{0}|_{\infty,1+\mu,J\times\Gamma}$$

$$+\sum_{k=1}^{n+1}|B^{k}|_{\infty,2,I\times\Gamma}+\sum_{k=1}^{n+1}|A^{k}|_{\infty,1,I\times\Omega}+|H^{n+1}|_{\infty,1,I\times\Omega}.$$

For the sake of simplicity, we put $\mathcal{M}(r)=\mathcal{M}_{R,R}(r)$ and $\mathcal{M}_{T}(r)=\mathcal{M}_{[0,T],[-\kappa,T+\kappa]}(r)$ with some fixed $\kappa>0$.

$C=C(\cdots\cdots)$ denotes a constant depending essentially only on the quantities appearing in parentheses. In a given context, the same letter C will generally be used to denote different constants depending on the same set of arguments.

Now, we introduce the assumptions.

(A.1) The $A^{i}{}_{a}{}^{jb}$ and $H^{i}{}_{a}{}^{b}$ are in $\mathcal{B}^{2}(\boldsymbol{R}\times\bar{\Omega})$; $H^{n+1}{}_{a}{}^{b}$ and $A^{l}{}_{a}{}^{b}$ in $\mathcal{B}^{l}(\boldsymbol{R}\times\bar{\Omega})$; $H^{0}{}_{a}{}^{b}$ and $B^{l}{}_{a}{}^{b}$ in $\mathcal{B}^{2}(\boldsymbol{R}\times\Gamma)$ for $i, j=1, \cdots, n$, $l=1, \cdots, n+1$, and $a, b=1, \cdots, m$.

(A.2) $A^{i}{}_{a}{}^{jb}(t, x)=\delta^{ij}\delta_{a}{}^{b}$ and other functions all vanish for $|t|>T_{0}$ with some $T_{0}>0$.

(A.3) ${}^{t}A^{ij}(t, x)=A^{ji}(t, x)$, ${}^{t}H^{i}(t, x)=H^{i}(t, x)$, ${}^{t}H^{0}(t, x')=H^{0}(t, x')$, ${}^{t}B^{i}(t, x')=-B^{i}(t, x')$ for any $(t, x)\in\boldsymbol{R}\times\bar{\Omega}$ and $(t, x')\in\boldsymbol{R}\times\Gamma$.

(A.4) There exist constants δ_{1} and $\delta_{2}>0$ such that

$(A^{ij}(t, \cdot)\partial_{j}\vec{u}, \partial_{i}\vec{u})+\langle B^{i}(t, \cdot)\partial_{i}\vec{u}, \vec{u}\rangle\geqq2\delta_{1}\|\partial^{1}\vec{u}\|^{2}-\delta_{2}\|\vec{u}\|^{2}$

for any $\vec{u}\in H^{2}(\Omega)$ and $t\in\boldsymbol{R}$.

(A.5) $\nu_{i}(x)B^{i}(t, x)=0$ for all $(t, x)\in\boldsymbol{R}\times\Gamma$.

(A.6) The inequality: $(\nu_{i}(x)H^{i}(t, x)+H^{0}(t, x))\vec{v}\cdot\vec{v}\geqq0$, holds for any constant vector $\vec{v}\in\boldsymbol{R}^{m}$ and $t\in\boldsymbol{R}$.

Now, we introduce compatibility conditions. Put

$$H^{(l)}(t_{0}, \bar{\partial}^{1})\vec{u}(x)=\partial_{t}^{l}[H(t, \bar{\partial}^{1})\vec{u}(x)]|_{t=t_{0}},$$

$$A^{(l)}(t_{0}, \bar{\partial}^{2})\vec{u}(x)=\partial_{t}^{l}[A(t, \bar{\partial}^{2})\vec{u}(x)]|_{t=t_{0}},$$

$$B^{(l)}(t_{0}, \bar{\partial}^{1})\vec{u}(x)=\partial_{t}^{l}[B(t, \bar{\partial}^{1})\vec{u}(x)]|_{t=t_{0}}.$$

For $k\geqq0$, we define $\vec{u}_{k+2}(x)$ successively by

$$\vec{u}_{k+2}(x)=\partial_{t}^{k}\vec{f}(0, x)-\sum_{l=0}^{k}\binom{k}{l}\{H^{(l)}(0, \bar{\partial}^{1})\vec{u}_{k+1-l}(x)+A^{(l)}(0, \bar{\partial}^{2})\vec{u}_{k-l}(x)\}. \tag{2.3}$$

DEFINITION 2.1. Let L be an integer $\geqq 2$. We shall say that $\vec{u}_0$, $\vec{u}_1$, $\vec{f}$ and $\vec{g}$ satisfy the compatibility condition of order $L-2$ if

$$\sum_{l=0}^{k} \binom{k}{l} \{B^{(l)}(0, \bar{\partial}^1)\vec{u}_{k-l}(x) + (\partial_t^l H^0)(0, x)\vec{u}_{k+1-l}(x)\} = (\partial_t^k \vec{g})(0, x) \qquad \text{on } \Gamma$$

for $k=0, 1, \cdots, L-2$.

Now, we state our main results.

THEOREM 2.2. *Assume that* $n \geqq 2$. *Let* $T>0$. *Assume that* (A.1)-(A.6) *are valid.* 1° *If* $\vec{u}_0 \in H^2(\Omega)$, $\vec{u}_1 \in H^1(\Omega)$, $\vec{f} \in C^1([0, T]; L^2(\Omega))$, $\vec{g} \in C^1([0, T]; H^{1/2}(\Omega))$ *and the compatibility condition of order* 0 *is satisfied, then there exists a unique solution* $\vec{u} \in E^2([0, T])$ *of* (N) *with initial data* $\vec{u}_0$, $\vec{u}_1$, *right member* $\vec{f}$ *and boundary data* $\vec{g}$. 2° *Let* μ *be a small positive number* $\in (0, 1)$. *Put*

$$\mathcal{E}(t, \vec{u}) = \int_0^t \{\|P(s)[\vec{u}(s, \cdot)]\|^2 + \langle Q(s)[\vec{u}(s, \cdot)]\rangle_{1/2}^2\} ds .$$

Then, there exists a constant $C=C(\delta_1, \delta_2, \Gamma, \mathcal{M}(1+\mu))>0$ *such that for any* $t \in [0, T]$ *and* $\vec{u} \in E^2([0, T])$ *the following three estimates hold*:

(a) $\|\partial_t \vec{u}(t, \cdot)\|^2 + \|\vec{u}(t, \cdot)\|_{\mathcal{J}(t)}^2 \leqq 2e^{Ct}\{\|\partial_t \vec{u}(0, \cdot)\|^2 + \|\vec{u}(0, \cdot)\|_{\mathcal{J}(0)}^2 + C\mathcal{E}(t, \vec{u})\}$,

(b) $\int_0^t \langle \partial_s \vec{u}(s, \cdot)\rangle_{-1/2}^2 ds \leqq Ce^{Ct}\{\|\bar{D}^1 \vec{u}(0, \cdot)\|^2 + \mathcal{E}(t, \vec{u})\}$,

(c) $\|\partial_t \vec{u}(t, \cdot)\|^2 + \|\vec{u}(t, \cdot)\|_{\mathcal{J}(t)}^2 \leqq e^{Ct}\{\|\partial_t \vec{u}(0, \cdot)\|^2 + \|\vec{u}(0, \cdot)\|_{\mathcal{J}(0)}^2\}$

$$+Ce^{Ct}\{\|\bar{D}^1 \vec{u}(0, \cdot)\|^2 + \mathcal{E}(t, \vec{u})\}^{1/2}\mathcal{E}(t, \vec{u})^{1/2} .$$

Here, the norm $\|\cdot\|_{\mathcal{J}(t)}$ *will be defined in* (3.30.a) *of* §3 *below.*

THEOREM 2.3. *Assume that* $n \geqq 2$. *Let* $T>0$ *and* L *be an integer* $\geqq 3$. *In addition to* (A.1)-(A.6), *we assume that* $A^i{}_a{}^{jb} \in \mathcal{B}^L([0, T] \times \bar{\Omega})$; $H^k{}_a{}^b$, $A^k{}_a{}^b \in \mathcal{B}^{L-1}([0, T] \times \bar{\Omega})$; $H^0{}_a{}^b$, $B^k{}_a{}^b \in \mathcal{B}^L([0, T] \times \Gamma)$. *If* $\vec{u}_0 \in H^L(\Omega)$, $\vec{u}_1 \in H^{L-1}(\bar{\Omega})$, $\vec{f} \in C^{L-1}([0, T]; L^2(\Omega)) \cap E^{L-2}([0, T])$, $\vec{g} \in C^{L-1}([0, T]; H^{1/2}(\Gamma)) \cap E^{L-2, 1/2}([0, T]; \Gamma)$ *and the compatibility condition of order* $L-2$ *are satisfied, then the solution* $\vec{u}$ *of* (N) *with initial data* $\vec{u}_0$, $\vec{u}_1$, *right member* $\vec{f}$ *and boundary data* $\vec{g}$ *belongs to* $E^L([0, T])$.

Finally, we shall prove that what coefficients are defined for all $t \in \boldsymbol{R}$ imposes no restrictions on us in some sense. We can extend coefficients as follows.

LEMMA 2.4. *Let* T *and* $\kappa>0$. *Let* $\tilde{A}^i{}_a{}^{jb}$, $\tilde{H}^l{}_a{}^b$, $\tilde{A}^k{}_a{}^b$, $\tilde{B}^k{}_a{}^b$, $i, j=1, \cdots, n$, $l=0, 1, \cdots, n+1$, $k=1, \cdots, n+1$, $a, b=1, \cdots, m$, *be functions satisfying the following properties*:

(a.1) $\tilde{A}^{i}{}_{a}{}^{jb}\in\mathcal{B}^2([0, T]\times\bar{\Omega})$, $\tilde{H}^{i}{}_{a}{}^{b}\in\mathcal{B}^2([-\kappa, T+\kappa]\times\bar{\Omega})$,
$\tilde{H}^{n+1}{}_{a}{}^{b}$, $\tilde{A}^{k}{}_{a}{}^{b}\in\mathcal{B}^1([0, T]\times\bar{\Omega})$, $\tilde{H}^{0}{}_{a}{}^{b}\in\mathcal{B}^2([-\kappa, T+\kappa]\times\Gamma)$,
$\tilde{B}^{k}{}_{a}{}^{b}\in\mathcal{B}^2([0, T]\times\Gamma)$.

(a.2) ${}^t\tilde{A}^{ij}(t, x)=\tilde{A}^{ji}(t, x)$, ${}^t\tilde{H}^{i}(t', x)=\tilde{H}^{i}(t', x)$, ${}^t\tilde{H}^{0}(t, x')=\tilde{H}^{0}(t, x')$,
${}^t\tilde{B}^{i}(t', x')=-\tilde{B}^{i}(t', x')$ *for any* $(t, x)\in[0, T]\times\bar{\Omega}$, $(t', x)\in[-\kappa, T+\kappa]\times\bar{\Omega}$.
$(t, x')\in[-\kappa, T+\kappa]\times\Gamma$ *and* $(t', x')\in[0, T]\times\Gamma$.

(a.3) *There exist constants* δ_1 *and* $\delta_2>0$ *such that*

$$(\tilde{A}^{ij}(t, \cdot)\partial_j\vec{u}, \partial_i\vec{u})+\langle\tilde{B}^{i}(t, \cdot)\partial_i\vec{u}, \vec{u}\rangle\geqq 3\delta_1\|\partial^1\vec{u}\|^2-(\delta_2/2)\|\vec{u}\|^2$$

for any $\vec{u}\in H^2(\Omega)$ *and* $t\in[0, T]$.

(a.4) *The inequality*: $(\nu_i(x)H^i(t, x)+H^0(t, x))\vec{u}\cdot\vec{u}\geqq 0$, *holds for any* $(t, x)\in[-\kappa, T+\kappa]\times\Gamma$ *and constant vector* $\vec{u}\in\boldsymbol{R}^m$.

(a.5) $\nu_i(x)B^i(t, x)=0$ *for any* $(t, x)\in[0, T]\times\Gamma$.

Here, we have put

$$\tilde{A}^{ij}=(\tilde{A}^{i}{}_{a}{}^{jb}),\ \tilde{H}^{l}=(\tilde{H}^{l}{}_{a}{}^{b}),\ \tilde{A}^{k}=(\tilde{A}^{k}{}_{a}{}^{b}),\ \tilde{B}^{k}=(\tilde{B}^{k}{}_{a}{}^{b}).$$

Then, there exist $A^{i}{}_{a}{}^{jb}$, $H^{l}{}_{a}{}^{b}$, $A^{k}{}_{a}{}^{b}$, $B^{k}{}_{a}{}^{b}$ *which satisfy* (A.1)-(A.6) *and have the following properties*:

$$\begin{aligned}&A^{i}{}_{a}{}^{jb}(t, x)=\tilde{A}^{i}{}_{a}{}^{jb}(t, x),\ H^{k}{}_{a}{}^{b}(t, x)=\tilde{H}^{k}{}_{a}{}^{b}(t, x),\ A^{k}{}_{a}{}^{b}(t, x)=\tilde{A}^{k}{}_{a}{}^{b}(t, x),\\&H^{0}{}_{a}{}^{b}(t, x')=\tilde{H}^{0}{}_{a}{}^{b}(t, x'),\ B^{k}{}_{a}{}^{b}(t, x')=\tilde{B}^{k}{}_{a}{}^{b}(t, x')\end{aligned}\tag{2.4}$$

for any $(t, x)\in[0, T]\times\bar{\Omega}$ *and* $(t, x')\in[0, T]\times\Gamma$.

Let $\tilde{\mathcal{M}}_T(r)$ *be the bound for functions with tilde defined in the same manner as* $\mathcal{M}_T(r)$ (*cf. Notation*). *Then, there exists a* $C=C(T, \Gamma)$ *such that*

$$\mathcal{M}(r)\leqq C\tilde{\mathcal{M}}_T(r)\tag{2.4}$$

for $r=0$ *and* $1\leqq r\leqq 2$.

Proof. By Lions' method, we shall extend functions. Let $\phi(t)\in C^\infty(\boldsymbol{R})$ so that $0\leqq\phi\leqq 1$, $\phi(t)=1$ for $t<T/3$ and $=0$ for $t>2T/3$. Take $a_l=l+1$, $l=0, 1, 2$ and choose b_l so that $\sum_{l=0}^{2}b_l(-a_l)^k=1$ for $k=0, 1, 2$, i.e., $b_0=6$, $b_1=-8$, $b_2=3$. For any function f defined on $[0, T]$, put

$$f^+(t, x)=\phi(t)f(t, x) \text{ for } t\geqq 0 \quad\text{and}\quad =\sum_{l=0}^{2}b_l\phi(-a_lt)f(-a_lt, x) \text{ for } t<0,$$

$$f^-(t, x)=(1-\phi(t))f(t, x) \text{ for } t\leqq T \quad\text{and}\quad -\sum_{l=0}^{2}b_l[(1-\phi)f(\cdot, x)](T-a_l(t-T)) \text{ for } t>T.$$

And then, set $E(f)(t, x)=f^{+}(t, x)+f^{-}(t, x)$. Given $t<0$ (resp. $t>T$), by the mean value theorem we have

$$\begin{aligned}&\delta^{aa'}\{(E(\tilde{A}^{i}{}_{a'}{}^{jb})(t, \cdot)\partial_{j}u_{b}, \partial_{i}u_{a})+\langle E(\tilde{B}^{i}{}_{a'}{}^{b})(t, \cdot)\partial_{i}u_{b}, u_{a}\rangle\}\\&\leqq(\tilde{A}^{ij}(t_{0}, \cdot)\partial_{j}\vec{u}, \partial_{i}\vec{u})+\langle\tilde{B}^{i}(t_{0}, \cdot)\partial_{i}\vec{u}, \vec{u}\rangle\\&\quad -C(T)|t|(\text{resp. } |T-t|)(\sum_{i,j=1}^{n}|\partial_{t}\tilde{A}^{ij}|_{\infty,0,[0,T]\times\Omega}\\&\quad+\sum_{i=1}^{n}|\bar{D}^{1}\tilde{B}^{i}|_{\infty,0,[0,T]\times\Gamma})\|\bar{\partial}^{1}\vec{u}\|^{2},\end{aligned}$$

where $t_0=0$ (resp. $t_0=T$). Here, we have used (3.9)-(3.11) in §3 below. By (a.3) we have that there exists a $\kappa'>0$ such that

$$\delta^{aa'}\{(E(\tilde{A}^{i}{}_{a'}{}^{jb})(t, \cdot)\partial_{j}u_{b}, \partial_{i}u_{a})+\langle E(\tilde{B}^{i}{}_{a'}{}^{b})(t, \cdot)\partial_{i}u_{b}, u_{a}\rangle\}\geqq 2\delta_{1}\|\partial^{1}\vec{u}\|^{2}-\delta_{2}\|\vec{u}\|^{2}$$

for any $t\in[-\kappa', T+\kappa']$ and $\vec{u}\in H^{2}(\Omega)$. Put $\kappa''=\min(\kappa, \kappa')$. Choose $\rho(t)\in C_{0}^{\infty}(\boldsymbol{R})$ so that $0\leqq\rho\leqq 1$, $\rho(t)=1$ for $t\in[-\kappa''/2, T+(\kappa''/2)]$ and $=0$ for $t<-\kappa''$ or $t>T+\kappa''$. Put

$$\begin{aligned}&A^{i}{}_{a}{}^{jb}(t, x)=\rho(t)E(\tilde{A}^{i}{}_{a}{}^{jb})(t, x)+2\delta_{1}(1-\rho(t))\delta^{ij}\delta_{a}{}^{b},\\&H^{i}{}_{a}{}^{b}(t, x)=\rho(t)\tilde{H}^{i}{}_{a}{}^{b}(t, x),\quad H^{0}{}_{a}{}^{b}(t, x)=\rho(t)\tilde{H}^{0}{}_{a}{}^{b}(t, x),\\&H^{n+1}{}_{a}{}^{b}(t, x)=\rho(t)E(\tilde{H}^{n+1}{}_{a}{}^{b})(t, x),\quad A^{k}{}_{a}{}^{b}(t, x)=\rho(t)E(\tilde{A}^{k}{}_{a}{}^{b})(t, x),\\&B^{k}{}_{a}{}^{b}(t, x)=\rho(t)E(\tilde{B}^{k}{}_{a}{}^{b})(t, x).\end{aligned}$$

Then, these functions without tildes satisfy all desired properties, which completes the proof of the lemma.

§3. Preliminaries.

First, we shall discuss the boundary value problem for $A(t, \bar{\partial}^{2})$ and $B(t, \bar{\partial}^{1})$ in Ω, t being regarded as a parameter; secondly we shall derive an existence theorem for $P(t_0)$ and $Q(t_0)$ with coefficients freezed at $t=t_0\in[0, T]$. First of all, we shall give three lemmas. The first one is well-known (cf. Hörmander [1, §2.5]).

LEMMA 3.1. *There exists a* $C=C(\Gamma)$ *such that*

(i) $|\langle u, v\rangle|\leqq C\langle u\rangle_{1/2}\langle v\rangle_{-1/2}$ *for any* $u\in H^{1/2}(\Gamma)$ *and* $v\in H^{-1/2}(\Gamma)$,

(ii) $\langle u\rangle_{1/2}\leqq C\|\bar{\partial}^{1}u\|$ *for any* $u\in H^{1}(\Omega)$.

LEMMA 3.2. *Let* $a(x)\in\mathscr{B}^{1}(\Gamma)$ *and* $u(x)\in H^{1/2}(\Gamma)$. *Then, there exists a* $C=C(\Gamma)$ *such that*

$$\langle au\rangle_{1/2}\leqq C|a|_{\infty,1,\Gamma}\langle u\rangle_{1/2},$$

where $|a|_{\infty,k,\Gamma}=\sum|b_l|_{\infty,k,B(\sigma_l)}$, $(b_l(y')=\phi_l'(\Psi_l(y',0))a(\Psi_l(y',0)))$.

Proof. Interpolating the two inequalities: $\langle au\rangle_0\leqq|a|_{\infty,0,\Gamma}\langle u\rangle_0$ and $\langle au\rangle_1\leqq C|a|_{\infty,1,\Gamma}\langle u\rangle_1$, we get the lemma.

LEMMA 3.3. *Let G be a domain in $\boldsymbol{R}^n$ and $P^{iajb}(x)$ functions in $C^0(G)$. Put $P^{ij}(x)=(P^{iajb}(x))$ where the superscripts a and b denote the row and column, respectively. Assume that there exist d_1, $d_2>0$ such that*

$$(3.1.a)\qquad (P^{ij}\partial_j\vec{u},\ \partial_i\vec{u})_G\geqq d_1\|\partial^1\vec{u}\|^2-d_2\|\vec{u}\|^2\qquad \text{for any } \vec{u}\in C_0^\infty(G)$$

and that

$$(3.1.b)\qquad {}^tP^{ij}(x)=P^{ji}(x)\qquad \text{for any } i,j=1,\cdots,n \text{ and } x\in G\,.$$

Then,

$$(3.2)\qquad P^{iajb}(x)\xi_i\xi_j\eta_a\eta_b\geqq d_1|\xi|^2|\eta|^2$$

for any $x\in G$, $\xi=(\xi_1,\cdots,\xi_n)\in\boldsymbol{R}^n$ and $\eta=(\eta_1,\cdots,\eta_m)\in\boldsymbol{R}^m$.

Proof. Let $x_0\in G$, $\xi\in\boldsymbol{R}^n$ and $\eta\in\boldsymbol{R}^m$ be taken arbitrary. For any $\sigma>0$, let us choose $\delta>0$ so that

$$(3.3)\qquad |P^{iajb}(x)-P^{iajb}(x_0)|<\sigma \quad\text{and}\quad x\in G$$

provided that $|x-x_0|<\delta$. Choose $\tilde{\chi}(x)\in C^\infty$ so that $\operatorname{supp}\chi\subset\{x\in\boldsymbol{R}^n\,|\,|x-x_0|<\delta\}$ $(\subset G)$ and put $\vec{u}(x)={}^t\eta[\exp(\sqrt{-1}\,x\cdot\xi)R]\chi(x)$ for any large R. By (3.1) and (3.3) we have

$$P^{iajb}(x_0)\Big[R^2\eta_a\eta_b\xi_i\xi_j\int_G|\chi(x)|^2dx+\eta_a\eta_b\int_G|\partial^1\chi(x)|^2dx\Big]$$

$$\geqq(d_1-\sigma)\Big[R^2|\xi|^2|\eta|^2\int_G|\chi(x)|^2dx+|\eta|^2\int_G|\partial^2\chi(x)|^2dx\Big]-d_2|\eta|^2\int_G|\chi(x)|^2dx.$$

Dividing by R^2 and letting $R\to\infty$, we have (3.2) from the arbitrariness of choice of σ.

Now, we shall discuss boundary value problem in Ω with parameter $t\in[0,T]$:

$$(3.4)\qquad A(t,\partial^2)\vec{u}+\lambda_0\vec{u}=\vec{f}(t,\cdot)\ \text{ in } \Omega,\ B(t,\partial^1)\vec{u}=\vec{g}(t,\cdot)\ \text{ on } \Gamma,$$

where λ_0 is a constant determined in (3.18) below. By using local coordinate systems defined in § 2, we can write for $\vec{u}\in H^2(\Omega)$ and $\vec{v}\in H^1(\Omega)$

$$\langle B^i(t,\cdot)\partial_i\vec{u},\vec{v}\rangle=\sum_{k=1}^{N}\int_{R^{n-1}}\phi_k(y',0)^2B^i(t,\Psi_k(y',0))Y^j_{ik}(y',0)\partial'_j\vec{U}_k(y',0)\cdot\vec{V}_k(y',0)J_k(y')dy' \tag{3.5}$$

where

$$\phi_k(y)=\phi_k(\Psi_k(y)),\quad \vec{U}_k(y)=\vec{u}(\Psi_k(y)),\quad \vec{V}_k(y)=\vec{v}(\Psi_k(y)),\quad \partial'_j=\partial/\partial y_j,$$
$$Y^j_{ik}(y)=(\partial\Phi^k_j/\partial x_i)(\Psi_k(y)),\quad J_k(y')=[\sum_{i=1}^{n}(\partial\Phi^k_n/\partial x_i)(\Psi_k(y',0))^2]^{1/2}. \tag{3.6}$$

Note that

$$dS_x=J_k(y')dy', \tag{3.7.a}$$

$$\nu_i(x)=Y^n_{ik}(y',0)/J_k(y') \quad \text{(cf. (2.1))}, \tag{3.7.b}$$

for any $x=\Psi_k(y',0)\in\mathcal{O}_k\cap\Gamma$. By (A.6)

$$B^i(t,\Psi_k(y',0))Y^n_{ik}(y',0)=0 \quad \text{for } (y',0)\in B(\sigma_k)'. \tag{3.8}$$

Substituting (3.8) into (3.5) and integrating in y_n, we have

$$\langle B^i(t,\cdot)\partial_i\vec{u},\vec{v}\rangle=\mathcal{B}_1(t,\vec{u},\vec{v})+\mathcal{C}_1(t,\vec{u},\vec{v}) \tag{3.9}$$

where

$$Q^p_k(t,y')=B^i(t,\Psi_k(y',0))Y^p_{ik}(y')J_k(y'),\qquad p=1,\cdots,n-1, \tag{3.10}$$

$$\mathcal{B}_1(t,\vec{u},\vec{v})=\sum_{k=1}^{N}\int_{R^n_+}\phi^2_k(y)\{Q^p_k(t,y')\partial'_n\vec{U}_k(y)\cdot\partial'_p\vec{V}_k(y)-Q^p_k(t,y')\partial'_p\vec{U}_k(y)\cdot\partial'_n\vec{V}_k(y)\}dy, \tag{3.11.a}$$

$$\mathcal{C}_1(t,\vec{u},\vec{v})=\sum_{k=1}^{N}\int_{R^n_+}\{\{\partial'_p(\phi^2_k(y)Q^p_k(t,y'))\}\partial'_n\vec{U}_k(y)\cdot\vec{V}_k(y)-(\partial'_n\phi^2_k(y))Q^p_k(t,y')\partial'_p\vec{U}_k(y)\cdot\vec{V}_k(y)\}dy. \tag{3.11.b}$$

By the assumption: ${}^tB^i+B^i=0$ on $\boldsymbol{R}\times\Gamma$, we see that ${}^tQ^p_k+Q^p_k=0$, which implies that

$$\mathcal{B}_1(t,\vec{u},\vec{v})=\mathcal{B}_1(t,\vec{v},\vec{u}) \quad \text{for any } \vec{u},\vec{v}\in H^1(\Omega). \tag{3.12}$$

Obviously, by Schwarz's inequality we have

$$|\mathcal{C}_1(t,\vec{u},\vec{v})|\leq C\mathcal{M}(1)\|\partial^1\vec{u}\|\|\vec{v}\|. \tag{3.13}$$

In the same manner, we can write

$$\langle B(t,\cdot)\vec{u},\vec{v}\rangle=\mathcal{B}_2(t,\vec{u},\vec{v})+\mathcal{C}_2(t,\vec{u},\vec{v}), \tag{3.14}$$

where

$$\mathcal{B}_2(t,\vec{u},\vec{v})=-\sum_{k=1}^{N}\int_{R^n_+}\phi^2_k(y)B(t,\Psi_k(y',0))\vec{U}_k(y)\cdot\partial'_n\vec{V}_k(y)J_k(y')dy, \tag{3.15.a}$$

$$(3.15.b)\qquad \mathcal{C}_2(t, \vec{u}, \vec{v})=-\sum_{k=1}^{N}\left\{\int_{\boldsymbol{R}_+^n}\partial_n'(\varphi_k^2(y))B(t, \Psi_k(y', 0))\vec{U}_k(y)\cdot\vec{V}_k(y)J_k(y')dy\right.$$
$$\left.+\int_{\boldsymbol{R}_+^n}\varphi_k^2(y)B(t, \Psi_k(y', 0))\partial_n'\vec{U}_k(y)\cdot\vec{V}_k(y)J_k(y')dy\right\}$$

By Schwarz's inequality we have

$$(3.16.a)\qquad |\mathcal{C}_2(t, \vec{u}, \vec{v})|\leqq C\mathcal{M}(1)\|\partial^1\vec{u}\|\|\vec{v}\|,$$

$$(3.16.b)\qquad |\mathcal{B}_2(t, \vec{u}, \vec{v})|\leqq C\mathcal{M}(1)\|\vec{u}\|\|\partial^1\vec{v}\|.$$

Let us define the sesquilinear form $\mathcal{A}(t, \cdot, \cdot)$ associated with (3.4) by

$$(3.17)\qquad \mathcal{A}(t, \vec{u}, \vec{v})=(A^{ij}(t, \cdot)\partial_j\vec{u}, \partial_i\vec{v})+((A^j(t, \cdot)\partial_j+A^{n+1}(t, \cdot))\vec{u}, \vec{v})$$
$$+\lambda_0(\vec{u}, \vec{v})+\sum_{l=1}^{2}(\mathcal{B}_l(t, \vec{u}, \vec{v})+\mathcal{C}_l(t, \vec{u}, \vec{v}))\qquad \text{for } \vec{u}, \vec{v}\in H^1(\Omega).$$

By (A.4), (3.9), (3.13), (3.16) and Schwarz's inequality we have

$$(3.18.a)\qquad |\mathcal{A}(t, \vec{u}, \vec{v})|\leqq C\mathcal{M}(1)\|\partial^1\vec{u}\|\|\partial^1\vec{v}\|,$$

$$(3.18.b)\qquad \mathcal{A}(t, \vec{u}, \vec{u})\geqq\delta_1\|\partial^1\vec{u}\|^2,$$

provided that $\lambda_0=\delta_1+\delta_2+C\mathcal{M}(1)^2(\delta_1^{-1}+1)$ with some constant $C>0$. Here, we have used the fact that $H^2(\Omega)$ is dense in $H^1(\Omega)$. From a point of view of (3.18), the well-known Lax and Milgram theorem yields that for any $\vec{f}\in L^2(\Omega)$ and $\vec{g}\in H^{1/2}(\Gamma)$ there exists a unique solution $\vec{u}\in H^1(\Omega)$ of variational equation:

$$(3.19)\qquad \mathcal{A}(t, \vec{u}, \vec{v})=(\vec{f}, \vec{v})+\langle\vec{g}, \vec{v}\rangle$$

for any $\vec{v}\in H^1(\Omega)$. In particular, by (3.18.b) and (3.19)

$$(3.20)\qquad \|\partial^1\vec{u}\|\leqq\delta_1^{-1}C(\Gamma)\{\|\vec{f}\|+\langle\vec{g}\rangle_{1/2}\}.$$

In view of (3.9) and (3.14), if we can prove that $\vec{u}\in H(\Omega)$, then by integration by parts we see that $\vec{u}$ is a solution of (3.4) in strong sense. In proving that $\vec{u}\in H^2(\Omega)$, we adopt a well-known method. Namely, the boundary is straightened locally and difference quotients are used. For a large $\lambda_1>0$, put

$$(3.21)\qquad \mathcal{A}_k(t, \vec{v}, \vec{w})=(P_k^{ij}(t, \cdot)\partial_j'\vec{v}, \partial_i'\vec{w})'$$
$$+(Q_k^p(t, \cdot)\partial_n'\vec{v}, \partial_p'\vec{w})'-(Q_k^p(t, \cdot)\partial_p'\vec{v}, \partial_n'\vec{w})'+\lambda_1(\vec{v}, \vec{w})'$$

for any $\vec{v}, \vec{w}\in H^1(\boldsymbol{R}_+^n)$ such that $\operatorname{supp}\vec{v}, \operatorname{supp}\vec{w}\subset Q^+(\sigma_k)$. Here, we have put $(\ ,\ )'=(\ ,\ )_{\boldsymbol{R}_+^n}$ and

$$(3.22)\qquad P_k^{ij}(t, y)=A^{i'j'}(t, \Psi_k(y))Y_{i'k}^i(y)Y_{j'k}^j(y).$$

Noting that the Jacobian of the map Ψ_k is 1 and changing the variables in the right hand-side of (3.21), by (A.4), (3.9)-(3.11) and (3.13) we see that there exist λ_1 and $c_1>0$ depending only on δ_1, δ_2 and Γ such that

(3.23) $$\mathcal{A}_k(t, \vec{v}, \vec{v})\geqq c_1\|\bar{\partial}_y^1\vec{v}\|_{R_+^n}^1$$

for any $\vec{v}\in H^1(\boldsymbol{R}_+^n)$ with $\operatorname{supp}\vec{v}\subset Q^+(\sigma_k)$ and $k=1, \cdots, N$ $(\bar{\partial}_y^1\vec{v}=(\partial_1'\vec{v}, \cdots, \partial_n'\vec{v}, \vec{v}))$. Choose $\sigma_k', \sigma_k''>0$ so that $\sigma_k''<\sigma_k'<\sigma_k$ and $\operatorname{supp}\phi_k'(\Psi_k(y))\subset\{y\in\boldsymbol{R}^n\mid|y'|<\sigma_k', |y_n|<\sigma_k'\}$, $\operatorname{supp}\phi_k(\Psi_k(y))\subset\{y\in\boldsymbol{R}^n\mid|y'|<\sigma_k'', |y_n|<\sigma_k''\}$. For any $h\neq 0$ such that $|h|<\min(\sigma_k-\sigma_k', \sigma_k'-\sigma_k'')$, put $[\vec{w}]_h=\{\vec{w}(y+he_p)-\vec{w}(y)\}/h$ where $e_p=(0, \cdots, 0, \underset{p\text{-th}}{\overset{\uparrow}{1}}, 0, \cdots, 0)$, $p=1, \cdots, n-1$. If we put $\vec{v}_k(y)=\phi_k(y)\vec{u}(\Psi_k(y))$, then by (3.19) and (3.21) we see that

(3.24) $$\mathcal{A}_k([\vec{v}_k]_h, \vec{w})\leqq\{\|\vec{f}(t, \cdot)\|+\langle\vec{g}(t, \cdot)\rangle_{1/2}+C(\Gamma, \mathcal{M}(1))\|\bar{\partial}^1\vec{u}\|\}\|\bar{\partial}_y^1\vec{w}\|_{R_+^n}$$

for any $\vec{w}\in H^1(\boldsymbol{R}_+^n)$ with $\operatorname{supp}\vec{w}\subset\{y\in\boldsymbol{R}_+^n\mid|y'|<\sigma_k', y_n<\sigma_k'\}$. Here, we have used the fact that $\|[\vec{w}]_h\|_{R_+^n}\leqq\|\bar{\partial}_y^1\vec{w}\|_{R_+^n}$. Putting $\vec{w}=[\vec{v}_k]_h$ in (3.23), by (3.23) we have

(3.25) $$\|\bar{\partial}_y^1[\vec{v}_k]_h\|_{R_+^n}\leqq C(\Gamma, \delta_1, \delta_2)\{\|\vec{f}(t, \cdot)\|+\langle\vec{g}(t, \cdot)\rangle_{1/2}+C(\mathcal{M}(1))\|\bar{\partial}^1\vec{u}\|\},$$

which implies that $\partial_p'\vec{v}_k\in H^1(\boldsymbol{R}_+^n)$, $p=1, \cdots, n-1$. By Lemma 3.3 we see that P_k^{nn} is non-singular. In fact, for any $\vec{v}\in C_0^\infty(Q^+(\sigma_k))$ we have

$$(P_k^{ij}(t, \cdot)\partial_j'\vec{v}, \partial_i'\vec{v})'=(A^{ij}(t, \cdot)\partial_j\vec{w}_k, \partial_i\vec{w}_k)\quad(\vec{w}_k(x)=\vec{v}(\Phi_k(x)))$$
$$\geqq\delta_1\|\partial^1\vec{w}_k\|^2-\delta_2\|\vec{w}_k\|^2\geqq c_2\delta_1\|\partial^1\vec{v}\|_{R_+^n}-\delta_2\|\vec{v}\|_{R_+^n}^2$$

where $c_2=C(\Gamma)>0$. Applying Lemma 3.3 implies that

(3.26) $$P_k^{ij}(t, y)\xi_i\xi_j\geqq c_2\delta_1\mathrm{I}_m|\xi|^2$$

for any $\xi=(\xi_1, \cdots, \xi_n)\in\boldsymbol{R}^n$ and $y\in Q^+(\sigma_k)$, where I_m is the $m\times m$ identity matrix. In particular, from this it follows that $P_k^{nn}(t, y)$ is non-singular for any $y\in Q^+(\sigma_k)$.

Since $\vec{u}$ satisfies (3.4) in the distribution sense as follows from (3.19), noting that the coefficient of the second derivative of $\vec{v}_k$ with respect to y_n is P_k^{nn}, we see directly that $\partial^2\vec{v}_k/\partial y_n^2\in L^2(\boldsymbol{R}_+^n)$ and then obtain

$$\sum_{|\alpha|=2}\|\partial_y^\alpha\vec{v}_k\|_{R_+^n}\leqq C(\Gamma, \delta_1, \delta_2)(1+\mathcal{M}(0))\{\|\vec{f}(t, \cdot)\|+\langle\vec{g}(t, \cdot)\rangle_{1/2}+C(\mathcal{M}(1))\|\bar{\partial}^1\vec{u}\|\}.$$

By easier arguments, we can also get the interior regularity of $\vec{u}$, i.e., $\phi_0\vec{u}\in H^2(\Omega)$ and

$$\sum_{|\alpha|=2}\|\partial_x^\alpha(\phi_0\vec{u})\|\leqq C(\Gamma, \delta_1, \delta_2)\{\|\vec{f}(t, \cdot)\|+C(\mathcal{M}(1))\|\bar{\partial}^1\vec{u}\|\}.$$

Recalling that $\vec{v}_k(y)=\phi_k(\Psi_k(y))\vec{u}(\Psi_k(y))$ and (2.2.c), we have that $\vec{u}\in H^2(\Omega)$ and

(3.27) $\|\partial^2\vec{u}\| \leqq C(\Gamma, \delta_1, \delta_2)(1+\mathcal{M}(0))\{\|\vec{f}(t, \cdot)\| + \langle g(t, \cdot)\rangle_{1/2} + C(\mathcal{M}(1))\|\partial^1\vec{u}\|\}$.

Since $\vec{u}$ depends actually on t, we now prefer to write $\vec{u}=\vec{u}(t, x)$. Then, by (3.20) and (3.27) we get

$$\|\partial^2\vec{u}(t, \cdot)-\partial^2\vec{u}(t', \cdot)\| \leqq C(\delta_1, \delta_2, \Gamma, \mathcal{M}(2))\{\|\vec{f}(t, \cdot)-\vec{f}(t', \cdot)\| + \langle \vec{g}(t, \cdot)-\vec{g}(t', \cdot)\rangle_{1/2} + |t-t'|\,\|\partial^2\vec{u}(t, \cdot)\|\}.$$

From this it follows that $\vec{u}\in C^0([0, T]; H^2(\Omega))$ if $\vec{f}\in C^0([0, T]; L^2(\Omega))$ and $\vec{g}\in C^0([0, T]; H^{1/2}(\Gamma))$. In the same manner, we can get the differentiabilities of higher order of $\vec{u}$ with respect to t and x under the suitable assumptions of differentiabilities of $\vec{f}$, $\vec{g}$ and the coefficients of $A(t, \partial^2)$ and $B(t, \partial^1)$.

Summing up, we have obtained

THEOREM 3.4. *Assume that* (A.1)-(A.5) *are valid.* 1° *Let* λ_0 *be the same constants as in* (3.18). *Then, for any* $\vec{f}\in C^0([0, T]; L^2(\Omega))$ *and* $\vec{g}\in C^0([0, T]; H^{1/2}(\Gamma))$ (3.4) *admits a unique solution* $\vec{u}\in C^0([0, T]; H^2(\Omega))$ *satisfying* (3.20) *and* (3.27). 2° *Let* L *and* K *be integers* $\geqq 0$. *In addition to* (A.1)-(A.5), *we assume that* $A^i{}_a{}^{jb}\in C^K([0, T]; \mathcal{B}^{L+1}(\bar{\Omega}))$, $B^k{}_a{}^b\in C^K([0, T]; \mathcal{B}^{L+1}(\Gamma))$, $A^k{}_a{}^b\in C^K([0, T]; \mathcal{B}^L(\bar{\Omega}))$ *for* $i, j=1, \cdots, n$, $k=1, \cdots, n+1$ *and* $a, b=1, \cdots, m$. *If* $\vec{f}\in C^K([0, T]; H^L(\Omega))$, $\vec{g}\in C^K([0, T]; H^{L+(1/2)}(\Gamma))$, *then* $\vec{u}\in C^K([0, T]; H^{L+2}(\Omega))$.

Now, we shall consider the following problem:

$$(3.28)\quad \begin{aligned} &P(t_0)[\vec{u}(t, x)]=\vec{f}(t, x) \text{ in } [t_1, t_2]\times\Omega,\ Q(t_0)[\vec{u}(t, x)]=0 \text{ on } [t_1, t_2]\times\Gamma, \\ &\vec{u}(t, x)=\vec{u}_0(x),\ \partial_t\vec{u}(t_1, x)=\vec{u}_1(x) \text{ in } \Omega. \end{aligned}$$

In what follows, t_0, t_1 and t_2 always refer to any fixed times in $[0, T]$ such that $t_1<t_2$. In this section, we shall employ essentially the same arguments as in Ikawa [3, § 2]. Since we would like to make the paper self-contained and Ikawa did not treat the case where operators are systems (he treated the case where $m=1$ and (1.2) is valid), we prove all lemmas briefly, below. Let $H^k(\Omega)\times H^{k-1}(\Omega)$, $k\geqq 1$, be Hilbert spaces equipped with norms:

$$(3.29)\quad |||\mathcal{U}|||_k^2=\|\partial^k\vec{u}_0\|^2+\|\partial^{k-1}\vec{u}_1\|^2$$

for $\mathcal{U}=(\vec{u}_0, \vec{u}_1)\in H^k(\Omega)\times H^{k-1}(\Omega)$. Put

$$(3.30.a)\quad (\vec{u}, \vec{v})_{\mathcal{J}(t)}=(A^{ij}(t, \cdot)\partial_j\vec{u}, \partial_i\vec{v})+\mathcal{B}_1(t, \vec{u}, \vec{v})+\mathcal{B}_2(t, \vec{u}, \vec{v}) + \mathcal{B}_2(t, \vec{v}, \vec{u})+\lambda_2(\vec{u}, \vec{v}),$$

$$(3.30.b)\quad (\mathcal{U}, \mathcal{V})_{\mathcal{H}(t)}=(\vec{u}_0, \vec{v}_0)_{\mathcal{J}(t)}+(\vec{u}_1, \vec{v}_1),$$

$$(3.30.c)\quad \|\vec{u}\|_{\mathcal{J}(t)}^2=(\vec{u}, \vec{u})_{\mathcal{J}(t)}, \qquad \|\mathcal{U}\|_{\mathcal{H}(t)}^2=(\mathcal{U}, \mathcal{U})_{\mathcal{H}(t)}.$$

Here, $\mathcal{U}=(\vec{u}_0, \vec{u}_1)$ and $\mathcal{V}=(\vec{v}_0, \vec{v}_1)\in H^1(\Omega)\times L^2(\Omega)$ and λ_2 is a constant determined in (3.31) below. By (3.12) and (A.2) we know that $(\ ,\ )_{\mathcal{H}(t)}$ and $(\ ,\ )_{\mathcal{J}(t)}$ are bilinear forms. By (3.9), (3.13), (3.16) and (A.4), we have

$$\min(1, \delta_1)|||\mathcal{U}|||_1^2\leqq\|\mathcal{U}\|_{\mathcal{H}(t)}^2\leqq C(1+\mathcal{M}(1))|||\mathcal{U}|||_1^2\,, \tag{3.31.a}$$

$$\delta_1\|\tilde{\partial}^1\vec{u}\|^2\leqq\|\vec{u}\|_{\mathcal{J}(t)}^2\leqq C(1+\mathcal{M}(1))\|\tilde{\partial}^1\vec{u}\|^2 \tag{3.31.b}$$

for any $\mathcal{U}\in H^1(\Omega)\times L^2(\Omega)$, $\vec{u}\in H^1(\Omega)$ and $t\in[0, T]$, provided that

$$\lambda_2=\delta_1+2\delta_2+(C\mathcal{M}(1)/\delta_1)^2 \tag{3.31.c}$$

with some $C>0$. Let $\mathcal{H}(t)$ denote the Hilbert space $H^1(\Omega)\times L^2(\Omega)$ equipped with inner product $(\ ,\ )_{\mathcal{H}(t)}$. Put

$$\mathcal{A}(t)\mathcal{U}=(\vec{u}_1,\ -H(t, \tilde{\partial}^1)\vec{u}_1-A(t, \tilde{\partial}^2)\vec{u}_0)\,, \tag{3.32.a}$$

$$\mathcal{B}(t)\mathcal{U}=B(t, \tilde{\partial}^1)\vec{u}_0+H^0(t, x)\vec{u}_1\,, \tag{3.32.b}$$

for $\mathcal{U}=(\vec{u}_0, \vec{u}_1)\in H^2(\Omega)\times H^1(\Omega)$.

LEMMA 3.5. *Assume that* (A.1)-(A.6) *are valid. Then, there exists a* $C=C(\delta_1, \delta_2, \mathcal{M}(1))$ *such that*

$$(\mathcal{A}(t)\mathcal{U}, \mathcal{U})_{\mathcal{H}(t)}+(\mathcal{U}, \mathcal{A}(t)\mathcal{U})_{\mathcal{H}(t)}\leqq C\|\mathcal{U}\|_{\mathcal{H}(t)}^2 \tag{3.33}$$

for any $\mathcal{U}\in\mathcal{D}(t)=\{\mathcal{U}\in H^2(\Omega)\times H^1(\Omega)\mid \mathcal{B}(t)\mathcal{U}=0 \text{ on } \Gamma\}$.

Proof. Noting that ${}^tH^j=H^j$ and ${}^tA^{ij}=A^{ji}$, by integration by part we have

$$(H(t, \tilde{\partial}^1)\vec{u}_1, \vec{u}_1)=\langle(\nu_jH^j)\vec{u}_1, \vec{u}_1\rangle+((H^{n+1}-\partial_jH^j)\vec{u}_1, \vec{u}_1)\,, \tag{3.34}$$

$$(A(t, \tilde{\partial}^2)\vec{u}_0, \vec{u}_1)=-\langle\nu_iA^{ij}\partial_j\vec{u}_0, \vec{u}_1\rangle+(A^{ij}\partial_j\vec{u}_0, \partial_i\vec{u}_1)+((A^i\partial_i+A^{n+1})\vec{u}_0, \vec{u}_1). \tag{3.35}$$

By (3.9), (3.14), (3.34) and (3.35) we have

$$\begin{aligned}(\mathcal{A}(t)\mathcal{U}, \mathcal{U})_{\mathcal{H}(t)}=&-\langle(\nu_jH^j+H^0)\vec{u}_1, \vec{u}_1\rangle+\langle\mathcal{B}(t)\mathcal{U}, \vec{u}_1\rangle\\ &-((H^{n+1}-\partial_jH^j)\vec{u}_1, \vec{u}_1)-((A^j\partial_j+A^{n+1})\vec{u}_0, \vec{u}_1)+\lambda_2(\vec{u}_1, \vec{u}_0)\\ &-\mathcal{C}_1(t, \vec{u}_0, \vec{u}_1)-\mathcal{C}_2(t, \vec{u}_0, \vec{u}_1)+\mathcal{B}_2(t, \vec{u}_1, \vec{u}_0)\,.\end{aligned} \tag{3.36}$$

Since $\mathcal{B}(t)\mathcal{U}=0$ and since $\langle(\nu_jH^j+H^0)\vec{u}_1, \vec{u}_1\rangle\geqq 0$ as follows from (A.6), applying (3.13) and (3.16) and using (3.29) and (3.31.a) we have the lemma.

COROLLARY 3.6. *Assume that* (A.1)-(A.6) *are valid. Let C be the same constant as in Lemma* 3.5. *Then,*

$$\|(\lambda I-\mathcal{A}(t))\mathcal{U}\|_{\mathcal{H}(t)}\geqq(|\lambda|-C)\|\mathcal{U}\|_{\mathcal{H}(t)} \tag{3.37}$$

for any $|\lambda|>C$ *ane* $\mathcal{U}\in\mathcal{D}(t)$. *Here,* I *is the identity operator.*

Proof. By Lemma 3.5, we have

$$\|(\lambda\mathrm{I}-\mathcal{A}(t))\mathcal{U}\|_{\mathcal{H}(t)}^2=((\lambda\mathrm{I}-\mathcal{A}(t))\mathcal{U},\ (\lambda\mathrm{I}-\mathcal{A}(t))\mathcal{U})_{\mathcal{H}(t)}$$
$$\geqq(|\lambda|^2-|\lambda|C)\|\mathcal{U}\|_{\mathcal{H}(t)}^2\geqq((|\lambda|-C)^2+C(|\lambda|-C))\|\mathcal{U}\|_{\mathcal{H}(t)}^2.$$

This implies (3.37) provided that $|\lambda|>C$.

LEMMA 3.7. *Assume that* (A.1)-(A.6) *are valid. Then, there exists a* $C=C(\delta_1,\ \delta_2,\ \mathcal{M}(1))>0$ *such that for any* $\lambda>C$ $\lambda\mathrm{I}-\mathcal{A}(t)$ *is a bijective map of* $\mathcal{D}(t)$ *onto* $\mathcal{H}(t)$. *If we denote its inverse by* $(\lambda\mathrm{I}-\mathcal{A}(t))^{-1}$, *then*

$$\|(\lambda\mathrm{I}-\mathcal{A}(t))^{-1}U\|_{\mathcal{H}(t)}\leqq(\lambda-C)^{-1}\|U\|_{\mathcal{H}(t)}\quad \textit{for } \lambda>C \textit{ and } U\in\mathcal{D}(t). \tag{3.38}$$

Proof. In view of Corollary 3.6, it suffices to prove the bijectiveness. Namely, given $\mathcal{V}=(\vec{v}_0,\ \vec{v}_1)\in\mathcal{H}(t)$ we shall prove a unique existence of $\mathcal{U}=(\vec{u}_0,\ \vec{u}_1)\in\mathcal{D}(t)$ such that $(\lambda\mathrm{I}-\mathcal{A}(t))\mathcal{U}=\mathcal{V}$. If we use the relation of first components, i.e., $\lambda\vec{u}_0-\vec{u}_1=\vec{v}_0$, we can rewrite the relation of second components as follows:

$$A(t,\ \bar{\partial}^2)\vec{u}_0+\lambda H(t,\ \bar{\partial}^1)\vec{u}_0+\lambda^2\vec{u}_0=\vec{f}\qquad \text{in } \Omega \tag{3.39.a}$$

where $\vec{f}=\vec{v}^1+H(t,\ \bar{\partial}^1)\vec{v}_0+\lambda\vec{v}_0\in L^2(\Omega)$. If we use the relation: $\lambda\vec{u}_0-\vec{u}_1=\vec{v}_0$ again, we see that the condition: $\mathcal{U}\in\mathcal{D}(t)$ means that

$$B(t,\ \bar{\partial}^1)\vec{u}_0+\lambda H^0(t,\ x)\vec{u}_0=\vec{g}\qquad \text{on } \Gamma \tag{3.39.b}$$

where $\vec{g}=H^0(t,\ x)\vec{v}_0\in H^1(\Omega)$. Thus, to prove the lemma it suffices to prove that there exists a constant C such that for any $\lambda>C$ the boundary value problem (3.39) always admits a unique solution $\vec{u}_0\in H^2(\Omega)$. To solve (3.39) let us introduce sesquilinear form $\mathcal{A}_\lambda$ associated with (3.39) as follows:

$$\mathcal{A}_\lambda(t,\ \vec{u},\ \vec{v})=(A^{ij}(t,\ \cdot)\partial_j\vec{u},\ \partial_i\vec{v})+((A^i(t,\ \cdot)\partial_i+A^{n+1}(t,\ \cdot))\vec{u},\ \vec{v})$$
$$+\sum_{l=1}^{2}(\mathcal{B}_l(t,\ \vec{u},\ \vec{v})+\mathcal{C}_l(t,\ \vec{u},\ \vec{v}))+\lambda(H(t,\ \bar{\partial}^1)\vec{u},\ \vec{v})$$
$$+\lambda\langle H^0(t,\ \cdot)\vec{u},\ \vec{v}\rangle+\lambda^2(\vec{u},\ \vec{v}).$$

By (3.34) and (A.5) we have

$$\lambda(H(t,\ \bar{\partial}^1)\vec{u},\ \vec{u})+\lambda\langle H^0(t,\ \cdot)\vec{u},\ \vec{u}\rangle\geqq-\lambda C\mathcal{M}(1)\|\vec{u}\|^2.$$

By (A.4), (3.9), (3.31), (3.16) and Schwarz's inequality we see that there exists a $C=C(\delta_1,\ \delta_2,\ \mathcal{M}(1))>0$ such that for any $\lambda>C$ and $\vec{u}\in H^1(\Omega)$

$$\mathcal{A}_\lambda(t,\ \vec{u},\ \vec{u})\geqq\delta_1\|\bar{\partial}^1\vec{u}\|^2.$$

Employing the same arguments as in the proof of Theorem 3.4, we see that (3.39) admits a unique solution $\vec{u}_0 \in H^2(\Omega)$ for any $\lambda > C$, which completes the proof.

To apply the well-known Hille-Yoshida theorem, the rest of out task is to prove that $\mathcal{D}(t)$ is dense in $\mathcal{H}(t)$. For this we need the following.

LEMMA 3.8. *Assume that* (A.1)-(A.5) *are valid. Then, for any integer* $l \geqq 1$ *and* $\vec{g} \in H^2(\Gamma)$, *there exists a* $\vec{u}^l \in H^2(\Omega)$ *such that*

$$B(t, \partial^1)\vec{u}^l = \vec{g} \quad \text{and} \quad \vec{u}^l = 0 \text{ on } \Gamma, \tag{3.40}$$

$$\|\partial^1 \vec{u}^l\| \leqq 1/l. \tag{3.41}$$

Proof. Let $P_k^{ij}(t, y)$ be the same functions as in (3.22). Note that $P_k^{nn}(t, y)$ is non-singular for any $y \in Q^+(\sigma_k)$, and then $(P_k^{nn}(t, y))^{-1}$ is also in $\mathcal{B}^2(\boldsymbol{R} \times Q^+(\sigma_k))$. Choose $\rho(s) \in C_0^\infty(\boldsymbol{R})$ so that $\rho(s) = 1$ for $|s| \leqq 1/2$ and $= 0$ for $|s| \geqq 1$. Put

$$\vec{U}_{kR}(y) = -y_n \rho(R y_n)\{P_k^{nn}(t, y)\}^{-1} \vec{g}_k(y') J_k(y')$$

for sufficiently large R, where $\vec{g}_k(y') = \phi_k^2(y', 0)\vec{g}(\Psi_k(y', 0))$ (cf. (3.6)). Then, we see easily that

$$\vec{U}_{kR}(y) \in H^2(\boldsymbol{R}_+^n), \ \operatorname{supp} \vec{U}_{kR} \subset \{y \in \boldsymbol{R}^n \mid |y_n| < R^{-1}, \ y' \in B(\sigma_k)'\}, \tag{3.42.a}$$

$$\|\partial_y^1 \vec{U}_{kR}\|_{R_+^n} \leqq C(\mathcal{M}(1)) \langle \vec{g} \rangle_1 R^{-1}, \tag{3.42.b}$$

$$\partial_n' \vec{U}_{kR}(y', 0) = -\{P_k^{nn}(t, y', 0)\}^{-1} \vec{g}_k(y') J_k(y'), \tag{3.42.c}$$

$$\partial_p' \vec{U}_{kR}(y', 0) = \vec{U}_{kR}(y', 0) = 0 \qquad \text{for } p = 1, \cdots, n-1. \tag{3.42.d}$$

Put $\vec{u}_{kR}(x) = \vec{U}_{kR}(\Phi_k(x))$ for $x \in \mathcal{O}_k$ and $= 0$ for $x \notin \mathcal{O}_k$. Set $\vec{u}^R(x) = \sum_{k=1}^N \vec{u}_{kR}(x)$. By (2.2.c), (3.8), (3.42.c) and (3.42.d) we have

$$\begin{aligned} B(t, \partial^1)\vec{u}^R(x) = \sum_{k=1}^N \{&\nu_{i'} A^{i'j'}(t, \Psi_k(y', 0)) Y_{j'k}^i(y', 0)\partial_j' \\ &+ B^i(t, \Psi_k(y', 0)) Y_{ik}^j(y', 0)\partial_j' + B(t, \Psi_k(y', 0))\} \vec{U}_{kR}(y)|_{y_n = 0} \\ = \sum_{k=1}^N \vec{g}_k(y') = \sum_{k=1}^N &\phi_k^2(x)\vec{g}(x) = \vec{g}(x) \qquad \text{for } x \in \Gamma. \end{aligned}$$

Thus, by (3.42.b) and (3.42.d) we see easily that $\vec{u}^R$ satisfies desired properties with suitable choice of sufficiently large R.

LEMMA 3.9. *Assume that* (A.1)-(A.5) *are valid. Then,* $\mathcal{D}(t)$ *is dense in* $\mathcal{H}(t)$.

Proof. Put $H_\nabla^2 = \{\vec{u} \in H^2(\Omega) \mid B(t, \partial^1)\vec{u} = 0 \text{ on } \Gamma\}$. Note that $H_\nabla^2 \times C_0^\infty(\Omega) \subset \mathcal{D}(t)$. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, it suffices to prove that H_∇^2 is dense in $H^1(\Omega)$.

Given $\varepsilon>0$ and $\vec{u}\in H^1(\Omega)$, we choose $\vec{v}\in H(\Omega)$ so that

$$\|\partial^1(\vec{u}-\vec{v})\|<\varepsilon/2. \tag{3.43}$$

By Lemma 3.8 we know that there exists a $\vec{w}\in H^2(\Omega)$ such that

$$\|\partial^1\vec{w}\|<\varepsilon/2 \quad \text{and} \quad B(t, \partial^1)\vec{w}=B(t, \partial^1)\vec{v} \qquad \text{on } \Gamma. \tag{3.44}$$

Thus, combining (3.43) and (3.44) implies that $\vec{v}-\vec{w}\in H^2_\gamma$ and that

$$\|\partial^1(\vec{u}-(\vec{v}-\vec{w}))\|<\varepsilon.$$

which completes the proof.

In view of Lemmas 3.7 and 3.9, an application of the Hille-Yoshida theorem (cf. Tanabe [12, Theorem 3.22]) yields the following.

THEEOREM 3.10. *Assume that* (A.1)-(A.6) *are valid. Let* $t_0, t_1, t_2\in[0, T]$ *such that* $t_1<t_2$. *Then, for any* $\mathcal{U}_0\in\mathcal{D}(t_0)$ *and* $\mathcal{F}(t)\in C^1([t_1, t_2]; \mathcal{H}(t_0))$ *there exists a unique* $\mathcal{U}(t)\in C^1([t_1, t_2]; \mathcal{H}(t_0))\cap C^0([t_1, t_2]; H^2(\Omega)\times H^1(\Omega))$ *such that*

$$\frac{d}{dt}\mathcal{U}(t)=\mathcal{A}(t_0)\mathcal{U}(t)+\mathcal{F}(t) \quad \textit{and} \quad \mathcal{U}(t)\in\mathcal{D}(t_0) \qquad \textit{for any } t\in[t_1, t_2], \tag{3.45}$$
$$\mathcal{U}(t_1)=\mathcal{U}_0.$$

If we put $\mathcal{U}_0=(\vec{u}_0, \vec{u}_1)$ and $\mathcal{F}(t)=(0, \vec{f}(t, \cdot))$, then the first component of $\mathcal{U}(t)$ of Theorem 3.10 is a solution of (3.28). Namely, we have proved

THEOREM 3.11. *Assume that* (A.1)-(A.6) *are valid. Let* $t_0, t_1, t_2\in[0, T]$ *such that* $t_1<t_2$. *If* $\vec{u}_0\in H^2(\Omega)$, $\vec{u}_1\in H^1(\Omega)$, $\vec{f}\in C^1([t_1, t_2]; L^2(\Omega))$ *and*

$$B(t_0, \partial^1)\vec{u}_0+H^0(t_0, x)\vec{u}_1=0 \qquad \textit{on } \Gamma, \tag{3.46}$$

then there exists a unique solution $\vec{u}\in E^2([t_1, t_2])$ *of the equations* (3.28).

§ 4. A proof an existence theorem.

In this section, under the assumptions that some kinds of estimates are valid we shall prove an existence theorem which can be applied to the case where the operator $P(t)$ satisfies (1.2) rather than (1.3). Throughout this section, it is assumed that the following two inequalities are valid:

(E.1) For any $T>0$, there exists a $C_1(T)\geqq 1$ such that

$$\|\bar{D}^1\vec{u}(t, \cdot)\|^2\leqq C_1(T)\Big\{\|\bar{D}^1\vec{u}(0, \cdot)\|^2 + \int_0^t(\|P(s)[\vec{u}(s, \cdot)]\|^2+\langle Q(s)[\vec{u}(s, \cdot)]\rangle^2_{1/2})ds\Big\}, \quad 0\leqq t\leqq T_1,$$

for any $T_1\in(0, T]$ and $\vec{u}\in E^2([0, T_1])$.

(E.2) For any $T>0$, there exist constants $C_2(T)$, $C_3(T)\geqq 1$ such that

$$\|\partial_t\vec{u}(t,\cdot)\|^2+\|\vec{u}(t,\cdot)\|^2_{\mathfrak{J}(t_0)}\leqq(\exp C_2(T)(t-t_1))$$
$$\times\left\{\|\partial_t\vec{u}(t_1,\cdot)\|^2+\|\vec{u}(t_1,\cdot)\|^2_{\mathfrak{J}(t_0)}+C_3(T)\int_{t_1}^{t}(\|P(t_0)[\vec{u}(s,\cdot)]\|^2\right.$$
$$\left.+\langle Q(t_0)[u(s,\cdot)]\rangle^2_{1/2})ds\right\},\quad t_1\leqq t\leqq t_2,$$

for any t_0, t_1, $t_2\in[0, T]$ $(t_1<t_2)$ and $\vec{u}\in E^2([t_1, t_2])$.

In what follows, as Gronwall's inequality we always refer to the following inequality.

Gronwall's inequality: *Let* $a(t)$ *and* $b(t)$ *be in* $C^0([\alpha, \beta])$ *such that* $a(t)$, $b(t)\geqq 0$ *and* $b(t)$ *is non-decreasing in* $[\alpha, \beta]$. *If* $a(t)\leqq c\int_{\alpha}^{t}a(s)ds+b(t)$ *for any* $t\in[\alpha, \beta]$ *with some constant* $c>0$, *then* $a(t)\leqq(\exp c(t-\alpha))b(t)$ *for any* $t\in[\alpha, \beta]$.

First, we derive the estimates for second derivatives from (E.1) and (E.2).

LEMMA 4.1. *Assume that* (E.2) *is valid. Let* t_0, t_1, $t_2\in[0, T]$ *such that* $t_1<t_2$. *If* $\vec{u}\in E^2([t_1, t_2])$ *satisfies*

$$\vec{f}(t, x)=P(t_0)[\vec{u}(t, x)]\in C^1([t_1, t_2]; L^2(\Omega)),\tag{4.1}$$
$$\vec{g}(t, x)=Q(t_0)[\vec{u}(t, x)]\in C^1([t_1, t_2]; H^{1/2}(\Gamma)),$$

then

$$\|\partial_t^2\vec{u}(t,\cdot)\|^2+\|\partial_t\vec{u}(t,\cdot)\|^2_{\mathfrak{J}(t_0)}\leqq(\exp C_2(T)(t-t_1))\tag{4.2}$$
$$\times\left\{\|\partial_t^2\vec{u}(t_1,\cdot)\|^2+\|\partial_t\vec{u}(t_1,\cdot)\|^2_{\mathfrak{J}(t_0)}+C_3(T)\int_{t_1}^{t}(\|\partial_s\vec{f}(s,\cdot)\|^2+\langle\partial_s\vec{g}(s,\cdot)\rangle^2_{1/2})ds\right\},$$

for any $t\in[t_1, t_2]$.

Proof. Let σ be any small positive number and assume that $\vec{u}\in C^\infty([t_1, t_2-\sigma]; H^2(\Omega))$. Applying (E.2) to $\partial_t\vec{u}$, we see that (4.2) is valid for $t\in[t_1, t_2-\sigma]$. To remove the assumption: $\vec{u}\in C^\infty([t_1, t_2-\sigma]; H^2(\Omega))$, we use the mollifier with respect to t. Let $\rho(t)\in C_0^\infty(\boldsymbol{R})$ such that $\rho\geqq 0$, $\int_{\boldsymbol{R}}\rho(t)dt=1$ and $\operatorname{supp}\rho\subset[-2, -1]$. For $v\in L^2((t_1, t_2)\times\Omega)$, put

$$v_\delta(t, x)=\rho_\delta*v=\int_{\boldsymbol{R}}\rho_\delta(t-s)v(s, x)ds,\quad \vec{v}_\delta=\rho_\delta*\vec{v}={}^t(\rho_\delta*v_1,\cdots,\rho_\delta*v_m)$$

for $0<\delta<\sigma/2$, where $\rho_\delta(s)=\delta^{-1}\rho(\delta^{-1}s)$. Note that $\vec{v}_\delta\in C^\infty([t_1, t_2-\sigma]; L^2(\Omega))$. Since the coefficients of operators $P(t_0)$ and $Q(t_0)$ are independent of t, we have that $\partial_t\vec{f}_\delta=P(t_0)[\partial_t\vec{u}_\delta]$ and $\partial_t\vec{g}_\delta=Q(t_0)[\partial_t\vec{u}_\delta]$. Since $\vec{u}\in E^2([t_1, t_2])$ and (4.1) are valid, $\partial_t\vec{u}_\delta$, $\partial_t^2\vec{u}_\delta$, $\partial_t\vec{f}_\delta$ and $\partial_t\vec{g}_\delta$ converge to $\partial_t\vec{u}$, $\partial_t^2\vec{u}$, $\partial_t\vec{f}$ and $\partial_t\vec{g}$ as $\delta\downarrow 0$ in $C^0([t_1, t_2-\sigma]; H^1(\Omega))$, $C^0([t_1, t_2-\sigma]; L^2(\Omega))$, $C^0([t_1, t_2-\sigma]; L^2(\Omega))$ and $C^0([t_1, t_2-\sigma]; H^{1/2}(\Gamma))$, respectively. Here, $C^0(I; X)$ has the uniform topology with respect to $t\in I$. Applying (E.2) to $\partial_t\vec{u}_\delta$ and letting $\delta\downarrow 0$, we have that (4.2) is valid for any $t\in[t_1, t_2-\sigma]$. By the arbitrariness of the choice of σ and the continuity of the second derivatives of $\vec{u}$ in t we have the lemma.

LEMMA 4.2. *Assume that* (E.1) *is valid. If* $\vec{u}\in E^2([0, T])$ *satisfies*

$$\vec{f}(t, x)=P(t)[\vec{u}(t, x)]\in C^1([0, T]; L^2(\Omega)),\qquad (4.3)$$
$$\vec{g}(t, x)=Q(t)[\vec{u}(t, x)]\in C^1([0, T]; H^{1/2}(\Gamma)),$$

then there exists a $C(T)>0$ *independent of* $\vec{u}$ *such that*

$$(4.4)\qquad \|\bar{D}^2\vec{u}(t, \cdot)\|^2\leqq C(T)\Big\{\|\bar{\partial}^2\vec{u}(0, \cdot)\|^2+\|\bar{\partial}^1\vec{u}(0, \cdot)\|^2+\|\vec{f}(0, \cdot)\|^2+\langle\vec{g}(0, \cdot)\rangle_{1/2}^2$$
$$+\int_0^t\sum_{k=0}^1(\|\partial_s^k\vec{f}(s, \cdot)\|^2+\langle\partial_s^k\vec{g}(s, \cdot)\rangle_{1/2}^2)ds\Big\},\quad 0\leqq t\leqq T.$$

Proof. Let σ be any positive small number. First, we assume that $\partial_t\vec{u}\in E^2([0, T-\sigma])$. Since $P(t)[\partial_t\vec{u}]=\partial_t\vec{f}-H^{(1)}(t, \bar{\partial}^1)\partial_t\vec{u}-A^{(1)}(t, \partial^2)\vec{u}$, $Q(t)[\partial_t\vec{u}]=\partial_t\vec{g}-\partial_t H^0(t, x)\partial_t\vec{u}-B^{(1)}(t, \bar{\partial}^1)u$, applying (E.1) implies that

$$(4.5)\qquad \|\bar{D}^1\partial_t\vec{u}(t, \cdot)\|^2\leqq C_1(T)\Big\{\|\bar{\partial}^2\vec{u}(0, \cdot)\|^2+\|\bar{\partial}^1\partial_t\vec{u}(0, \cdot)\|^2+\|\vec{f}(0, \cdot)\|^2$$
$$+C(\mathcal{M}(2))\int_0^t\|\bar{D}^2\vec{u}(s, \cdot)\|^2ds+\int_0^t(\|\partial_s\vec{f}(s, \cdot)\|^2+\langle\partial_s\vec{g}(s, \cdot)\rangle_{1/2}^2)ds\Big\},$$

for any $t\in[0, T-\sigma]$. Here, we have used the relation: $\partial_t^2\vec{u}(0, x)=\vec{f}(0, x)$ $H(0, \bar{\partial}^1)\partial_t\vec{u}(0, x)-A(0, \bar{\partial}^2)\vec{u}(0, x)$ and also used Lemmas 3.1 and 3.2 to evaluate boundary terms. If we apply Theorem 3.4 to equations: $A(t, \partial^2)\vec{u}+\lambda_0\vec{u}=\vec{f}-\partial_t^2\vec{u}-H(t, \bar{\partial}^1)\partial_t\vec{u}+\lambda_0\vec{u}$ in Ω and $B(t, \bar{\partial}^1)\vec{u}=\vec{g}-H^0(t, \cdot)\partial_t\vec{u}$ on Γ, we have

$$(4.6)\qquad \|\bar{\partial}^2\vec{u}(t, \cdot)\|^2\leqq C\{\|\vec{f}(t, \cdot)\|^2+\langle\vec{g}(t, \cdot)\rangle_{1/2}^2+\|\bar{D}^1\partial_t\vec{u}(t, \cdot)\|^2+\|\bar{D}^1\vec{u}(t, \cdot)\|^2\}.$$

Note that

$$(4.7.a)\qquad \|\vec{f}(t, \cdot)\|^2\leqq\|\vec{f}(0, \cdot)\|^2+\int_0^t\sum_{k=0}^1\|\partial_s^k(\vec{f}(s, \cdot)\|^2ds,$$

$$(4.7.b)\qquad \langle\vec{g}(t, \cdot)\rangle_{1/2}^2\leqq\langle\vec{g}(0, \cdot)\rangle_{1/2}^2+\int_0^t\sum_{k=0}^1\langle\partial_s^k\vec{g}(s, \cdot)\rangle_{1/2}^2ds,$$

$$(4.7.c)\qquad \|\bar{D}^1\vec{u}(t, \cdot)\|^2\leqq\|\bar{D}^1\vec{u}(0, \cdot)\|^2+\int_0^t\|\bar{D}^2\vec{u}(s, \cdot)\|^2ds.$$

Substituting (4.7.a and b) into (4.6) and combining (4.5), (4.6) and (4.7.c), we have

$$\|\bar{D}^2\vec{u}(t,\cdot)\|^2 \leqq C(T)\Big\{\|\bar{\partial}^2\vec{u}(0,\cdot)\|^2+\|\bar{\partial}^1\partial_t\vec{u}(0,\cdot)\|^2+\|\vec{f}(0,\cdot)\|^2+\langle\vec{g}(0,\cdot)\rangle_{1/2}^2 + \sum_{k=0}^{1}\int_0^t(\|\partial_s^k\vec{f}(s,\cdot)\|^2+\langle\partial_s^k\vec{g}(s,\cdot)\rangle_{1/2}^2)ds+\int_0^t\|\bar{D}^2\vec{u}(s,\cdot)\|^2ds\Big\} \tag{4.8}$$

for any $t\in[0, T-\sigma]$. Applying Gronwall's inequality to (4.8), we see that (4.4) is valid for any $t\in[0, T-\sigma]$.

Now, we remove the additional assumption: $\partial_t\vec{u}\in E^2([0, T])$. Let us use the same mollifier with respect to t as in the proof of Lemma 4.1, i.e., $\vec{v}_\delta$ and $\rho_\delta*$ are the same notations. We employ the same arguments as in the proof of Proposition 2.6 in Ikawa [2]. Put $C_\delta\vec{u}=\rho_\delta*(P(\cdot)[\vec{u}])-P(\cdot)[\vec{u}_\delta]$ and $\Gamma_\delta\vec{u}=\rho_\delta*(Q(\cdot)[\vec{u}])-Q(\cdot)[\vec{u}_\delta]$. The estimate already proved implies that

$$\|\bar{D}^2\vec{u}_\delta(t,\cdot)\|^2 \leqq C(T)\Big\{\|\bar{\partial}^2\vec{u}_\delta(0,\cdot)\|^2+\|\bar{\partial}^1\partial_t\vec{u}_\delta(0,\cdot)\|^2+\|\vec{f}_\delta(0,\cdot)\|^2+\langle\vec{g}_\delta(0,\cdot)\rangle_{1/2}^2 + \sum_{k=0}^{1}\int_0^t(\|\partial_s^k\vec{f}_\delta(s,\cdot)\|^2+\langle\partial_s^k\vec{g}_\delta(s,\cdot)\rangle_{1/2}^2)ds+\|C_\delta\vec{u}(0,\cdot)\|^2+\langle\Gamma_\delta\vec{u}(0,\cdot)\rangle_{1/2}^2 + \sum_{k=0}^{1}\int_0^t(\|\partial_s^kC_\delta\vec{u}(s,\cdot)\|^2+\langle\partial_s^k\Gamma_\delta\vec{u}(s,\cdot)\rangle_{1/2}^2)ds\Big\},\quad 0\leqq t\leqq T-\sigma, \tag{4.9}$$

provided that $0<\delta<\sigma/2$. If we note the identity:

$$\partial_t\{\rho_\delta*(av)-a(\rho_\delta*v)\}=\int_{\boldsymbol{R}}\partial_s\{\rho_\delta(t-s)(a(t,x)-a(s,x))\}(v(s,x)-v(t,x))ds + \int_{\boldsymbol{R}}\rho_\delta(t-s)\{\dot{a}(s,x)-\dot{a}(t,x)\}v(s,x)ds\quad(\dot{a}=\partial_t a),$$

we can prove that

$$\|C_\delta\vec{u}(0,\cdot)\|^2+\langle\Gamma_\delta\vec{u}(0,\cdot)\rangle_{1/2}^2 + \sum_{k=0}^{1}\int_0^t(\|\partial_s^kC_\delta\vec{u}(s,\cdot)\|^2+\langle\partial_s^k\Gamma_\delta\vec{u}(s,\cdot)\rangle_{1/2}^2)ds \longrightarrow 0 \tag{4.10}$$

as $\delta\downarrow 0$ provided that $\vec{u}\in H^2((0, T)\times\Omega)$. In the present case, $\vec{u}\in H^2((0, T)\times\Omega)$, since $\vec{u}\in E^2([0, T])\subset H^2((0, T)\times\Omega)$. Since (4.3) is valid, letting $\delta\downarrow 0$ in (4.9) and noting (4.10) we see that (4.4) is valid for any $t\in[0, T-\sigma]$. The arbitrariness of the choice of σ and the continuity of $\vec{u}$ in t imply the lemma.

Now, using Theorem 3.11 and Lemma 4.1, we prove an existence theorem for $P(t_0)$ and $Q(t_0)$ with the inhomogeneous boundary condition.

LEMMA 4.3. *Assume that* (A.1)-(A.6) *and* (E.2) *are valid. Let* $t_0, t_1, t_2\in[0, T]$ *such that* $t_1<t_2$. *If* $\vec{u}_0\in H^2(\Omega)$, $\vec{u}_1\in H^1(\Omega)$, $\vec{f}\in C^1([t_1, t_2]; L^2(\Omega))$ *and* $\vec{g}\in C^1([t_1, t_2]; H^{1/2}(\Gamma))$ *and they satisfy the compatibility condition of order* 0:

(4.11) $$B(t_0, \bar{\partial}^1)\vec{u}_0(x)+H^0(t_0, x)\vec{u}_1(x)=\vec{g}(t_1, x) \quad on\ \Gamma,$$

then, there exists a unique solution $\vec{u}\in E^2([t_1, t_2])$ *to the equations*:

(4.12) $$P(t_0)[\vec{u}(t, x)]=\vec{f}(t, x)\ in\ [t_1, t_2]\times\Omega, \quad Q(t_0)[\vec{u}(t, x)]=\vec{g}(t, x)\ on\ [t_1, t_2]\times\Gamma,$$
$$\vec{u}(t_1, x)=\vec{u}_0(x), \quad \partial_t\vec{u}(t_1, x)=\vec{u}_1(x) \quad in\ \Omega.$$

Proof. First, we assume that $\vec{g}\in C^\infty([t_1, t_2]; H^\infty(\Gamma))$. Let $\vec{w}(t, x)\in C^\infty([t_1, t_2]; H^2(\Omega))$ be a function satisfying:

(4.13) $$B(t_0, \bar{\partial}^1)\vec{w}=\vec{g} \quad \text{and} \quad \vec{w}=0 \qquad \text{on } [t_1, t_2]\times\Gamma.$$

Employing the same arguments as in the proof of Lemma 3.8, we see the existence of such $\vec{w}$. In particular, by (4.13) we have

(4.14) $$Q(t_0)[\vec{w}]=B(t_0, \bar{\partial}^1)\vec{w}+H^0(t_0, \cdot)\partial_t\vec{w}=\vec{g} \qquad \text{on } [t_1, t_2]\times\Gamma$$

(because, what $\vec{w}=0$ on $[t_1, t_2]\times\Gamma$ implies that $\partial_t\vec{w}=0$ on $[t_1, t_2]\times\Gamma$). Let $\vec{v}$ be a solution to the equations:

(4.15) $$P(t_0)[\vec{v}]=\vec{f}-P(t_0)[\vec{w}] \ \text{ in } [t_1, t_2]\times\Omega, \qquad Q(t_0)[\vec{v}]=0 \ \text{ on } [t_1, t_2]\times\Gamma,$$
$$\vec{v}(t_1, x)=\vec{u}_0(x)-\vec{w}(t_1, x), \qquad \partial_t\vec{v}(t_1, x)=\vec{u}_1(x)-\partial_t\vec{w}(t_1, x) \ \text{ in } \Omega.$$

By (4.11) and (4.14), we know that $\vec{u}_0-\vec{w}(t_1, \cdot)$ and $\vec{u}_1-\partial_t\vec{w}(t_1, \cdot)$ satisfy (3.46). Applying Theorem 3.11 implies the existence of $\vec{v}\in E^2([t_1, t_2])$ satisfying (4.15). If we put $\vec{u}=\vec{v}+\vec{w}$, then $\vec{u}\in E^2([t_1, t_2])$ and satisfies (4.12).

Now, we shall remove the additional assumption: $\vec{g}\in C^\infty([t_1, t_2]; H^\infty(\Gamma))$. Since $C^\infty([t_1, t_2]; H^\infty(\Gamma))$ is dense in $C^1([t_1, t_2]; H^{1/2}(\Gamma))$, there exist $\vec{g}^k\in C^\infty([t_1, t_2]; H^\infty(\Gamma))$ such that

(4.16) $$\sup_{t_1\leq s\leq t_2} \langle\partial_s^l(\vec{g}^k(s, \cdot)-\vec{g}(s, \cdot))\rangle_{1/2} \longrightarrow 0 \qquad \text{as } k\to\infty \text{ for } l=0, 1.$$

Let $\vec{v}_0^k\in H^2(\Omega)$ be functions satisfying:

(4.17.a) $$-\partial_i(A^{ij}(t_0, \cdot)\partial_j\vec{v}_0^k)+\lambda_0\vec{v}_0^k=0 \qquad \text{in } \Omega,$$

(4.17.b) $$B(t_0, \bar{\partial}^1)\vec{v}_0^k=\vec{g}^k(t_1, \cdot)-\vec{g}(t_1, \cdot) \qquad \text{on } \Gamma.$$

Theorem 3.4 guarantees the existence of $\vec{v}_0^k$ and implies the estimates:

(4.18) $$\|\bar{\partial}^2\vec{v}_0^k\|\leqq C\langle\vec{g}^k(t_1, \cdot)-\vec{g}(t_1, \cdot)\rangle_{1/2}.$$

Put $\vec{u}_0^k=\vec{u}_0+\vec{v}_0^k\in H^2(\Omega)$. Then, by (4.11), (4.16), (4.17.b) and (4.18) we have

(4.14.a) $$B(t_0, \bar{\partial}^1)\vec{u}_0^k(x)+H^0(t_0, x)\vec{u}_1(x)=\vec{g}^k(t_1, x) \qquad \text{on } \Gamma,$$

(4.19.b) $$\|\bar{\partial}^2(\vec{u}_0^k-\vec{u}_0)\| \longrightarrow 0 \qquad \text{as } k\to\infty.$$

As is already proved, (4.12) has a unique solution $\vec{u}^k \in E^2([t_1, t_2])$ with initial data $\vec{u}_0^k$, $\vec{u}_1$, right member $\vec{f}$ and boundary data $\vec{g}^k$. If we apply (E.2), Lemma 4.1 and Theorem 3.4 to $\vec{u}^k - \vec{u}^{k'}$ and use (3.31.b), we have

$$\text{(4.20)} \qquad \sup_{t_1 \leq s \leq t_2} \|\bar{D}^2(\vec{u}^k(t, \cdot) - \vec{u}^{k'}(t, \cdot)\|^2$$
$$\leq C\Big\{\|\bar{\partial}^2(\vec{u}_0^k - \vec{u}_0^{k'})\|^2 + \sup_{t_1 \leq s \leq t_2} \langle \vec{g}^k(s, \cdot) - \vec{g}^{k'}(s, \cdot)\rangle_{1/2}^2$$
$$+ \sum_{l=0}^{1} \int_{t_1}^{t_2} \langle \partial_s^l(\vec{g}^k(s, \cdot) - g^{k'}(s, \cdot))\rangle_{1/2}^2 ds\Big\}.$$

Here, we have used the relation: $\partial_t^2 \vec{u}^k(0, x) = \vec{f}(0, x) - H(t_0, \partial^1)\vec{u}_1(x) - A(t_0, \partial^2)\vec{u}_0^k(x)$. Combining (4.16), (4.19.b) and (4.20), we see that $\{\vec{u}^k\}$ is a Cauchy sequence in $E^2([t_1, t_2])$. As a result, the limit $\vec{u}$ of $\{\vec{u}^k\}$ exists in $E^2([t_1, t_2])$ as $k \to \infty$, and then we see that $\vec{u}$ is a desired solution of (4.12), which completes the proof.

Under these preparations, we prove an existence theorem to (N). First, we prove it with zero initial data.

LEMMA 4.4. *Assume that* (A.1)-(A.6), (E.1) *and* (E.2) *are valid. If* $\vec{f} \in C^1([0, T]; L^2(\Omega))$ *and* $\vec{g} \in C^1([0, T]; H^{1/2}(\Gamma))$ *satisfy*:

$$\text{(4.21)} \qquad \vec{f}(0, x) = 0 \text{ in } \Omega \quad \text{and} \quad \vec{g}(0, x) = 0 \text{ on } \Gamma,$$

then there exists a unique solution $\vec{u} \in E^2([0, T])$ *to* (N) *with zero initial data, right member* $\vec{f}$ *and boundary data* $\vec{g}$.

Proof. The uniqueness follows from (E.1). We only prove the existence. Our proof is essentially the same as in Ikawa [3, Lemma 4.1]. To make our paper self-contained, we prove the lemma. In view of well-known Lions' method of the extension of functions (cf. the proof of Lemma 2.4), we may assume that $\vec{f} \in C^1([0, 2T]; L^2(\Omega))$ and $\vec{g} \in C^1([0, 2T]; H^{1/2}(\Gamma))$. Let $\Delta_k: 0 = t_0 < t_1 < \cdots < t_k = 2T$, be the subdivision of $[0, 2T]$ into k equal parts. Let $\vec{u}_k(t, x)$ be the Cauchy's polygonal line for this subdivision, which is constructed as follows: Let $\vec{u}_{k0}(t, x)$, defined for $t \in [t_0, t_1]$, be a solution to equations:

$$\text{(4.22.a)}_0 \qquad P(t_0)[\vec{u}_{k0}(t, x)] = \vec{f}(t, \vec{u}) \quad \text{in } [t_0, t_1] \times \Omega,$$

$$\text{(4.22.b)}_0 \qquad Q(t_0)[\vec{u}_{k0}(t, x)] = \vec{g}(t, x) \quad \text{on } [t_0, t_1] \times \Gamma,$$

$$\text{(4.22.c)}_0 \qquad \vec{u}_{k0}(0, x) = \partial_t \vec{u}_{k0}(0, x) = 0 \quad \text{in } \Omega,$$

and for $l \geq 1$ $\vec{u}_{kl}(t, x)$, defined for $t \in [t_l, t_{l+1}]$, be a solution to equations:

$$\text{(4.22.a)}_l \qquad P(t_l)[\vec{u}_{kl}(t, x)] = \vec{f}(t, x) \quad \text{in } [t_l, t_{l+1}] \times \Omega,$$

$(4.22.\mathrm{b})_l$ $\quad Q(t_l)[\vec{u}_{kl}(t, x)]$

$$=\vec{g}(t, x)+\frac{t_{l+1}-t}{t_{l+1}-t_l}\{Q(t_l)[\vec{u}_{kl-1}(\cdot, x)]|_{t=t_l}-Q(t_{l-1})[\vec{u}_{kl-1}(\cdot, x)]|_{t=t_l}\} \quad \text{on } [t_l, t_{l+1}]\times\Gamma,$$

$(4.22.\mathrm{c})_l$ $\quad \vec{u}_{kl}(t_l, x)=\vec{u}_{kl-1}(t_l, x), \quad \partial_t\vec{u}_{kl}(t_l, x)=\partial_t\vec{u}_{kl-1}(t_l, x) \quad$ in Ω.

Then, $\vec{u}_k(t, x)$ is defined for $t\in[0, 2T]$ by $\vec{u}_k(t, x)=\vec{u}_{kl}(t, x)$ if $t\in[t_l, t_{l+1}]$. The existence of each $\vec{u}_{kl}$ is assured by Lemma 4.3 since the compatibility condition (4.11) is satisfied at each t_l. Consequently, we see that $\vec{u}_k\in C^0([0, 2T]; H^2(\Omega))\cap C^1([0, 2T]; H^1(\Omega))$ and $\vec{u}_{kl}\in E^2([t_l, t_{l+1}])$ for all l, from which it follows that $\vec{u}_k\in H^2(\Omega_{2T})$, $\Omega_{2T}=(0, 2T)\times\Omega$.

First, we shall prove that

$$(4.23) \qquad \|\bar{D}^2\vec{u}_{kl}(t, \cdot)\|^2\leqq C(T)\int_0^t\sum_{h=0}^{1}(\|\partial_s^h\vec{f}(s, \cdot)\|^2+\langle\partial_s^h\vec{g}(s, \cdot)\rangle_{1/2}^2)ds$$

for any $t\in[t_l, t_{l+1}]$, $l=0, 1, \cdots, k-1$ and $k\geqq 1$. For notational convenience, the same letter C is used to denote constants independent of l and k in the proof. Put $\mathcal{V}_{kl}(t)=(\vec{u}_{kl}(t, \cdot), \partial_t\vec{u}_{kl}(t, \cdot))$ and $\mathcal{V}'_{kl}(t)=(\partial_t\vec{u}_{kl}(t, \cdot), \partial_t^2\vec{u}_{kl}(t, \cdot))$ and use the norm: $\|\cdot\|_{\mathcal{H}(t_l)}$ (cf. (3.30)). Applying Theorem 3.4 to $(4.22)_{l-1}$ at $t=t_l$ implies that

$$\|\bar{\partial}^2\vec{u}_{kl-1}(t_l, \cdot)\|^2\leqq C\{\|\vec{f}(t_l, \cdot)\|^2+\langle\vec{g}(t_l, \cdot)\rangle_{1/2}^2 +\|\bar{\partial}^1\partial_t\vec{u}_{kl-1}(t_l, \cdot)\|^2+\|\partial_t^2\vec{u}_{kl-1}(t_l, \cdot)\|^2+\|\bar{D}^1\vec{u}_{kl-1}(t_l, \cdot)\|^2\}.$$

Thus, by (3.31.a) we see that

$$(4.24) \qquad \|\bar{D}^2\vec{u}_{kl-1}(t_l, \cdot)\|^2\leqq C\{\|\vec{f}(t_l, \cdot)\|^2+\langle\vec{g}(t_l, \cdot)\rangle_{1/2}^2 +\|\mathcal{V}_{kl-1}(t_l)\|_{\mathcal{H}(t_{l-1})}^2+\|\mathcal{V}'_{kl-1}\|_{\mathcal{H}(t_{l-1})}^2\}$$

for $l\geqq 1$. Since $t_{l+1}-t_l=t_l-t_{l-1}=2T/k$, by (4.24) and the mean value theorem we have

$$(4.25) \qquad \sum_{h=0}^{1}\int_{t_l}^{t}\left\langle\partial_s^h\left\{\frac{t_{l+1}-s}{t_{l+1}-t_l}[Q(t_l)[\vec{u}_{kl-1}]|_{t=t_l}-Q(t_{l-1})[\vec{u}_{kl-1}]|_{t=t_l}]\right\}\right\rangle_{1/2}^2 ds \leqq(GT/k)\{\|\vec{f}(t_l, \cdot)\|^2+\langle\vec{g}(t_l, \cdot)\rangle_{1/2}^2+\|\mathcal{V}_{kl-1}(t_l)\|_{\mathcal{H}(t_{l-1})}^2+\|\mathcal{V}'_{kl-1}(t_l)\|_{\mathcal{H}(t_{l-1})}^2\}.$$

Since $\partial_t^2\vec{u}_{kl}(t_l, \cdot)=\vec{f}(t_l, \cdot)-H(t_l, \bar{\partial}^1)\partial_t\vec{u}_{kl-1}(t_l, \cdot)-A(t_l, \bar{\partial}^2)\vec{u}_{kl-1}(t_l, \cdot)$ and $\partial_t^2\vec{u}_{kl-1}(t_l, \cdot) =\vec{f}(t_l, \cdot)-H(t_{l-1}, \bar{\partial}^1)\partial_t\vec{u}_{kl-1}(t_l, \cdot)-A(t_{l-1}, \bar{\partial}^2)\vec{u}_{kl-1}(t_l, \cdot)$ as follows from (4.22.a and c)$_l$, by (4.24) we see

$$(4.26) \qquad \|\partial_t^2\vec{u}_{kl}(t_l, \cdot)\|^2-\|\partial_t^2\vec{u}_{kl-1}(t_l, \cdot)\|^2 \leqq(CT/k)\{\|\vec{f}(t_l, \cdot)\|^2+\langle\vec{g}(t_l, \cdot)\rangle_{1/2}^2+\|\mathcal{V}_{kl-1}(t_l)\|_{\mathcal{H}(t_{l-1})}^2+\|\mathcal{V}'_{kl-1}(t_l)\|_{\mathcal{H}(t_{l-1})}^2\}.$$

By (4.26) and $(4.22.\mathrm{c})_l$ we have

$$(4.27)\quad \|\mathcal{U}_{kl}(t_l)\|^2_{\mathcal{H}(t_l)}+\|\mathcal{U}'_{kl}(t_l)\|^2_{\mathcal{H}(t_l)}$$
$$\leqq(1+(CT/k))\{\|\mathcal{U}_{kl-1}(t_l)\|^2_{\mathcal{H}(t_{l-1})}+\|\mathcal{U}'_{kl-1}(t_l)\|^2_{\mathcal{H}(t_{l-1})}\}$$
$$+(CT/k)\{\|\vec{f}(t_l,\ \cdot)\|^2+\langle\vec{g}(t_l,\ \cdot)\rangle^2_{1/2}\}.$$

Applying (E.2) and Lemma 4.1 to $(4.22)_l$ and substituting (4.25) and (4.27) into the resulting estimate, we have

$$(4.28)\quad \|\mathcal{U}_{kl}(t)\|^2_{\mathcal{H}(t_l)}+\|\mathcal{U}'_{kl}(t)\|^2_{\mathcal{H}(t_l)}$$
$$\leqq(\exp C(t-t_l))(1+(CT/k))\{\|\mathcal{U}_{kl-1}(t_l)\|^2_{\mathcal{H}(t_{l-1})}+\|\mathcal{U}'_{kl-1}(t_l)\|^2_{\mathcal{H}(t_{l-1})}\}$$
$$+(\exp C(t-t_l))\Big\{C\int_{t_l}^{t}\sum_{h=0}^{1}(\|\partial_s^h\vec{f}(s,\ \cdot)\|^2+\langle\partial_s^h g(s,\ \cdot)\rangle^2_{1/2})ds$$
$$+(CT/k)(\|\vec{f}(t_l,\ \cdot)\|^2+\langle\vec{g}(t_l,\ \cdot)\rangle^2_{1/2})\Big\},\quad t_l\leqq t\leqq t_{l+1}\quad (l\geqq 1).$$

Applying (E.2) and Lemma 4.1 to $(4.22)_0$, we also have

$$(4.29)\quad \|\mathcal{U}_{k0}(t)\|^2_{\mathcal{H}(0)}+\|\mathcal{U}'_{k0}(t)\|^2_{\mathcal{H}(0)}$$
$$\leqq Ce^{Ct}\sum_{h=1}^{1}\int_0^t(\|\partial_s^h\vec{f}(s,\ \cdot)\|^2+\langle\partial_s^h\vec{g}(s,\ \cdot)\rangle^2_{1/2})ds,\quad 0\leqq t\leqq t_1\,,$$

When $l\geqq 1$, repeated use of (4.28) implies that

$$(4.30)\quad \|\mathcal{U}_{kl}(t)\|^2_{\mathcal{H}(t_l)}+\|\mathcal{U}'_{kl}(t)\|^2_{\mathcal{H}(t_l)}$$
$$\leqq Ce^{Ct}(1+(CT/k))^l\int_0^t\sum_{h=0}^{1}(\|\partial_s^h\vec{f}(s,\ \cdot)\|^2+\langle\partial_s^h\vec{g}(s,\ \cdot)\rangle^2_{1/2})ds$$
$$+(CT/k)e^{Ct}\sum_{h=0}^{l-1}(1+(CT/k))^h(\|\vec{f}(t_{l-h},\ \cdot)\|^2+\langle\vec{g}(t_{l-h},\ \cdot)\rangle^2_{1/2})$$

for $t\in[t_l,\ t_{l+1}]$. Here, at the finial step we have used (4.29). To treat the second term of the right-hand side of (4.30), we use the inequalities:

$$(4.31)\quad \|\vec{f}(t_{l'},\ \cdot)\|^2+\langle\vec{g}(t_{l'},\ \cdot)\rangle^2_{1/2}\leqq\|\vec{f}(t_{l'-1},\ \cdot)\|^2+\langle\vec{g}(t_{l'-1},\ \cdot)\rangle^2_{1/2}$$
$$+\int_{t_{l'-1}}^{t_{l'}}\sum_{h=0}^{1}(\|\partial_s^h\vec{f}(s,\ \cdot)\|^2+\langle\partial_s^h\vec{g}(s,\ \cdot)\rangle^2_{1/2})ds$$

Repeated use of (4.31) implies that

$$(4.32)\quad \sum_{h=0}^{l-1}(1+(CT/k))^h(\|\vec{f}(t_{l-h},\ \cdot)\|^2+\langle\vec{g}(t_{l-h},\ \cdot)\rangle^2_{1/2})\leqq$$
$$\sum_{l'=0}^{l-1}\Big\{\Big(\sum_{l''=0}^{l'}(1+(CT/k))^{l''}\Big)\int_{t_{l-l'-1}}^{t_{l-l'}}\sum_{h=0}^{1}(\|\partial_s^h\vec{f}(s,\ \cdot)\|^2+\langle\partial_s^h\vec{g}(s,\ \cdot)\rangle^2_{1/2}ds\Big\}.$$

Here, we have used (4.21). Substituting the inequality:

$$\sum_{l''=0}^{l'}(1+(CT/k))^{l''}\leqq(CT/k)^{-1}(1+(CT/k))^{l'+1}$$

into (4.32), by (4.31) and (4.32) we have

$$(4.33)\qquad \|\mathcal{V}_{kl}(t)\|^2_{\mathcal{H}(t_l)}+\|\mathcal{V}'_{kl}(t)\|^2_{\mathcal{H}(t_l)}$$

$$\leqq Ce^{Ct}(1+(CT/k))^l\int_0^t \sum_{h=0}^{1}(\|\partial_s^h\vec{f}(s,\ \cdot)\|^2+\langle\partial_s^h\vec{g}(s,\ \cdot)\rangle^2_{1/2})ds \quad \text{for } l\geqq 1.$$

Applying Theorem 3.4 to (4.22.a and b)$_l$, by (3.31.a) we have

$$(4.34)\qquad \|\bar{\partial}^2\vec{u}_{kl}(t,\ \cdot)\|^2\leqq C\{\|\mathcal{V}_{kl}(t)\|^2_{\mathcal{H}(t_l)}+\|\mathcal{V}'_{kl}(t)\|^2_{\mathcal{H}(t_l)}+\|\vec{f}(t,\ \cdot)\|^2$$

$$+\langle\vec{g}(t,\ \cdot)\rangle^2_{1/2}+\|\bar{\partial}^2\vec{u}_{kl-1}(t_l,\ \cdot)\|^2+\|\bar{\partial}^1\partial_t\vec{u}_{kl-1}(t_l,\ \cdot)\|^2\}.$$

Substituting (4.24) into the right-hand side of (4.34), we have

$$(4.35)\qquad \|\bar{\partial}^2\vec{u}_{kl}(t,\ \cdot)\|^2\leqq C\{\|\mathcal{V}_{kl}(t)\|^2_{\mathcal{H}(t_l)}+\|\mathcal{V}'_{kl}(t)\|^2_{\mathcal{H}(t_l)}$$

$$+\|\vec{f}(t,\ \cdot)\|^2+\langle\vec{g}(t,\ \cdot)\rangle^2_{1/2}+\|\vec{f}(t_l,\ \cdot)\|^2+\langle\vec{g}(t_l,\ \cdot)\rangle^2_{1/2}$$

$$+\|\mathcal{V}_{kl-1}(t_l)\|^2_{\mathcal{H}(t_{l-1})}+\|\mathcal{V}'_{kl-1}(t_l)\|^2_{\mathcal{H}(t_{l-1})}\}.$$

Noting (4.7) and combining (4.33) and (4.35), we have

$$\|\bar{\partial}^2\vec{f}_{kl}(t,\ \cdot)\|^2+\|\mathcal{V}_{kl}(t)\|^2_{\mathcal{H}(t_l)}+\|\mathcal{V}'_{kl}(t)\|^2_{\mathcal{H}(t_l)}$$

$$\leqq Ce^{Ct}(1+(CT/k))^l\int_0^t \sum_{h=0}^{1}(\|\partial_s^h\vec{f}(s,\ \cdot)\|^2+\langle\partial_s^h\vec{g}(s,\ \cdot)\rangle^2_{1/2})ds.$$

Since $(1+(CT/k))^l\leqq e^{CT}$ for $l\leqq k$, by (3.31.a) we have (4.23).

Especially, from (4.23) it follows that $\{\vec{u}_{kl}\}$ is a bounded set in $H^2(\Omega_{2T})$. Consequently, there exists a subsequence $\{k_l\}$ of $\{k\}$ and $\vec{u}\in H^2(\Omega_{2T})$ such that $\vec{u}_{k_l}$ converges to $\vec{u}$ weakly in $H^2(\Omega_{2T})$ as $l\to\infty$. For the sake of notational simplicity, we also denote this subsequence by $\{k\}$. Thus, we have proved

$$(4.36)\qquad \vec{u}_k \longrightarrow \vec{u} \text{ weakly in } H^2(\Omega_{2T}) \text{ as } k\to\infty.$$

By (4.36) we see

$$(4.37)\qquad P(t)[\vec{u}(t,\ x)]=\vec{f}(t,\ x),\qquad Q(t)[\vec{u}(t,\ x)]=\vec{g}(t,\ x).$$

Here, the first and second parts of equalities are valid as elements of $L^2(\Omega_{2T})$ and $L^2((0,\ 2T)\times\Gamma)$, respectively. In fact, if we define $P_k[\vec{u}]$ and $Q_k[\vec{u}]$ corresponding to the subdivision Δ_k by: $P_k[\vec{u}]=P(t_l)[\vec{u}]$ and $Q_k[\vec{u}]=Q(t_l)[\vec{u}]$ for $t_l\leqq t\leqq t_{l+1}$, then for any $\phi\in L^2(\Omega_{2T})$ we have

$$|(P(\cdot)[\vec{u}]-\vec{f},\ \phi)_{\Omega_{2T}}|\leqq|((P_k-P(\cdot))[\vec{u}_k],\ \phi)_{\Omega_{2T}}|+|(P(\cdot)[\vec{u}_k-\vec{u}],\ \phi)_{\Omega_{2T}}|$$

$$\leqq(C/k)\|\bar{D}^2\vec{u}_k\|_{\Omega_{2T}}\|\phi\|_{\Omega_{2T}}+|(P(\cdot)[\vec{u}_k-\vec{u}],\ \phi)_{\Omega_{2T}}|.$$

In view of (4.23) and (4.26), letting $k\to\infty$, we have the first part. Since the map: $H^2(\Omega_{2T})\ni\vec{w}\mapsto Q(t)\vec{w}\in L^2((0,\ 2T)\times\Omega)$, is strongly continuous (cf. Hörmander

[1, § 2.5]), (4.36) implies that $Q(t)\vec{u}_k$ converges to $Q(t)\vec{u}$ weakly in $L^2((0, 2T)\times\Omega)$ as $k\to\infty$. On the other hand, by Lemmas 3.1 and 3.2, $(4.22.\mathrm{b})_l$ and (4.23) we have that $\langle Q(t)[\vec{u}_k]-\vec{g}(t, \cdot)\rangle_0^2\leqq C/k$ for any $t\in[0, 2T]$, which implies that $Q(t)[\vec{u}_k]$ converges to $\vec{g}$ strongly in $L^2((0, 2T)\times\Omega)$. Thus, we have the second part of the equality (4.37).

Now, we shall prove that $\vec{u}\in E^2([0, T])$ (we adjust the value on sets having measure 0 if necessary). To do this, first we should remark that the inequality:

$$\sum_{\substack{h+|\alpha|\leq 2\\ 0\leq h\leq 1}}\|\partial_x^\alpha\partial_t^h u\|_{\Omega_t}^2\leqq Kt^2 \qquad (\Omega_t=(0, t)\times\Omega,\ t\in[0, 2T]), \tag{4.38}$$

holds with $K=C(T)\sum_{h=0}^{1}\sup_{0\leq s\leq 2T}(\|\partial_s^h\vec{f}(s, \cdot)\|^2+\langle\partial_s^h\vec{g}(s, \cdot)\rangle_{1/2}^2)$. In fact, since $\partial_x^\alpha\partial_t^h\vec{u}(t, x)=\partial_x^\alpha\partial_t^h\vec{u}_{kl}(t, x)$ for $t_l\leqq t\leqq t_{l+1}$ provided that $|\alpha|+h\leqq 2$, $0\leqq h\leqq 1$, integrating (4.23) in t, we have that $\|\partial^2\vec{u}_k\|_{\Omega_t}^2+\|\bar{\partial}^1\partial_t\vec{u}_k\|_{\Omega_t}^2\leqq Kt^2$. By (4.36) we have (4.38).

To prove $\vec{u}\in E^2([0, T])$, we use the mollifier $\rho_\delta*$ which is the same as in the proof of Lemma 4.2. Note that $\rho_\delta*\vec{u}=\vec{u}_\delta\in C^\infty([0, T]; H^2(\Omega))$ for $0<\delta<T/2$. By (4.37) we have

$$P(t)[\vec{u}_\delta]=\vec{f}_\delta-C_\delta\vec{u} \text{ in } [0, T]\times\Omega, \qquad Q(t)[\vec{u}_\delta]=\vec{g}_\delta-\Gamma_\delta\vec{u} \text{ on } [0, T]\times\Gamma$$

for $0<\delta<T/2$, where $\vec{f}_\delta=\rho_\delta*\vec{f}$, $\vec{g}_\delta=\rho_\delta*\vec{g}$, $C_\delta\vec{u}=\rho_\delta*P(\cdot)[\vec{u}]-P(\cdot)[\rho_\delta*\vec{u}]$ and $\Gamma_\delta\vec{u}=\rho_\delta*Q(\cdot)[\vec{u}]-Q(\cdot)[\rho_\delta*\vec{u}]$. Applying Lemma 4.2, we have

$$\begin{aligned}\sup_{0\leq s\leq T}\|\bar{D}^2(\vec{u}_\delta-\vec{u}_{\delta'})(t, \cdot)\|^2\leqq C\{&\|\bar{\partial}^2(\vec{u}_\delta-\vec{u}_{\delta'})(0, \cdot)\|^2\\ &+\|\partial^1\bar{\partial}_t(\vec{u}_\delta-\vec{u}_{\delta'})(0, \cdot)\|^2+\|\vec{f}_\delta(0, \cdot)-\vec{f}_{\delta'}(0, \cdot)\|^2\\ &+\langle\vec{g}_\delta(0, \cdot)-\vec{g}_{\delta'}(0, \cdot)\rangle_{1/2}^2+\|C_\delta\vec{u}(0, \cdot)\|^2+\|C_{\delta'}\vec{u}(0, \cdot)\|^2\\ &+\langle\Gamma_\delta\vec{u}(0, \cdot)\rangle_{1/2}^2+\langle\Gamma_{\delta'}\vec{u}(0, \cdot)\rangle_{1/2}^2\\ &+\int_0^T\sum_{h=0}^{1}(\|\partial_s^h(\vec{f}_\delta-\vec{f}_{\delta'})(s, \cdot)\|^2+\langle\partial_s^h(\vec{g}_\delta-\vec{g}_{\delta'})(s, \cdot)\rangle_{1/2}^2)ds\\ &+\int_0^T\sum_{h=0}^{1}(\|\partial_s^h C_\delta\vec{u}(s, \cdot)\|^2+\|\partial_s^h C_{\delta'}\vec{u}(s, \cdot)\|^2\\ &\qquad+\langle\partial_s^h\Gamma_\delta\vec{u}(s, \cdot)\rangle_{1/2}^2+\langle\partial_s^h\Gamma_{\delta'}\vec{u}(s, \cdot)\rangle_{1/2}^2)ds\}.\end{aligned} \tag{4.39}$$

By (4.38) we see that

$$\|\partial_x^\alpha\partial_t^h\vec{u}_\delta(0, \cdot)\|\leqq C\delta^{1/2} \tag{4.40}$$

for any α and h such that $|\alpha|+h\leqq 2$ and $0\leqq h\leqq 1$. Noting (4.10) we see that $\{\vec{u}_\delta\}$ is a Cauchy sequence in $E^2([0, T])$, and then $\vec{u}\in E^2([0, T])$. By (4.40) we also see that $\vec{u}(0, x)=\partial_t\vec{u}(0, x)=0$. Thus, we have proved the lemma.

Now, we can prove the following existence theorem of complete form by using Lemmas 4.2 and 4.4.

THEOREM 4.5. *Assume that* (A.1)-(A.6), (E.1) *and* (E.2) *are valid. If* $\vec{u}_0\in H^2(\Omega)$, $\vec{u}_1\in H^1(\Omega)$, $\vec{f}\in C^1([0,T];L^2(\Omega))$ *and* $\vec{g}\in C^1([0,T];H^{1/2}(\Gamma))$ *and they satisfy the compatibility condition of order* 0. *i.e.*,

$$B(0,\bar{\partial}^1)\vec{u}_0(x)+H^0(0,x)\vec{u}_1(x)=\vec{g}(0,x) \quad \text{on } \Gamma, \tag{4.41}$$

then (N) *admits a unique solution in* $E^2([0,T])$ *with initial data* $\vec{u}_0$, $\vec{u}_1$ *right member* $\vec{f}$ *and boundary data* $\vec{g}$.

Proof. The uniqueness follows from (E.1). We only prove the existence. First, let us assume that $\vec{u}_0\in H^3(\Omega)$, $\vec{u}_1\in H^2(\Omega)$ and $\vec{f}\in C^1([0,T];H^1(\Omega))$. Put

$$\vec{u}_2(x)=\vec{f}(0,x)-H(0,\bar{\partial}^1)\vec{u}_1(x)-A(0,\bar{\partial}^2)\vec{u}_0(x)\in H^1(\Omega), \tag{4.42}$$

and $\vec{u}_k={}^t(u_{1k},\cdots,u_{mk})$, $k=0,1,2$. By u'_{ak} we denote the extensions of u_{ak} from Ω to $\boldsymbol{R}^n$. Let P be a strictly hyperbolic operator with respect to t of order 3 having constant coefficients. Let $U_a(t,x)\in E^3(\boldsymbol{R};\boldsymbol{R}^n)$ be solutions to Cauchy problems: $PU_a=0$ in $\boldsymbol{R}\times\boldsymbol{R}^n$ and $\partial_t^kU_a(0,x)=u'_{ak}(x)$ on $\boldsymbol{R}^n$, $k=0,1,2$. Put $\vec{U}={}^t(U_1,\cdots,U_m)$. Let $\vec{v}\in E^2([0,T])$ be a solution to the equations:

$$P(t)[\vec{v}]=\vec{f}-P(t)[\vec{U}] \text{ in } [0,T]\times\Omega, \quad Q(t)[\vec{v}]=\vec{g}-Q(t)[\vec{U}] \text{ on } [0,T]\times\Gamma,$$

$$\vec{v}(0,x)=\partial_t\vec{v}(0,x)=0 \text{ in } \Omega.$$

By the definitions of $\vec{u}_2$ and $\vec{U}$ and (4.41) we see that $\vec{f}-P(\cdot)[\vec{U}]\in C^1([0,T];L^2(\Omega))$, $\vec{g}-Q(\cdot)[U]\in C^1([0,T];H^{1/2}(\Gamma))$ and (4.21) are satisfied, and then the existence of $\vec{v}$ is assured by Lemma 4.4. Obviously, if we put $\vec{u}=\vec{v}+\vec{U}$, then $\vec{u}$ is in $E^2([0,T])$ and satisfies (N) with initial data $\vec{u}_0$, $\vec{u}_1$, right member $\vec{f}$ and boundary data $\vec{g}$.

Now, we shall construct approximations of $\vec{u}_0$, $\vec{u}_1$, $\vec{f}$ and $\vec{g}$. We can choose sequences $\{\vec{f}^k\}\subset C^\infty([0,T];H^\infty(\Omega))$, $\{\vec{g}^k\}\subset C^\infty([0,T];H^\infty(\Gamma))$, $\{\vec{v}_0^k\}\subset H^\infty(\Omega)$ and $\{\vec{u}_1^k\}\subset H^\infty(\Omega)$ so that $\vec{f}^k$ (resp. $\vec{g}^k$; resp, $\vec{v}_0^k$; resp. $\vec{u}_1^k$) converges to $\vec{f}$ (resp. $\vec{g}$; resp. $\vec{u}_0$; resp. $\vec{u}_1$) in $C^1([0,T];L^2(\Omega))$ (resp. $C^1([0,T];H^{1/2}(\Gamma))$; resp. $H^2(\Omega)$; resp. $H^1(\Omega)$) as $k\to\infty$. Let $\vec{w}_0^k\in H^3(\Omega)$ be a solution of the equations:

$$-\partial_i(A^{ij}(0,\cdot)\partial_j\vec{w}_0^k)+\lambda_0\vec{w}_0^k=0 \quad \text{in } \Omega,$$

$$B(0,\bar{\partial}^1)\vec{w}_0^k=\vec{g}^k(0,\cdot)-H^0(0,\cdot)\vec{u}_1^k-B(0,\bar{\partial}^1)\vec{v}_0^k \quad \text{in } \Gamma.$$

Since the coefficients of the operator $B(0,\bar{\partial}^1)$ and the $A^{ij}(0,x)$ are in $\mathscr{B}^2(\bar{\Omega})$, Theorem 3.4 guarantees the existence of $\vec{w}_0^k\in H^3(\Omega)$. By Theorem 3.4, (4.41), Lemmas 3.1 and 3.2, we see

$$\|\bar{\partial}^2\vec{w}_0^k\|\leq C\{\langle\vec{g}^k(0,\cdot)-\vec{g}(0,\cdot)\rangle_{1/2}+\|\bar{\partial}^1(\vec{u}_1^k-\vec{u}_1)\|+\|\bar{\partial}^2(\vec{v}_0^k-\vec{u}_0)\|\}.$$

Thus, if we put $\vec{u}_0^k=\vec{v}_0^k+\vec{w}_0^k$, then $\vec{u}_0^k$, $\vec{u}_1^k$ and $\vec{g}^k$ satisfy (4.41). Consequently, there exists a solution $\vec{u}^k \in E^2([0, T])$ to (N) with initial data $\vec{u}_0^k$, $\vec{u}_1^k$, right member $\vec{f}^k$ and boundary data $\vec{g}^k$. Applying Lemma 4.2 to $\vec{u}^k - \vec{u}^{k'}$ implies that $\{\vec{u}^k\}$ is a Cauchy sequence in $E^2([0, T])$, from which it follows that the limit $\vec{u} \in E^2(0, T])$ exists and satisfies (N) with initial data $\vec{u}_0$, $\vec{u}_1$, right member $\vec{f}$ and boundary data $\vec{g}$. This completes the proof of the theorem.

§ 5. A priori estimate in a half-space.

In this section, we derive some "*a priori estimate*" for the following problem:

(5.1) $$\mathcal{P}[\vec{u}] = \vec{f} \quad \text{in } \boldsymbol{R} \times \boldsymbol{R}_+^n, \qquad Q[\vec{u}] = \vec{g} \quad \text{on } \boldsymbol{R} \times \boldsymbol{R}_0^n,$$

where $\boldsymbol{R}_+^n = \{x \in \boldsymbol{R}^n \mid x_n > 0\}$, $\boldsymbol{R}_0^n = \{x \in \boldsymbol{R}^n \mid x_n = 0\}$,

$\mathcal{P}[\vec{u}] = D_t^2 \vec{u} + 2S^i(t, x) D_i D_t \vec{u} + P^{ij}(t, x) D_i D_j \vec{u}$, $D = -\sqrt{-1}\,\partial/\partial t$, $D_i = -\sqrt{-1}\,\partial/\partial x_i$,

$Q[u] = -P^{nj}(t, x) D_j \vec{u} + Q^p(t, x') D_p \vec{u} + S^0(t, x') D_t \vec{u}$, $x' = (x_1, \cdots, x_{n-1})$.

In this section and Appendix, the functions in general are assumed to be complex—valued and we use the following notations. Let γ always refer to any real number $\geqq 1$. Put $G_R = \{(t, x) \in \boldsymbol{R}^{n+1} \mid |x'| < R,\ t \in \boldsymbol{R},\ 0 < x_n < R\}$ and $G_R' = \{(t, x) \in \boldsymbol{R}^n \mid |x'| < R,\ t \in \boldsymbol{R}\}$. For any integer $L \geqq 0$, $s \in \boldsymbol{R}$, scalar functions u, v and vector valued functions $\vec{u}$, $\vec{v}$, put

$$\mathcal{H}_{\gamma,R}^L = \Big\{ \vec{u} = {}^t(u, \cdots, u_m) \mid u_i \in H_{\text{loc}}^2(\boldsymbol{R} \times \boldsymbol{R}_+^n),\ \text{supp}\, u_i \subset G_R\,,$$

$$|\vec{u}|_{L,\gamma}^2 = \sum_{k+|\alpha| \leq L} \int_{\boldsymbol{R}^1 \times \boldsymbol{R}_+^n} e^{-2\gamma t} |\partial_t^k \partial_x^\alpha \vec{u}(t, x)|^2 dt dx < \infty \Big\}\,,$$

$$\langle D' \rangle^s u(x') = (2\pi)^{1-n} \int_{\boldsymbol{R}^{n-1}} (1 + |\xi'|^2)^{s/2} \hat{u}(\xi') d\xi'$$

($\xi' = (\xi_1, \cdots, \xi_{n-1})$ and the $\hat{u}$ denotes the Fourier transform of $u(x')$),

$$\langle D' \rangle^s \vec{u}(x') = {}^t(\langle D' \rangle^s u_1(x'), \cdots, \langle D' \rangle^s u_m(x'))\,,$$

$$\langle u \rangle_{s,\gamma}^2 = \int_{\boldsymbol{R}^n} e^{-2\gamma t} |\langle D' \rangle^s u(t, x')|^2 dt dx'\,,$$

$$(u, v)_\gamma = \int_{\boldsymbol{R} \times \boldsymbol{R}_+^n} e^{-2\gamma t} u(t, x) \overline{v(t, x)} dt dx\,,$$

$$\langle u, v \rangle_\gamma = \int_{\boldsymbol{R}^n} e^{-2\gamma t} u(t, x') \overline{v(t, x')} dt dx'\,,$$

$$(\vec{u}, \vec{v})_\gamma = \sum_{a=1}^m (u_a, v_a)_\gamma\,, \qquad \langle \vec{u}, \vec{v} \rangle_\gamma = \sum_{a=1}^m \langle u_a, v_a \rangle_\gamma\,,$$

$$\|\vec{u}\|_{\boldsymbol{R}_+^n} = \|\vec{u}\|, \quad `u(x')' = u(x', 0), \quad `\vec{u}' = {}^t(`u_1', \cdots, `u_m')\,.$$

Put

$$B(l, R)=\sum_{i,j=1}^{n}|P^{ij}|_{\infty,l,G_R}+\sum_{i=1}^{n}|S^i|_{\infty,l,G_R}+|S^0|_{\infty,l,G'_R}+\sum_{p=1}^{n-1}|Q^p|_{\infty,l,G'_R}.$$

Throughout this section, it is assumed that:

(A.5.1) The P^{ij}, S^i are $m\times m$ matrices of real-valued functions in $\mathcal{B}^2(\bar{G}_R)$ and the S^0, Q^p are $m\times m$ matrices of real-valued functions in $\mathcal{B}^2(\bar{G}'_R)$.

(A.5.2) ${}^tP^{ij}=P^{ji}$, ${}^tS^i=S^i$ on $\bar{G}_R$ and ${}^tQ^p=-Q^p$ on $\bar{G}'_R$.

(A.5.3) There exist d_3, $d_4>0$ such that

$$\int_{R^n_+}P^{ij}(t, x)D_j\vec{u}(x)\cdot\overline{D_i\vec{u}(x)}dx-\int_{R^n_+}Q^p(t, x')D_p\vec{u}(x)\cdot\overline{D_n\vec{u}(x)}dx$$

$$-\int_{R^n_+}D_n\vec{u}(x)\cdot\overline{Q^p(t, x')D_p\vec{u}(x)}dx\geqq d_3\|\partial^1\vec{u}\|^2-d_4\|\vec{u}\|^2$$

for any $\vec{u}\in H^1(R^n_+)$ such that $\vec{u}(x)=0$ for $x\notin\{x\in R^n_+|0<x_n<R,\ |x'|<R\}$ and $t\in R$.

(A.5.4) $(S^0(t, x')-S^n(t, x', 0))\vec{v}\cdot\vec{v}\geqq 0$ for any $(t, x')\in G'_R$ and constant vector $\vec{v}\in R^m$.

(A.5.5) There exists a constant $d_5>0$ such that $P^{nn}(t, x)\vec{v}\cdot\vec{v}\geqq d_5|\vec{v}|^2$ for any $(t, x)\in G_R$ and constant vector $\vec{v}\in R^m$.

We begin with the following Green's formula.

LEMMA 5.1. *Assume that* (A.5.1)-(A.5.4) *are valid. For any* $\vec{u}\in\mathcal{H}^2_{\gamma,R}$, *the following two identities are valid*:

(5.2)
$$\sqrt{-1}\{(\mathcal{P}[\vec{u}], D_t\vec{u})_\gamma-(D_t\vec{u}, \mathcal{P}[\vec{u}])_\gamma\}$$
$$\cong 2\gamma\{(D_t\vec{u}, D_t\vec{u})_\gamma+(P^{ij}D_j\vec{u}, D_i\vec{u})_\gamma\}-\{\langle`\mathcal{Q}[\vec{u}]', `D_t\vec{u}'\rangle_\gamma+\langle`D_t\vec{u}', `\mathcal{Q}[\vec{u}]'\rangle_\gamma\}$$
$$+\{\langle Q^p`D_p\vec{u}', `D_t\vec{u}'\rangle_\gamma+\langle`D_t\vec{u}', Q^p`D_p\vec{u}'\rangle_\gamma\}$$
$$+\langle(S^0-`S^{n}')`D_t\vec{u}', `D_t\vec{u}'\rangle_\gamma+\langle`D_t\vec{u}', (S^0-`S^{n}')`D_t\vec{u}'\rangle_\gamma$$

where $A\cong B$ means that $|A-B|\leqq C(B(1, R))|\vec{u}|^2_{2,\gamma}$.

(5.3)
$$\sqrt{-1}\{(\mathcal{P}[\vec{u}], D_n\vec{u})_\gamma-(D_n\vec{u}, \mathcal{P}[\vec{u}])_\gamma\}$$
$$\cong\langle`D_t\vec{u}', `D_t\vec{u}'\rangle_\gamma+\langle`P^{nn}D_n\vec{u}', `D_n\vec{u}'\rangle_\gamma-\langle`P^{pq}D_q\vec{u}', `D_p\vec{u}'\rangle_\gamma$$
$$+2\langle`S^pD_p\vec{u}', `D_t\vec{u}'\rangle_\gamma+2\gamma\{(D_t\vec{u}, D_n\vec{u})_\gamma+(D_n\vec{u}, D_t\vec{u})_\gamma+2(S^iD_i\vec{u}, D_n\vec{u})_\gamma\}.$$

Proof. By integration by parts we have

(5.4.a) $$(u, D_tv)_\gamma=2\sqrt{-1}\gamma(u, v)_\gamma+(D_tu, v)_\gamma,$$

(5.4.b) $$(u, D_n v)_\gamma = -\sqrt{-1}\langle `u', `v'\rangle_\gamma + (D_n u, v)_\gamma,$$

(5.4.c) $$(u, D_p v)_\gamma = (D_p u, v)_\gamma.$$

Using (5.4) and (A.5.3), we see easily (5.2) and (5.3). So, we may omit the details of the proof.

LEMMA 5.2. *Assume that* (A.5.1)-(A.5.4) *are valid. For any* $\vec{u}\in\mathcal{H}^2_{\gamma,R}$ *the identity*:

(5.5) $$\langle Q^p`D_p\vec{u}', `D_t\vec{u}'\rangle_\gamma + \langle `D_t\vec{u}', Q^p`D_p\vec{u}'\rangle_\gamma$$
$$\cong -2\gamma\{(Q^p D_p\vec{u}, D_n\vec{u})_\gamma + (D_n\vec{u}, Q^p D_p\vec{u})_\gamma\}$$

holds where $\cong$ *is the same as in Lemma* 5.1.

Proof. We have by (5.4.a) and (A.5.2)

$$\langle Q^p`D_p\vec{u}', `D_t\vec{u}'\rangle_\gamma$$
$$\cong -2\gamma(Q^p D_p\vec{u}, D_n\vec{u})_\gamma + \sqrt{-1}(Q^p D_p D_t\vec{u}, D_n\vec{u})_\gamma + \sqrt{-1}(D_n\vec{u}, Q^p D_p D_t\vec{u})_\gamma.$$

In the same manner, we have

$$\langle `D_t\vec{u}', Q^p`D_p\vec{u}'\rangle_\gamma$$
$$\cong -2\gamma(D_n\vec{u}, Q^p D_p\vec{u})_\gamma - \sqrt{-1}(D_n\vec{u}, Q^p D_p D_t\vec{u})_\gamma - \sqrt{-1}(Q^p D_p D_t\vec{u}, D_n\vec{u})_\gamma.$$

Combining these identities, we have the lemma,

LEMMA 5.3. *Assume that* (A.5.1)-(A.5.5) *are valid. Then, the following two estimates are valid.*

(i) *There exists a* $\gamma_0\geqq 1$ *depending only on* d_3, d_4, d_5 *and* $\boldsymbol{B}(1, R)$ *such that*

$$\gamma|\vec{u}|^2_{1,\gamma}\leqq C(d_3)\{\gamma^{-1}|\mathcal{P}[\vec{u}]|^2_{0,\gamma} + \langle `\mathcal{Q}[\vec{u}]'\rangle_{1/2,\gamma}\langle `D_t\vec{u}'\rangle_{-1/2,\gamma}\}$$

for any $\gamma\geqq\gamma_0$ *and* $\vec{u}\in\mathcal{H}^2_{\gamma,R}$.

(ii) *For any* $\gamma\geqq 1$ *and* $\vec{u}\in\mathcal{H}^2_{\gamma,R}$,

$$\langle `D_t\vec{u}'\rangle^2_{0,\gamma} + d_5\langle `D_n\vec{u}'\rangle^2_{0,\gamma}\leqq C(\boldsymbol{B}(0, R))\sum_{p=1}^{n-1}\langle `D_p\vec{u}'\rangle^2_{0,\gamma}$$
$$+C(\boldsymbol{B}(1, R))\gamma|\vec{u}|^2_{1,\gamma} + \gamma^{-1}|\mathcal{P}[\vec{u}]|^2_{0,\gamma}.$$

Proof. (i) By (5.2) and Lemma 5.2, we have

(5.6) $$2\gamma\{(D_t\vec{u}, D_t\vec{u})_\gamma + (P^{ij}D_j\vec{u}, D_i\vec{u})_\gamma - (Q^p D_p\vec{u}, D_n\vec{u})_\gamma - (D_n u, Q^p D_p\vec{u})_\gamma\}$$
$$+\langle (S^0 - `S^{n}')`D_t\vec{u}', `D_t u'\rangle_\gamma + \langle `D_t\vec{u}', (S^0 - `S^{n}')`D_t\vec{u}'\rangle_\gamma$$
$$\cong\sqrt{-1}\{(\mathcal{P}[\vec{u}], D_t\vec{u})_\gamma + (D_t\vec{u}, \mathcal{P}[\vec{u}])_\gamma\} + \langle `\mathcal{Q}[\vec{u}]', `D_t\vec{u}'\rangle_\gamma + \langle `D_t\vec{u}', `\mathcal{Q}[\vec{u}]'\rangle_\gamma.$$

Noting the fact that $e^{-2\gamma t}=-(2\gamma)^{-1}(d/dt)e^{-2\gamma t}$, by integration by parts we have that $|\vec{u}|_{0,\gamma}^2\leqq\gamma^{-1}|D_t\vec{u}|_{0,\gamma}|\vec{u}|_{0,\gamma}$, that is,

(5.7) $$|\vec{u}|_{0,\gamma}\leqq\gamma^{-1}|D_t\vec{u}|_{0,\gamma}.$$

Applying (5.7), the assumptions (A.5.3) and (A.5.4) to the left-hand side of (5.6), we have

$$2\min(d_3, 1/2)\gamma|\vec{u}|_{1,\gamma}^2\leqq C(\boldsymbol{B}(1, R))|\vec{u}|_{1,\gamma}^2+2|\mathscr{P}[\vec{u}]|_{0,\gamma}|\vec{u}|_{1,\gamma}$$
$$+2\langle`Q[\vec{u}]'\rangle_{1/2,\gamma}\langle`D_t\vec{u}'\rangle_{-1/2,\gamma},$$

provided that $1-(d_4+1)\gamma^{-1}\geqq\min(d_3, 1/2)$. From this we have (i). (ii) By (5.3) we have

$$\langle`D_t\vec{u}', `D_t\vec{u}'\rangle_\gamma+\langle`P^{nn}D_n\vec{u}', `D_n\vec{u}'\rangle_\gamma\leqq C(\boldsymbol{B}(0, R), \sigma)\sum_{p=1}^{n-1}\langle`D_p\vec{u}'\rangle_{0,\gamma}^2$$
$$+\sigma\langle`D_t\vec{u}'\rangle_{0,\gamma}^2+C(\boldsymbol{B}(0, R))\gamma|\vec{u}|_{2,\gamma}^1+\gamma^{-1}|\mathscr{P}[\vec{u}]|_{0,\gamma}^2+C(\boldsymbol{B}(1, R))|\vec{u}|_{1,\gamma}^2$$

for any $\sigma\in(0, 1)$. Using the assumption (A.5.5), we have (ii).

THEOREM 5.4. *Assume that*

(A.5.6) *the* P^{ij}, S^i *are matrices of real-valued functions in* $C^2([0, R]; \mathscr{B}^\infty(\bar{G}_R'))$ *and the* S^0, Q^p *are matrices of real-valued functions in* $\mathscr{B}^\infty(\bar{G}_R')$.

Assume that (A.5.2)-(A.5.5) *are valid, Let* $\gamma_0\geqq1$ *be the same constant as in Lemma* 5.3. *Let* $R'\in(0, R)$ *and* $\mu\in(0, 1)$. *Then, there exists a* $C=C(R, R', \mu, \boldsymbol{B}(1+\mu, R), d_3, d_4, d_5)>0$ *such that*

$$\gamma|\vec{u}|_{1,\gamma}^2+\langle`D_t\vec{u}'\rangle_{-1/2,\gamma}^2\leqq C\{\gamma^{-1}|\mathscr{P}[u]|_{1,\gamma}^2+\langle`Q[\vec{u}]'\rangle_{1/2,\gamma}^2\}$$

for any $\gamma\geqq\gamma_0$ *and* $\vec{u}\in\mathscr{H}_{\gamma,R'}^2$.

Proof. We shall use the same notations as in Appendix and regard x_n as a parameter. Choose $\phi_0(\tau, \xi', \gamma)\in C^\infty(\boldsymbol{R}^{n+1})$ so that $0\leqq\phi_0\leqq1$,

$$\phi_0(\tau, \xi', \gamma)=\begin{cases}1 & \text{for } \tau^2+\gamma^2+|\xi'|^2\geqq1,\\ 0 & \text{for } \tau^2+\gamma^2+|\xi'|^2\leqq1/2.\end{cases}$$

Let $\sigma\in(0, 1)$ be a small number determined later. Choose $\phi_1(\tau, \xi', \gamma)\in C^\infty(\boldsymbol{R}^{n+1}-\{(0, 0, 0)\})$ so that $0\leqq\phi_1\leqq1$,

(5.8) $$\phi_1(\tau, \xi', \gamma)=\begin{cases}1 & \text{for } 2\sigma(\tau^2+\gamma^2)\geqq|\xi'|^2,\\ 0 & \text{for } \sigma(\tau^2+\gamma^2)\leqq|\xi'|^2.\end{cases}$$

Let Φ_0, Φ_1 and Φ_2 be weighted pseudo-differential operators with symbols $1-\phi_0$, $\phi_0\phi_1$, $\phi_0(1-\phi_1)$ in the sense of Appendix, respectively. Take R_1, R_2 so that

$R'<R_1<R_2<R$ and choose $\chi_1(x),\ \chi_2(x)\in C^\infty$ so that

$$\chi_1(x)=\begin{cases}1 & \text{for } |x|\leqq R',\\ 0 & \text{for } |x|\geqq R_1,\end{cases}\qquad \chi_2(x)=\begin{cases}1 & \text{for } |x|\leqq R_2,\\ 0 & \text{for } |x|\geqq R.\end{cases}$$

Put $\Phi_l\vec{u}={}^t(\Phi_l u_1, \cdots, \Phi_l u_m)$. Note that $\chi_1\vec{u}=\vec{u}$ and that $\vec{u}=\sum\Phi_l\vec{u}$.

First, we shall evaluate $\langle D_t\Phi_1\vec{u}\rangle_{-1/2,\gamma}$. Noting that $P^{nn}(t, x)$ is non-singular for $(t, x)\in G_R$ as follows from (A.5.5), we have

$$\mathcal{P}[\chi_1\vec{u}]=\vec{f}_1 \quad \text{in } \boldsymbol{R}\times\boldsymbol{R}^n_+, \tag{5.9}$$

where

$$\begin{aligned}\vec{f}_1={}&\chi_1P^{nn}\Phi\{\chi_2(P^{nn})^{-1}\mathcal{P}[\vec{u}]\}+P^{nn}\{2(D_n\chi_1)\Phi_1D_n\vec{u}+(D_n^2\chi_1)\Phi_1\vec{u}\}\\ &+(P^{np}+P^{pn})\{(D_p\chi_1)D_n\Phi_1\vec{u}+(D_n\chi_1)D_p\Phi_1\vec{u}+(D_pD_n\chi_1)\Phi_1\vec{u}\}\\ &+P^{pq}\{(D_p\chi_1)D_q\Phi_1\vec{u}+(D_q\chi_1)D_p\Phi_1\vec{u}+(D_pD_q\chi_1)\Phi_1\vec{u}\}-2S^i(D_i\chi_1)D_t\Phi_1\vec{u}\\ &-\chi_1P^{nn}\{[\Phi_1D_p,\ \chi_2(P^{nn})^{-1}(P^{np}+P^{pn})]D_n\vec{u}+[\Phi_1D_p,\ \chi_2(P^{nn})^{-1}P^{pq}]D_q\vec{u}\\ &-2[\Phi_1D_t,\ \chi_2(P^{nn})^{-1}S^i]D_i\vec{u}-[\Phi_1D_t,\ \chi_2(P^{nn})^{-1}]D_t\vec{u}\}\\ &+\chi_1P^{nn}\Phi_1\{(D_p(\chi_2(P^{nn})^{-1}(P^{np}+P^{pn})))D_n\vec{u}+(D_p(\chi_2(P^{nn})^{-1}P^{pq}))D_q\vec{u}\\ &-2(D_t(\chi_2(P^{nn})^{-1}S^i))D_i\vec{u}-(D_t(\chi_2(P^{nn})^{-1}))D_t\vec{u}\}\,.\end{aligned}\tag{5.10}$$

Here, $[A, B]\vec{u}$ means the commutator of A and B, that is, $[A, B]\vec{u}=A(B\vec{u})-B(A\vec{u})$. Applying Theorem Ap. 5 with $s=1$ in Appendix below to (5.10), we have

$$|\vec{f}_1|_{0,\gamma}\leqq C|\mathcal{P}[\vec{u}]|_{0,\gamma}+C(\boldsymbol{B}(1+\mu,\ R))|\vec{u}|_{1,\gamma}. \tag{5.11}$$

By Lemma 5.3-(ii), (5.9) and (5.11), we have

$$\begin{aligned}\langle{}'D_t(\chi_1\Phi_1\vec{u})'\rangle^2_{0,\gamma}\leqq{}& C(\boldsymbol{B}(0,\ R))\sum_{p=1}^{n-1}\langle{}'D_p(\chi_1\Phi_1\vec{u})'\rangle^2_{0,\gamma}\\ &+(C(\boldsymbol{B}(1,\ R))\gamma+C(\boldsymbol{B}(1+\mu,\ R))\gamma^{-1})|\vec{u}|^2_{1,\gamma}+C\gamma^{-1}|\mathcal{P}[\vec{u}]|^2_{0,\gamma}.\end{aligned}\tag{5.12}$$

Since

$$\sum_{p=1}^{n-1}\langle{}'D_p(\chi_1\Phi_1\vec{u})'\rangle^2_{0,\gamma}\leqq C\sigma\langle{}'D_t(\chi_1\Phi_1\vec{u})'\rangle_{0,\gamma}+C|\vec{u}|^2_{1,\gamma} \tag{5.13}$$

as follows from (5.8), substituting (5.13) into (5.12) and taking $\sigma>0$ so small, we have

$$\langle{}'D_t(\chi_1\Phi_1\vec{u})'\rangle^2_{0,\gamma}\leqq C\{\gamma|\vec{u}\|^2_{1,\gamma}+\gamma^{-1}|\mathcal{P}[\vec{u}]|^2_{0,\gamma}\} \tag{5.14}$$

where $C=C(R,\ R',\ \mu,\ \boldsymbol{B}(1+\mu,\ R),\ d_3,\ d_4,\ d_5)$.

Now, we shall evaluate $\langle{}'\chi_1\langle D'\rangle^{-1/2}\Phi_2u'\rangle_{0,\gamma}$. Put

$$\psi(\tau, \xi', \gamma)=(\tau^2+|\xi'|^2+\gamma^2)^{1/4}(1+|\xi'|^2)^{-1/4}\phi_0(\tau, \xi', \gamma)(1-\phi_1(\tau, \xi', \gamma)),$$

and let Ψ be a weighted pseudo-differential operator with symbol ψ. By (5.8),

$$\psi\in S_\gamma^0, \quad \langle D'\rangle^{-1/2}\Phi_2\vec{u}=\Lambda_\gamma^{-1/2}\Psi\vec{u} \tag{5.15}$$

where $\Psi\vec{u}={}^t(\Psi\vec{u}_1, \cdots, \Psi\vec{u}_m)$ and $\Lambda_\gamma^{-1/2}$ is the weighted pseudo-differential operator with symbol $(\tau^2+|\xi'|^2+\gamma^2)^{-1/4}$. Using (5.15), we have

$$\mathcal{P}[\chi_1\langle D'\rangle^{-1/2}\Phi_2\vec{u}]=\vec{f}_2 \quad \text{in} \quad \boldsymbol{R}\times\boldsymbol{R}_+^n \tag{5.16}$$

where $\vec{f}_2$ is a function given by replacing Φ_1 by $\Lambda_\gamma^{-1/2}\Psi$ in (5.10). Applying Theorem Ap. 5 with $s=1/2$ to $\vec{f}_2$ implies that

$$|\vec{f}_2|_{0,\gamma}\leqq C|\mathcal{P}[\vec{u}]|_{0,\gamma}+C(\boldsymbol{B}(1, R))|\vec{u}|_{1,\gamma}. \tag{5.17}$$

By Lemma 5.3-(ii), (5.16) and (5.17), we have

$$\langle 'D_t\chi_1\langle D'\rangle^{-1/2}\Phi_2\vec{u}'\rangle_{0,\gamma}^2\leqq C\Big\{\sum_{p=1}^{n-1}\langle 'D_p\chi_1\langle D'\rangle^{-1/2}\Phi_2\vec{u}'\rangle_{1,\gamma}^2 \tag{5.18}$$
$$+\gamma|\vec{u}|_{1,\gamma}^2+\gamma^{-1}|\mathcal{P}[\vec{u}]|_{0,\gamma}^2\Big\},$$

where $C=C(R, R', \boldsymbol{B}(1, R), d_3, d_4, d_5)$. Noting (5.8), by Lemmas 3.1 and 3.2 we have

$$\sum_{p=1}^{n-1}\langle 'D_p\chi_1\langle D'\rangle^{-1/2}\Phi_2\vec{u}'\rangle_{0,\gamma}^2\leqq C\langle '\vec{u}'\rangle_{1/2,\gamma}^2\leqq C|\vec{u}|_{1,\gamma}^2.$$

Since $[\chi_1, \langle D'\rangle^{-1/2}]v=\langle D'\rangle^{-1/2}[\langle D'\rangle^{1/2}, \chi_1]\langle D'\rangle^{-1/2}v$, employing the same arguments as in the proof of Theorem Ap. 5 in Appendix below, by Theorems Ap. 1 and Ap. 2 (also in Appendix below) we have

$$\|[\chi_1, \langle D'\rangle^{-1/2}]v\|_{\boldsymbol{R}^{n-1}}\leqq C\|\langle D'\rangle^{-1/2}v\|_{\boldsymbol{R}^{n-1}}.$$

Consequently, we have

$$\langle 'D_t\chi_1\langle D'\rangle^{-1/2}\Phi_2\vec{u}'\rangle_{0,\gamma}\geqq\langle '\langle D'\rangle^{-1/2}D_t\chi_1\Phi_2\vec{u}'\rangle_{0,\gamma}-\langle '[\chi_1, \langle D'\rangle^{-1/2}]D_t\Phi_2\vec{u}'\rangle_{0,\gamma}$$
$$\geqq\langle 'D_t\chi_1\Phi_2\vec{u}'\rangle_{-1/2,\gamma}-C\langle '\langle D'\rangle^{-1/2}D_t\Phi_2\vec{u}'\rangle_{0,\gamma}.$$

Noting the fact that $|\xi'|^2\geqq 2\sigma(\tau^2+\gamma^2)$ on $\operatorname{supp}\phi_0(1-\phi_1)$, by Lemma 3.1 we have that $\langle '\langle D'\rangle^{-1/2}D_t\Phi_2\vec{u}'\rangle_{0,\gamma}\leqq C|\vec{u}|_{1,\gamma}$. Substituting these estimates into (5.18) implies that

$$\langle 'D_t(\chi_1\Phi_2\vec{u})'\rangle_{-1/2,\gamma}^2\leqq C\{\gamma|\vec{u}|_{1,\gamma}^2+\gamma^{-1}|\mathcal{P}[\vec{u}]|_{0,\gamma}^2\} \tag{5.19}$$

where $C=C(R, R', \boldsymbol{B}(1, R), d_3, d_4, d_5)$. Since $\operatorname{supp}(1-\phi_0)\subset\{(\tau, \xi', \gamma)\in\boldsymbol{R}^{n+1}|\tau^2+|\xi'|^2+\gamma^2\leqq 1\}$, we have

$$\langle 'D_t\chi_1\Phi_0\vec{u}'\rangle_{-1/2,\gamma}^2\leqq\langle '\vec{u}'\rangle_{0,\gamma}^2\leqq C|\vec{u}|_{1,\gamma}^2. \tag{5.20}$$

Noting that $D_t\vec{u}=\sum D_t(\chi_1\Phi_l u)$, by (5.14), (5.19) and (5.20) we have

$$\langle{}'D_t\vec{u}'\rangle^2_{-1/2,\gamma}\leqq C\{\gamma|\vec{u}|^2_{1,\gamma}+\gamma^{-1}|\mathscr{P}[\vec{u}]|^2_{0,\gamma}\}. \tag{5.21}$$

On the other hand, by Lemma 5.3-(i) we have

$$\gamma|\vec{u}|^2_{1,\gamma}\leqq C(d_3)\{\gamma^{-1}|\mathscr{P}[\vec{u}]|^2_{0,\gamma}+\langle{}'Q[\vec{u}]'\rangle_{1/2,\gamma}\langle{}'D_t\vec{u}'\rangle_{-1/2,\gamma}\} \tag{5.22}$$

for any $\gamma\leqq\gamma_0$. Substituting (5.21) into (5.22) and using Schwarz's inequality, we have

$$\gamma|\vec{u}|^2_{1,\gamma}\leqq C(d_3)\{\gamma^{-1}|\mathscr{P}[\vec{u}]|^2_{0,\gamma}+C\langle{}'Q[\vec{u}]'\rangle^2_{1/2,\gamma}\} \tag{5.23}$$

where $C=C(R, R', \mu, \boldsymbol{B}(1+\mu, R), d_3, d_4, d_5)$. Substituting (5.23) into (5.21) and combining (5.23) and the resulting inequalities, we have the theorem.

We would like to get the estimates of the same type as in Theorem 5.4 under the assumptions (A.5.1) instead of (A.5.6). To do this, we need to construct the approximation functions of the coefficients of $\mathscr{P}$ and Q.

LEMMA 5.5. *Let $\varepsilon>0$ be any number. If the coefficients P^{ij}, S^k and Q^p ($i, j=1, \cdots, n$, $k=0, 1, \cdots, n$ and $p=1, \cdots, n-1$) of the operators $\mathscr{P}$ and Q satisfy* (A.5.1)-(A.5.5), *then there exist $\sigma_0>0$ and $m\times m$ matrices P^{ij}_σ, S^k_σ and Q^p_σ having the following properties for any $\sigma\in(0, \sigma_0)$:*

(a) *The P^{ij}_σ, S^i_σ are matrices of real-valued functions in $C^2([0, R]; \mathscr{B}^\infty(\overline{G'_{R-\varepsilon}}))$ and the S^i_σ, Q^p_σ are matrices of real-valued functions in $\mathscr{B}^\infty(\overline{G'_{R-\varepsilon}})$.*

(b) (A.5.2) *are valid for the functions with subscript σ.*

(c) $$\int_{\boldsymbol{R}^n_+}P^{ij}_\sigma(t, x)D_j\vec{u}(x)\cdot\overline{D_i\vec{u}(x)}dx-\int_{\boldsymbol{R}^n_+}Q^p_\sigma(t, x')D_p\vec{u}(x)\cdot\overline{D_n\vec{u}(x)}dx$$
$$-\int_{\boldsymbol{R}^n_+}D_n\vec{u}(x)\cdot\overline{Q^p_\sigma(t, x')D_p\vec{u}(x)}dx\geqq(d_3-\varepsilon)\|\partial^1\vec{u}\|^2-d_4\|\vec{u}\|^2$$
for any $\vec{u}\in H^1(\boldsymbol{R}^n_+)$ such that $\vec{u}(x)=0$ for $x\notin\{x\in\boldsymbol{R}^n_+|0<x_n<R-\varepsilon, |x'|<R-\varepsilon\}$.

(d) *$(S^0_\sigma(t, x')-S^n_\sigma(t, x', 0))\vec{v}\cdot\vec{v}\geqq-\varepsilon|\vec{v}|^2$ for any $(t, x')\in G'_{R-\varepsilon}$ and constant vector $\vec{v}\in\boldsymbol{R}^m$.*

(e) *$P^{nn}_\sigma(t, x)\vec{v}\cdot\vec{v}\geqq(d_5-\varepsilon)|\vec{v}|^2$ for any $(t, x)\in G'_{R-\varepsilon}\times[0, R]$ and constant vector $\vec{v}\in\boldsymbol{R}^m$.*

(f) *If we put*
$$\boldsymbol{B}_\sigma(l, R-\varepsilon)=\Big\{\sum_{i,j=1}^n|P^{ij}_\sigma|_{\infty,l,G'_{R-\varepsilon}\times[0,R]}+\sum_{i=1}^n|S^i_\sigma|_{\infty,l,G'_{R-\varepsilon}\times[0,R]}$$
$$+|S^0_\sigma|_{\infty,l,G'_{R-\varepsilon}}+\sum_{p=1}^{n-1}|Q^p_\sigma|_{\infty,l,G'_{R-\varepsilon}}\Big\},$$
then $\boldsymbol{B}_\sigma(l, R-\varepsilon)\leqq\boldsymbol{B}(l, R)$ for $l\in[0, 2]$.

(g) $\lim_{\sigma\downarrow 0}|P_\sigma^{ij}-P^{ij}|_{\infty,0,G'_{R-\varepsilon}\times[0,R]}=0$, $\lim_{\sigma\downarrow 0}|S_\sigma^i-S^i|_{\infty,0,G'_{R-\varepsilon}\times[0,R]}=0$,

$\lim_{\sigma\downarrow 0}|Q_\sigma^p-Q^p|_{\infty,0,G'_{R-\varepsilon}}=0$, $\lim_{\sigma\downarrow 0}|S_\sigma^0-S^0|_{\infty,0,G'_{R-\varepsilon}}=0$.

Proof. Choose $\chi(t, x')\in C_0^\infty(\boldsymbol{R}^n)$ ($x'=(x_1, \cdots, x_{n-1})$) so that $\chi\geqq 0$, $\iint\chi(t, x')dtdx'=1$, $\operatorname{supp}\chi(t, x')\subset\{(t, x')\in\boldsymbol{R}^n\,|\,t^2+|x'|^2\leqq 1\}$. For any function $u=u(t, x', x_n)$, put

$$(u)_\sigma=\sigma^{-n}\int_{\boldsymbol{R}^n}\chi((t-s)/\sigma, (x'-y')/\sigma)u(s, y', x_n)dsdy'$$

(this operation is the usual convolution with respect to (t, x')). Put $P^{ij}=(P^i{}_a{}^{jb})$, $S^k=(S^k{}_a{}^b)$ and $Q^p=(Q^p{}_a{}^b)$ where the subscript a and superscript b denote the row and column, respectively. Let P_σ^{ij}, S_σ^k and Q_σ^p are $m\times m$ matrices whose (a, b) components are $(P^i{}_a{}^{jb})_\sigma$, $(S^k{}_a{}^b)_\sigma$ and $(Q^p{}_a{}^b)_\sigma$, respectively. Then, we see easily (f) and (g). Since

$$(S_\sigma^i(t, x')-S_\sigma^n(t, x', 0))\vec{v}\cdot\bar{\vec{v}}\geqq(S^0(t, x')-S^n(t, x', 0))\vec{v}\cdot\bar{\vec{v}}$$
$$-\{|S_\sigma^0-S^0|_{\infty,0,G'_{R-\varepsilon}}+|S_\sigma^n(\cdot, \cdot, 0)-S^n(\cdot, \cdot, 0)|_{\infty,0,G'_{R-\varepsilon}}\}|\vec{v}|^2,$$

by (A.5.4) and (g) we have (d). In the same manner, we have (c) by (A.5.3) and (g). Other assertions also follows immediately.

The following is the main result in this section.

THEOREM 5.6. *Assume that* (A.5.1)-(A.5.5) *are valid. Let* $R'\in(0, R)$ *and* $\mu\in(0, 1)$. *Then, there exist a* $\gamma_1\geqq 1$ *depending only on* d_3, d_4, d_5 *and* $\boldsymbol{B}(1, R)$ *and a* $C=C(R, R', \mu, \boldsymbol{B}(1+\mu, R), d_3, d_4, d_5)>0$ *such that*

$$\gamma|\vec{u}|_{1,\gamma}^2-\langle{}'D_t\vec{u}'\rangle_{-1/1,\gamma}^2\leqq C\{\gamma^{-1}|\mathcal{P}[\vec{u}]|_{0,\gamma}^2+\langle{}'\mathcal{Q}[\vec{u}]'\rangle_{1/2,\gamma}^2\}$$

for any $\gamma\geqq\gamma_1$ *and* $\vec{u}\in\mathcal{H}_{\gamma,R'}^2$.

Proof. Let $\varepsilon>0$ be any small positive number. Choose $R''>0$ so that $R'<R''<R$. Without loss of generality, we may assume that $R''<R-\varepsilon$, $d_3-\varepsilon\geqq d_3/2$, $d_4+\varepsilon\leqq 2d_4$ and $d_5-\varepsilon\geqq d_5/2$. Let P_σ^{ij}, S_σ^k and Q_σ^p be matrices given in Lemma 5.5. Put

$$\mathcal{P}_{\sigma,\varepsilon}[\vec{u}]=D_t^2\vec{u}+2S_\sigma^i(t, x)D_iD_t\vec{u}-P_\sigma^{ij}(t, x)D_iD_j\vec{u}-2\varepsilon\mathrm{I}_mD_nD_t\vec{u},$$
$$\mathcal{Q}_\sigma[u]=-P_\sigma^{nj}(t, x)D_j\vec{u}+Q_\sigma^p(t, x')D_p\vec{u}+S_\sigma^0(t, x')D_t\vec{u},$$

where I_m is the $m\times m$ identity matrix. By Lemma 5.5, the coefficients of $\mathcal{P}_{\sigma,\varepsilon}$ and $\mathcal{Q}_\sigma$ satisfy (A.5.2)-(A.5.6). Noting (f) in Lemma 5.5 and replacing R by R'' in Theorem 5.4, by Theorem 5.4 we have that there exist a $\gamma_1\geqq 1$ depending only on d_3, d_4, d_5 and $\boldsymbol{B}(1, R)$ and a $C=C(R'', R', \mu, \boldsymbol{B}(1+\mu, R''), d_3, d_4, d_5)$ such that

$$(5.24)\qquad \gamma|\vec{u}|_{1,\gamma}^2+\langle 'D_t\vec{u}'\rangle_{-1/2,\gamma}^2\leqq C\{\gamma^{-1}|\mathcal{P}_{\sigma,\varepsilon}[\vec{u}]|_{0,\gamma}^2+\langle 'Q_\sigma[\vec{u}]'\rangle_{1/2,\gamma}^2\}$$

for any $\gamma\geqq\gamma_1$ and $\vec{u}\in\mathcal{H}_{\gamma,R'}^2$. Since $\vec{u}\in\mathcal{H}_{\gamma,R'}^2$, letting $\sigma\downarrow 0$ in (5.24), by (g) in Lemma 5.5 we have

$$(5.25)\qquad \gamma|\vec{u}|_{1,\gamma}^2+\langle 'D_t\vec{u}'\rangle_{-1/2,\gamma}^2\leqq C\{\gamma^{-1}|\mathcal{P}[\vec{u}]|_{0,\gamma}^2+\langle 'Q[\vec{u}]'\rangle_{1/2,\gamma}^2+\varepsilon|\vec{u}|_{2,\gamma}^2\}.$$

Since $\vec{u}\in\mathcal{H}_{\gamma,R'}^2$, letting $\varepsilon\downarrow 0$ in (5.25), we have the theorem.

§6. A priori estimate with zero initial data.

First, using a partition of unity, we prove "a priori estimate" of the same type as in Theorem 5.6 for $\boldsymbol{R}\times\Omega$. To do this, we introduce some notations. For any integer $M\geqq 0$, $s\in\boldsymbol{R}$ and $\gamma\geqq 1$, put

$$(6.1)\qquad \mathcal{H}_\gamma^M(\boldsymbol{R}\times\Omega)=\Big\{\vec{u}={}^t(u_1,\cdots,u_m)\,|\,u_i\in H_{\mathrm{loc}}^M(\boldsymbol{R}\times\Omega),$$

$$|\vec{u}|_{M,\gamma,\Omega}^2=\int_{\boldsymbol{R}\times\Omega}e^{-2\gamma t}|\bar{D}^M\vec{u}(t,x)|^2dtdx<\infty\Big\},$$

$$(6.2)\qquad \langle\vec{u}\rangle_{s,\gamma,\Gamma}^2=\int_{\boldsymbol{R}}e^{-2\gamma t}\langle\vec{u}(t,\cdot)\rangle_s^2dt.$$

THEOREM 6.1. *Assume that* (A.1)-(A.6) *are valid. Then, there exist* $\gamma_2\geqq 1$ *and* $C>0$ *depending only on* δ_1, δ_2, Γ *and* $\mathcal{M}(1+\mu)$ *snch that*

$$\gamma|\vec{u}|_{1,\gamma,\Omega}^2+\langle\partial_t\vec{u}\rangle_{-1/2,\gamma,\Gamma}^2\leqq C\{\gamma^{-1}|P(\cdot)[\vec{u}]|_{0,\gamma,\Omega}^2+\langle Q(\cdot)[\vec{u}]\rangle_{1/2,\gamma,\Gamma}^2\}$$

for any $\gamma\geqq\gamma_2$ *and* $\vec{u}\in\mathcal{H}_\gamma^2(R\times\Omega)$.

Proof. Let ϕ_l be the same as in §2. Put $\phi_l\vec{u}=\vec{v}_l$,

$$P_0(\cdot)[\vec{u}]=\partial_t^2\vec{u}+2H^i\partial_i\partial_t\vec{u}-\partial_i(A^{ij}\partial_j\vec{u}),\qquad Q_0(\cdot)[\vec{u}]=\nu_iA^{ij}\partial_j\vec{u}+B^i\partial_i\vec{u}+H^0\partial_t\vec{u}.$$

We have

$$(6.3)\qquad P_0(\cdot)[\vec{v}_l]=\phi_lP(\cdot)[\vec{u}]+F_l\quad\text{in }\boldsymbol{R}\times\Omega,$$

$$Q_0(\cdot)[\vec{v}_l]=\phi_lQ(\cdot)[\vec{u}]+G_l\quad\text{on }\boldsymbol{R}\times\Gamma,$$

where

$$(6.4.a)\qquad \vec{F}_l=2H^i(\partial_i\phi_l)\partial_t\vec{u}-\partial_i(A^{ij}(\partial_j\phi_l)\vec{u})-(\partial_i\phi_l)A^{ij}\partial_j\vec{u}-\phi_l(A^j\partial_j\vec{u}+A^{n+1}\vec{u}),$$

$$(6.4.b)\qquad \vec{G}_l=\nu_iA^{ij}(\partial_j\phi_l)\vec{u}+B^i(\partial_i\phi_l)\vec{u}-\phi_lB^{n+1}\vec{u}.$$

First, we shall consider the case where $l=0$. Since ϕ_0 vanishes near Γ, by integration by parts we have

(6.5) $$\int_{-\infty}^{\infty} e^{-1\gamma t}(\phi_0 P(t)[\vec{u}(t,\ \cdot)]+F_0(t,\ \cdot),\ \partial_t\vec{v}_0(t,\ \cdot))dt$$

$$=\gamma\int_{-\infty}^{\infty} e^{-2\gamma t}\{\|\partial_t\vec{v}_0(t,\ \cdot)\|^2+(A^{ij}(t,\ \cdot)\partial_j\vec{v}_0(t,\ \cdot),\ \partial_i\vec{v}_0(t,\ \cdot))\}dt$$

$$-\frac{1}{2}\int_{-\infty}^{\infty} e^{-2\gamma t}(\partial_j H^j(t,\ \cdot)\partial_t\vec{v}_0(t,\ \cdot),\ \partial_t\vec{v}_0(t,\ \cdot))dt$$

$$-\frac{1}{2}\int_{-\infty}^{\infty} e^{-2\gamma t}(\partial_t A^{ij}(t,\ \cdot)\partial_j\vec{v}_0(t,\ \cdot),\ \partial_i\vec{v}_0(t,\ \cdot))dt.$$

In view of the assumption (A.4), we have

$$(A^{ij}(t,\ \cdot)\partial_j\vec{v}_0(t,\ \cdot),\ \partial_i\vec{v}_0(t,\ \cdot))\geqq 2\delta_1\|\partial^1\vec{v}_0\|^2-\delta_2\|\vec{v}_0\|^2.$$

Noting the fact: $|\vec{v}_0|_{0,\gamma,\Omega}\leqq\gamma^{-1}|\partial_t\vec{v}_0|_{0,\gamma,\Omega}$, by (6.4) and (6.5) we see

(6.6) $$\gamma|\vec{v}_0|_{1,\gamma,\Gamma}^2\leqq C\{\gamma^{-1}|P(\cdot)[\vec{u}]|_{0,\gamma,\Omega}^2+|\vec{u}|_{1,\gamma,\Omega}^2\}$$

for any $\gamma\geqq 2(1+\delta_2)$ where $C=C(\mathcal{M}(1),\ \delta)$.

Now, we shall treat the case where $1\leqq l\leqq N$. Using the change of variables, we shall reduce the problem to the half-space case. Let us use the notations defined in (3.6) and in § 2. Put

$$\vec{w}_l(t,\ y)=\vec{v}_l(t,\ \Psi_l(y)),\quad \vec{f}_l(t,\ x)=\phi_l(x)P(t)[\vec{u}(t,\ x)],\quad \vec{g}_l(t,\ x)=\phi_l(x)Q(t)[\vec{u}(t,\ x)],$$

$$\vec{h}_l(t,\ y)=\vec{f}_l(t,\ \Psi_l(y))+\vec{F}_l(t,\ \Psi_l(y))+A^{i'j'}(t,\ \Psi_l(y)(Y_{i'l}^i(y)(\partial_i' Y_{j'l}^j(y))\partial_j\vec{w}_l(t,\ y),$$

$$\vec{h}_l'(t,\ y')=\vec{g}_l(t,\ \Psi_l(y',\ 0))+\vec{G}_l(t,\ \Psi_l(y',\ 0)).$$

Noting (3.7.b) and (3.8), we can rewrite (6.3) as follows:

(6.7.a) $$\partial_t^2\vec{w}_l+2H^j(t,\ \Psi_l(y))Y_{jl}^i(y)\partial_t\partial_i'\vec{w}_l-A^{i'j'}(t,\ \Psi_l(y))Y_{i'l}^i(y)Y_{j'l}^j(y)\partial_i'\partial_j'\vec{w}_l$$
$$=\vec{h}_l(t,\ y)\quad\text{in } \boldsymbol{R}\times\boldsymbol{R}_+^n,$$

(6.7.b) $$-\{Y_{i'l}^n(y',\ 0)/J_l(y')\}A^{i'j'}(t,\ \Psi_l(y',\ 0))Y_{j'l}^j(y',\ 0)\partial_j'\vec{w}_l$$
$$+B^j(t,\ \Psi_l(y',\ 0))Y_{jl}^p(y',\ 0)\partial_p'\vec{w}_l+H^0(t,\ \Psi_l(y',\ 0))\partial_t\vec{w}_l$$
$$=\vec{h}_l'(t,\ y')\quad\text{on } \boldsymbol{R}\times\boldsymbol{R}_0^n.$$

Put

$$S^i(t,\ y)=H^j(t,\ \Psi_l(y))Y_{jl}^i(y),\qquad S^0(t,\ y')=H^0(t,\ \Psi_l(y',\ 0))J_l(y',\ 0),$$

$$P^{ij}(t,\ y)=A^{i'j'}(t,\ \Psi_l(y))Y_{i'l}^i(y)Y_{j'l}^j(y),$$

$$Q^p(t,\ y')=B^j(t,\ \Psi_l(y',\ 0))Y_{jl}^p(y',\ 0)J_l(y').$$

Choose $R'<\sigma_l$ so that $\operatorname{supp}\vec{w}_l\subset\{(t,\ y)|\,|y|<R'\}$ and put $R=\sigma_l$. Then, we can rewrite (6.3) in the form of (5.1). And then, $\vec{f}(t,\ y)=-\vec{h}_l(t,\ y)$ and $\vec{g}(t,\ y')=-\sqrt{-1}J_l(y')\vec{h}_l'(t,\ y)$. Noting that the Jacobian of the map: $x=\Psi_l(y)$, is equal

to 1, it follows immediately from (A.1), (A.3)-(A.6) that the assumptions (A.5.1) -(A.5.4) in § 5 are valid. Note that P^{ij} is equal to P_l^{ij} defined in (3.22). (3.26) implies that the assumption (A.5.5) in § 5 is also valid. Consequently, we can apply Theorem 5.6, and then the inequality:

$$\text{(6.8)} \qquad \gamma|\vec{w}_l|_{1,\gamma}^2+\langle{}'\partial_t\vec{w}_l{}'\rangle_{-1/2,\gamma}^2\leqq C\{\gamma^{-1}|\vec{h}_l|_{0,\gamma}^2+\langle\vec{h}_l'\rangle_{1/2,\gamma}^2\},$$

holds for any $\gamma\geqq\gamma_1$. Noting (2.2.c) and combining (6.6) and (6.8), we have the theorem under the suitable choice of γ_2.

Now, we introduce the operators:

$$P_\varepsilon(t)[\vec{u}]=P(t)[\vec{u}]+2\varepsilon\sum_{i=1}^n\nu_i(x)\partial_i\partial_t\vec{u}, \qquad 0<\varepsilon\leqq 1.$$

By the usual energy method, we can see that for $P_\varepsilon(t)$ (E.1) and (E.2) in § 4 are valid (cf. Lemma 6.4 below), and then the existence theorem of solutions to (N) is valid in the case where the operators are $P_\varepsilon(t)$, $0<\varepsilon\leqq 1$. And also, Theorem 6.1 is valid for $P_\varepsilon(t)$ as will be stated in Corollary 6.2 below. Using these facts, in §§ 6 and 7 we prove that the constants in the inequalities (E.1) and (E.2) are independent of ε. And then, we can prove the assertions 2° of Theorem 2.2 in § 7 below. To prove the existence theorem for the original operator $P(t)$, we consider the set $\{\vec{u}_\varepsilon\}_{0<\varepsilon<1}$ where each $\vec{u}_\varepsilon$ is a solution to the equations: $P_\varepsilon(t)[\vec{u}_\varepsilon]=\vec{f}$ in $[0, T]\times\Omega$, $Q(t)[\vec{u}_\varepsilon]=\vec{g}$ on $[0, T]\times\Gamma$, $\vec{u}_\varepsilon(0, x)=\vec{u}_0(x)$ and $\partial_t\vec{u}_\varepsilon(0, x)=\vec{u}_1(x)$ in Ω. By means of Theorem 2.2-2° and Lemma 4.2 we see that $\{\vec{u}_\varepsilon\}_{0<\varepsilon<1}$ is a bounded set in $H^2((0, T)\times\Omega)$. Hence, a weak limit $\vec{u}$ exists as $\varepsilon\downarrow 0$. Our task is to prove that $\vec{u}\in E^2([0, T])$. If we prove this, $\vec{u}$ becomes actually a solution of the original problem (N). In the final part of § 7, we shall prove this fact. The technical point lies in proving the right continuity of $\vec{u}$ at $t=0$. Theorem 2.2-2°-(c) is used essentially to prove it.

COROLLARY 6.2. *Assume that* (A.1)-(A.6) *are valid. Then, there exist* $\gamma_3\geqq 1$ *and* $C>0$ *depending only on* δ_1, δ_2, Γ *and* $\mathcal{M}(1+\mu)$ *such that*

$$\gamma|\vec{u}|_{1,\gamma,\Omega}^2+\langle\partial_t\vec{u}\rangle_{-1/2,\gamma,\Gamma}^2\leqq C\{\gamma^{-1}|P_\varepsilon(\cdot)[\vec{u}]|_{0,\gamma,\Omega}^2+\langle Q(\cdot)[\vec{u}]\rangle_{1/2,\gamma,\Gamma}^2\}$$

for any $\varepsilon\in(0, 1]$, $\gamma\in\gamma_3$ *and* $\vec{u}\in H_\gamma^2(\boldsymbol{R}\times\Omega)$.

To use the energy method, we need the following formula.

LEMMA 6.3. *Assume that* (A.1)-(A.6) *are valid. For any* $\lambda\geqq 0$, $t>0$, $\varepsilon\in[0, 1]$ *and* $\vec{u}$, $\vec{v}\in E^2([0, t])$, *the following identity is valid:*

$$\frac{d}{ds}\{(\partial_s\vec{u}(s,\cdot),\ \partial_s\vec{v}(s,\cdot))+(A^{ij}(s,\cdot)\partial_j\vec{u}(s,\cdot),\ \partial_i\vec{v}(s,\cdot))+\lambda(\vec{u}(s,\cdot),\ \vec{v}(s,\cdot))$$
$$+\mathscr{B}_1(s,\ \vec{u}(s,\cdot),\ \vec{v}(s,\cdot))+\mathscr{B}_2(s,\ \vec{u}(s,\cdot),\ \vec{v}(s,\cdot))+\mathscr{B}_2(s,\ \vec{v}(s,\cdot),\ \vec{u}(s,\cdot))$$
$$+\langle(\nu_i(\cdot)H^i(s,\cdot)+\varepsilon I_m+H^0(s,\cdot))\partial_s\vec{u}(s,\cdot),\ \partial_s\vec{v}(s,\cdot)\rangle$$
$$+\langle\partial_s\vec{u}(s,\cdot),\ (\nu_i(\cdot)H^i(s,\cdot)+\varepsilon I_m+H^0(s,\cdot))\partial_s\vec{v}(s,\cdot)\rangle$$
$$\equiv(P_\varepsilon(s)[\vec{u}(s,\cdot)],\ \partial_s\vec{v}(s,\cdot))+(\partial_s\vec{u}(s,\cdot),\ P_\varepsilon(s)[\vec{v}(s,\cdot)])$$
$$+\langle Q(s)[\vec{u}(s,\cdot)],\ \partial_s\vec{v}(s,\cdot)\rangle+\langle\partial_s\vec{u}(s,.),\ Q(s)[\vec{v}(s,\cdot)]\rangle$$

for any $s\in(0,t)$. *Here,* $P_0(t)=P(t)$ *and* $A\equiv B$ *means that*

$$|A-B|\leqq C(\lambda,\ \mathscr{M}(1))\|\bar{D}^1\vec{u}(s,\cdot)\|\|\bar{D}^1\vec{v}(s,\cdot)\|$$

Proof. By (3.9), (3.11.a), (3.12) and (3.13), we have

(6.9)
$$\langle B^i(s,\cdot)\partial_i\vec{u}(s,\cdot),\ \partial_s\vec{v}(s,\cdot)\rangle+\langle\partial_s\vec{u}(s,\cdot),\ B^i(s,\cdot)\partial_i\vec{v}(s,\cdot)\rangle$$
$$\equiv\frac{d}{ds}\mathscr{B}_1(s,\ \vec{u}(s,\cdot),\ \vec{v}(s,\cdot)).$$

By (3.14)-(3.16) we have also

(6.10)
$$\langle B(s,\cdot)\vec{u}(s,\cdot),\ \partial_s\vec{v}(s,\cdot)\rangle+\langle\partial_s\vec{u}(s,\cdot),\ B(s,\cdot)\vec{v}(s,\cdot)\rangle$$
$$\equiv\frac{d}{ds}\{\mathscr{B}_2(s,\ \vec{u}(s,\cdot),\ \vec{v}(s,\cdot))+\mathscr{B}_2(s,\ \vec{v}(s,\cdot),\ \vec{u}(s,\cdot))\}.$$

By integration by parts and the assumption: ${}^tA^{ij}=A^{ji}$, we have

(6.11)
$$\frac{d}{ds}(A^{ij}(s,\cdot)\partial_j\vec{u}(s,\cdot),\ \partial_i\vec{v}(s,\cdot))$$
$$\equiv\langle\nu_i(\cdot)A^{ij}(s,\cdot)\partial_j\vec{u}(s,\cdot),\ \partial_s\vec{v}(s,\cdot)\rangle-(\partial_i(A^{ij}(s,\cdot)\partial_j\vec{u}(s,\cdot)),\ \partial_s\vec{v}(s,\cdot))$$
$$+\langle\partial_s\vec{u}(s,\cdot),\ \nu_i(\cdot)A^{ij}(s,\cdot)\partial_j\vec{v}(s,\cdot)\rangle-(\partial_s\vec{u}(s,\cdot),\ \partial_i(A^{ij}(s,\cdot)\partial_j\vec{v}(s,\cdot))).$$

Combining (6.9)-(6.11) and using the definitions of $P_\varepsilon(t)$ and $Q(t)$, we can immediately obtain the lemma.

Using Lemma 6.3, we have the following lemma.

LEMMA6.4. *Assume that the assumptions* (A.1)-(A.6) *are valid and that* $0<\varepsilon\leqq 1$. *Then, the following two estimates are valid.*

(i) *There exists a* $C=C(\lambda_2,\ \mathscr{M}(1),\ \delta_1)>0$ *such that*

$$\|\partial_t\vec{u}(t,\cdot)\|^2+\|\vec{u}(t,\cdot)\|^2_{\mathfrak{Z}(t)}+\varepsilon\int_0^t\langle\partial_s\vec{u}(s,\cdot)\rangle_0^2 ds$$

$$\leqq e^{Ct}\Big\{\|\partial_t\vec{u}(0,\ \cdot)\|^2+\|\vec{u}(0,\ \cdot)\|^2_{\mathcal{J}(0)}$$

$$+\int_0^t\{\|P_\varepsilon(s)[\vec{u}(s,\ \cdot)]\|^2+\varepsilon^{-1}\langle Q(s)[\vec{u}(s,\ \cdot)]\rangle^2_{1/2}\}ds\Big\},\qquad 0\leqq t\leqq T,$$

for any $\vec{u}\in E^2([0,\ T])$ *and* $T>0$.

(ii) *There exists a* $C=C(\lambda_2,\ \mathcal{M}(1),\ \delta_1)>0$ *such that*

$$\|\partial_t\vec{u}(t,\ \cdot)\|^2+\|\vec{u}(t,\ \cdot)\|^2_{\mathcal{J}(t_0)}\leqq(\exp C(t-t_1))\Big[\|\partial_t\vec{u}(t_1,\ \cdot)\|^2$$

$$+\|\vec{u}(t_1,\ \cdot)\|^2_{\mathcal{J}(t_0)}+\int_{t_1}^t\{\|P_\varepsilon(t_0)[\vec{u}(s,\ \cdot)]\|^2+\varepsilon^{-1}\langle Q(t_0)[\vec{u}(s,\ \cdot)]\rangle^2_{1/2}\rangle\}ds\Big]$$

for $t\in[t_1,\ t_2]$ *and* $\vec{u}\in E^2([t_1,\ t_2])$, *where* $t_0,\ t_1,\ t_2$ *are any numbers* $\in[0,\ T]$ *such that* $t_1<t_2$.

Proof. (i) Putting $\vec{u}=\vec{v}$ in Lemma 6.3 and integrating the resulting identity, we have

(6.12)
$$\|\partial_t\vec{u}(t,\ \cdot)\|^2+\|\vec{u}(t,\ \cdot)\|^2_{\mathcal{J}(t)}+2\varepsilon\int_0^t\langle\partial_s\vec{u}(s,\ \cdot)\rangle^2_0ds$$

$$\leqq\|\partial_t\vec{u}(0,\ \cdot)\|^2+\|\vec{u}(0,\ \cdot)\|^2_{\mathcal{J}(0)}+\int_0^t\|P_\varepsilon(s)[\vec{u}(s,\ \cdot)]\|^2ds$$

$$+2\int_0^t|\langle Q(s)[\vec{u}(s,\ \cdot)],\ \partial_s\vec{u}(s,\ \cdot)\rangle|ds$$

$$+(\min(1,\ \delta_1))^{-1}C(\lambda_2,\ \mathcal{M}(1))\int_0^t\{\|\partial_s\vec{u}(s,\ \cdot)\|^2+\|\vec{u}(s,\ \cdot)\|^2_{\mathcal{J}(s)}\}ds.$$

Here, we have used (3.31.b). Since by Schwarz's inequality we have

(6.13)
$$2\int_0^t|\langle Q(s)[\vec{u}(s,\ \cdot)],\ \partial_s\vec{u}(s,\ \cdot)\rangle|ds$$

$$\leqq\varepsilon\int_0^t\langle\partial_s\vec{u}(s,\ \cdot)\rangle^2_0ds+\varepsilon^{-1}\int_0^t\langle Q(s)[\vec{u}(s,\ \cdot)]\rangle^2_0ds,$$

substituting (6.13) into (6.12) and applying Gronwall's inequality to the resulting inequality, we have (i).

(ii) Since the coefficients of $P_\varepsilon(t_0)$ and $Q(t_0)$ are independent of t, an obvious modification of the proof of Lemma 6.3 yields that

(6.14)
$$\frac{d}{ds}\{\|\partial_s\vec{u}(s,\ \cdot)\|^2+\|\vec{u}(s,\ \cdot)\|^2_{\mathcal{J}(t_0)}\}+2\varepsilon\langle\partial_s\vec{u}(s,\ \cdot)\rangle^2_0$$

$$\leqq\|P_\varepsilon(t_0)[\vec{u}(s,\ \cdot)]\|^2+\varepsilon\langle\partial_s\vec{u}(s,\ \cdot)\rangle^2_0+\varepsilon^{-1}\langle Q(s)[\vec{u}(s,\ \cdot)]\rangle^2_0$$

$$+C(\lambda_2,\ \mathcal{M}(1),\ \delta_1)\{\|\partial_s\vec{u}(s,\ \cdot)\|^2+\|\vec{u}(s,\ \cdot)\|^2_{\mathcal{J}(t_0)}\}.$$

Here, we have used Schwarz's inequality and (3.31.b). Integrating (6.14) from t_1

to t and applying Gronwall's inequality to the resulting inequality, we have (ii).

In view of Lemma 6.4, (E.1) and (E.2) in §4 hold for $P_\varepsilon(t)$ $(0<\varepsilon\leqq 1)$. Thus, by Theorem 4.5 we have the following existence theorem for P_ε and Q.

LEMMA 6.5. *Assume that* (A.1)-(A.6) *are valid and that* $0<\varepsilon\leqq 1$. *Let* $T>0$ *be any positive number. If* $\vec{u}_0\in H^2(\Omega)$, $\vec{u}_1\in H^1(\Omega)$, $\vec{f}\in C^1([0, T]; L^2(\Omega))$, $\vec{g}\in C^1([0, T]; H^{1/2}(\Gamma))$ *and* (4.41) *is valid, then there exists a unique solution* $\vec{u}\in E^2([0, T])$ *to the equations*:

$$(6.15.a)\qquad P_\varepsilon(t)[\vec{u}]=\vec{f}\quad in\ [0, T]\times\Omega,\qquad Q(t)[\vec{u}]=\vec{g}\quad on\ [0, T]\times\Gamma,$$

$$(6.15.b)\qquad \vec{u}(0, x)=\vec{u}_0(x),\qquad \partial_t\vec{u}(0, x)=\vec{u}_1(x)\quad in\ \Omega.$$

In addition, we assume that $\vec{f}\in C^1([0, \infty); L^2(\Omega))$, $\vec{g}\in C^1([0, \infty); H^{1/2}(\Gamma))$ *and that* $\vec{f}$ *and* $\vec{g}$ *vanish for* $t>T_1$ *with some* $T_1>0$. *Put* $T_2=\max(T_0, T_1)$ (*cf.* (A.2)). *Then, there exists a unique* $\vec{u}\in E^2([0, \infty))$ *satisfying* (6.15) *for any* $T>0$ *and the estimate*:

$$\|\bar{D}^2\vec{u}(t, \cdot)\|^2\leqq C(\exp(t-T_2))\|\bar{D}^2\vec{u}(T_2, \cdot)\|^2$$

for any $t>T_2$ *with some* $C=C(\Gamma)>0$,

Proof. By Lemma 6.4 and Theorem 4.5, the first assertion can be seen immediately. If $\vec{f}\in C^1([0, \infty); L^2(\Omega))$ and $\vec{g}\in C^1([0, \infty); H^{1/2}(\Gamma))$, for any $T>0$ (6.15) admits a unique solution $\vec{u}_T\in E^2([0, T])$. By the uniqueness we know that $\vec{u}_T(t, x)=\vec{u}_{T'}(t, x)$ for $0\leqq t\leqq T$ provided that $T'>T$. So, if we put $\vec{u}(t, x)=\vec{u}_T(t, x)$ for $t\in[0, T]$, then $\vec{u}$ is well-defined, satisfies (6.15) for any $T>0$ and $\vec{u}\in E^2([0, \infty))$. By (A.2) we have

$$(6.16)\qquad \partial_t^2\vec{u}-\sum_{i=1}^n\partial_i^2\vec{u}=0\quad \text{in } [T_2, \infty)\times\Omega,\qquad \sum_{i=1}^n\nu_i(x)\partial_i\vec{u}=0\quad \text{on } [T_2, \infty)\times\Gamma.$$

Multiplying the first equality of (6.16) by $\partial_t\vec{u}$ and integrating the resulting formula, by Gronwall's inequality we have

$$(6.17)\qquad \|\bar{D}^1\vec{u}(t, \cdot)\|^2\leqq(\exp(t-T_2))\|\bar{D}^1\vec{u}(T_2, \cdot)\|^2\quad \text{for}\quad t>T_2.$$

Employing the same arguments as in the proof of Lemma 4.1, by (6.17) we have

$$(6.18)\qquad \|\bar{D}^1\partial_t\vec{u}(t, \cdot)\|^2\leqq(\exp(t-T_2))\|\bar{D}^1\partial_t\vec{u}(T_2, \cdot)\|^2\quad \text{for}\quad t>T_2.$$

Finally, regarding t as a parameter and applying Theorem 3.4, we have

$$(6.19)\qquad \|\bar{\partial}^2\vec{u}(t, \cdot)\|^2\leqq C\{\|\partial_t^2\vec{u}(t, \cdot)\|^2+\|\bar{D}^1\vec{u}(t, \cdot)\|^2\}.$$

Combining (6.17)-(6.19), we have the lemma.

Under these preparations, we shall prove the following key lemma.

LEMMA 6.6. *Assume that* (A.1)-(A.6) *are valid and that* $0<\varepsilon\leqq 1$. *Then, there exists a* $C=C(\delta_1, \delta_2, \mathcal{M}(1+\mu), \Gamma)>0$ *such that*

$$(6.20)\qquad \int_0^t \{\|\bar{D}^1\vec{u}(s,\cdot)\|^2+\langle\partial_s\vec{u}(s,\cdot)\rangle_{-1/2}^2\}ds$$

$$\leqq Ce^{Ct}\int_0^t \{\|P_\varepsilon(s)[\vec{u}(s,\cdot)]\|^2+\langle Q(s)[\vec{u}(s,\cdot)]\rangle_{1/2}^2\}ds,\qquad 0\leqq t\leqq T,$$

for any $T>0$ *and* $\vec{u}\in C^\infty([0, T]; H^2(\Omega))$ *satisfying the conditions*: $\partial_t^k\vec{u}(0, x)=0$ *in* Ω *for any* $k\geqq 0$. *Here, C is a constant independent of* ε.

Proof. Let t_0 be any time in $[0, T]$ and fixed. Put

$$\vec{f}(s, x)=\begin{cases} P_\varepsilon(s)[\vec{u}(s, x)] & \text{for } 0\leqq s\leqq t_0,\\ 0 & \text{for } s<0,\end{cases}$$

$$\vec{g}(s, x)=\begin{cases} Q(s)[\vec{u}(s, x)] & \text{for } 0\leqq s\leqq t_0,\\ 0 & \text{for } s<0.\end{cases}$$

By the assumption for $\vec{u}$ we know that $\vec{f}\in C^1((-\infty, t_0]; L^2(\Omega))$ and $\vec{g}\in C^1((-\infty, t_0]; H^{1/2}(\Gamma))$. Let $a_0=1$ and $a_1=2$, and choose b_0, b_1 so that $b_0(-a_0)^k+b_1(-a_1)^k=1$ for $k=0$ and 1, i.e., $b_0=3$ and $b_1=-2$. Put

$$(6.21.\text{a})\qquad F(t, x)=\begin{cases} \vec{f}(t, x) & \text{for } t\leqq t_0,\\ \sum_{l=0}^{1} b_l\vec{f}(t_0-a_l(t-t_0), x) & \text{for } t>t_0,\end{cases}$$

$$6.21.\text{b})\qquad G(t, x)=\begin{cases} \vec{g}(t, x) & \text{for } t\leqq t_0,\\ \sum_{l=0}^{1} b_l\vec{g}(t_0-a_l(t-t_0), x) & \text{for } t>t_0.\end{cases}$$

Then, we see easily that

$$(6.22.\text{a})\qquad F\in C^1(R; L^2(\Omega)),\qquad G\in C^1(R; H^{1/2}(\Gamma)),$$

$$(6.22.\text{b})\qquad F(t, x)=P_\varepsilon(t)[\vec{u}(t, x)] \text{ and } G(t, x)=Q(t)[\vec{u}(t, x)] \text{ for } 0\leqq t\leqq t_0,$$

$$(6.22.\text{c})\qquad F(t, x)=0 \text{ and } G(t, x)=0 \text{ for } t\leqq 0 \text{ or } t\geqq 2t_0.$$

Let $\vec{v}\in E^2([0, \infty))$ be a solution of the equations:

$$(6.23.\text{a})\qquad P_\varepsilon(t)[\vec{v}]=F \text{ in } [0, \infty)\times\Omega,\qquad Q(t)[\vec{v}]=G \text{ on } [0, \infty)\times\Gamma,$$

$$(6.23.\text{b})\qquad \vec{v}(0, x)=\partial_t\vec{v}(0, x)=0 \text{ in } \Omega.$$

In view of (6.22.c), the existence of $\vec{v}$ is assured by Lemma 6.5. Noting also (6.22.c), by Lemma 6.5 we know that

$$(6.24)\qquad \|\bar{D}^2\vec{v}(t,\cdot)\|^2\leqq C(\exp(t-T_2))\|\bar{D}^2\vec{v}(T_2,\cdot)\|^2\qquad \text{for } t>T_2$$

where $T_2=\max(2t_0, T_0)$. Put $\vec{v}_0(t, x)=\vec{v}(t, x)$ for $t\geqq 0$ and $=0$ for $t<0$. By (6.23.b) and the fact that $F(0, x)=0$ in Ω, we see that $\partial_t^2\vec{v}(0, x)=0$ in Ω, and then $\vec{v}_0\in E^2(\boldsymbol{R})$. By (6.24) we know that $\vec{v}_0\in\mathcal{H}_\gamma^2(\boldsymbol{R}\times\Omega)$ for any $\gamma>1$. Since $P_\varepsilon(t)[\vec{v}_0]=F$ in $\boldsymbol{R}\times\Omega$ and $Q(t)[\vec{v}_0]=G$ on $\boldsymbol{R}\times\Gamma$ as follows from (6.22.c) and (6.23.a), by Corollary 6.2 we have

$$\gamma|\vec{v}_0|_{1,\gamma,\Omega}^2+\langle\partial_t\vec{v}_0\rangle_{-1/2,\gamma,\Gamma}^2\leqq C\{\gamma^{-1}|F|_{0,\gamma,\Omega}^2+\langle G\rangle_{1/2,\gamma,\Gamma}^2\} \tag{6.25}$$

for any $\gamma\geqq\gamma_3$. Noting that F and G vanish for $t<0$, by (6.21) we have

$$|F|_{0,\gamma,\Omega}^2\leqq 19\int_0^{t_0}\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2ds, \qquad \langle G\rangle_{0,\gamma,\Gamma}^2\leqq 19\int_0^{t_0}\langle Q(s)[\vec{u}(s, \cdot)]\rangle_{1/2}^2ds.$$

On the other hand, by (6.22.b) and the uniqueness of solutions for (6.15) we have that $\vec{v}_0(t, x)=\vec{u}(t, x)$ for $0\leqq t\leqq t_0$. Therefore, the lemma follows from (6.25).

To prove the estimates of the same type as in Lemma 6.6 for any $\vec{u}\in E^2([0, T])$ satisfying the conditions: $\vec{u}(0, x)=\partial_t\vec{u}(0, x)=0$ in Ω, we need the following lemma about the approximation of $\vec{u}$.

LEMMA 6.7. *Let $\vec{u}\in E^2([0, T])$ such that $\vec{u}(0, x)=\partial_t\vec{u}(0, x)=0$ in Ω. Then, there exist $\vec{u}^k\in C^\infty([0, T]; H^2(\Omega))$, $k=1, 2, \cdots$, satisfying the following properties:*

(a) $\partial_t^l\vec{u}^k(0, x)=0$ *for any* $k\geqq 1$ *and* $l\geqq 0$,

(b) $\|\bar{D}^1(\vec{u}^k(t, \cdot)-\vec{u}(t, \cdot))\|\to 0$ *as* $k\to\infty$ *for any* $t\in[0, T]$,

(c) $\int_0^T\|\bar{D}^2(\vec{u}^k(t, \cdot)-\vec{u}(t, \cdot))\|^2dt\to 0$ *as* $k\to\infty$.

Proof. By Lions' method we see that there exists a $\vec{v}\in E^2(\boldsymbol{R})$ such that $\vec{u}(t, x)=\vec{v}(t, x)$ for $0\leqq t\leqq T$. Without loss of generality, we may assume that $\vec{v}$ vanishes for $t<-T$ and $t>2T$. Choose $\rho(t)\in C_0^\infty(\boldsymbol{R})$ so that $\operatorname{supp}\rho\subset[0, 1]$ and $\int_{\boldsymbol{R}}\rho(s)ds=1$. Put

$$u_a^k(t, x)=k\int_0^\infty\rho(k(t-s))v_a(s, x)ds \quad\text{and}\quad \vec{u}^k={}^t(u_1^k, \cdots, u_m^k)$$

where $\vec{v}={}^t(v_1, \cdots, v_m)$. Since $\operatorname{supp}\rho\subset[0, 1]$, (a) follows immediately.

Since $\vec{v}(0, x)=\partial_t\vec{v}(0, x)=0$ in Ω as follows from the assumptions: $\vec{u}(0, x)=\partial_t\vec{u}(0, x)=0$ in Ω, by integration by parts we have

$$\partial_t^l\partial_x^\alpha u_a^k(t, x)=k\int_0^\infty\rho(k(t-s))(\partial_s^l\partial_y^\alpha v_a)(s, y)dsdy$$

for any l and α such that $l+|\alpha|\leqq 2$. If we put $v_a^{l\alpha}(t, x)=\partial_s^l\partial_x^\alpha v_a(t, x)$ for $t\geqq 0$ and $=0$ for $t<0$ and set $\vec{v}^{l\alpha}={}^t(v_1^{l\alpha}, \cdots, v_m^{l\alpha})$, then we see that $\vec{v}^{l\alpha}\in E^0(\boldsymbol{R})$ for $l+|\alpha|\leqq 1$ and that $v^{l\alpha}\in L^2(\boldsymbol{R}\times\Omega)$ for $l+|\alpha|\leqq 2$. Since $\partial_t^l\partial_x^\alpha\vec{u}(t, x)=\vec{v}^{l\alpha}(t, x)$ for

$0 \leqq t \leqq T$ and

$$\partial_t^l \partial_x^\alpha u_a^k(t, x) = k \int_{-\infty}^{\infty} \rho(k(t-s)) v_a^{l\alpha}(s, x) ds, \qquad a = 1, \cdots, m,$$

(b) and (c) follow immediately.

The following is the main result in this section.

THEOREM 6.8. *Assume that* (A.1)-(A.6) *are varid and that* $0 \leqq \varepsilon \leqq 1$. *Then, for any* $T > 0$ *there exists a* $C = C(\delta_1, \delta_2, \Gamma, \mathcal{M}(1+\mu)) > 0$ *such that*

(6.26) $$\|\bar{D}^1 \vec{u}(t, \cdot)\|^2 + \int_0^t \{\langle \partial_s \vec{u}(s, \cdot)\rangle_{-1/2}^2 + \|\bar{D}^1 \vec{u}(s, \cdot)\|^2\} ds$$

$$\leqq C e^{Ct} \int_0^t \{\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2 + \langle Q(s)[\vec{u}(s, \cdot)]\rangle_{1/2}^2\} ds, \qquad 0 \leqq t \leqq T,$$

for any $\vec{u} \in E^2([0, T])$ *satisfying the conditions*: $\vec{u}(0, x) = \partial_t \vec{u}(0, x) = 0$ *in* Ω.

Proof. When $\varepsilon \in (0, 1]$, using Lemmas 6.6 and 6.7, we see that

(6.27) $$\int_0^t \{\langle \partial_s \vec{u}(s, \cdot)\rangle_{-1/2}^2 + \|\bar{D}^1 \vec{u}(s, \cdot)\|^2\} ds$$

$$\leqq C e^{Ct} \int_0^t \{\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2 + \langle Q(s)[\vec{u}(s, \cdot)]\rangle_{1/2}^2\} ds, \qquad 0 \leqq t \leqq T,$$

for any $\vec{u} \in E^2([0, T])$ satisfying the conditions: $\vec{u}(0, x) = \partial_t \vec{u}(0, x) = 0$ in Ω. Since $\vec{u} \in E^2([0, T])$ and the constant C in (6.27) independent of ε, letting $\varepsilon \downarrow 0$ in (6.27), we have that (6.27) is also valid for $\varepsilon = 0$. Putting $\vec{u} = \vec{v}$ and $\lambda = \lambda_2$ in Lemma 6.3 and integrating the resulting identity from 0 to t, we have

(6.28) $$\|\partial_t \vec{u}(t, \cdot)\|^2 + \|\vec{u}(t, \cdot)\|_{\mathcal{G}(t)}^2 \leqq \|\partial_t \vec{u}(0, \cdot)\|^2 + \|\vec{u}(0, \cdot)\|_{\mathcal{G}(0)}^2$$

$$+ 2\int_0^t |(P_\varepsilon(s)[\vec{u}(s, \cdot)], \partial_s \vec{u}(s, \cdot))| ds + 2\int_0^t |\langle Q(s)[\vec{u}(s, \cdot)], \partial_s \vec{u}(s, \cdot)\rangle| ds$$

$$+ C(\lambda_2, \mathcal{M}(1)) \int_0^t \|\bar{D}^1 \vec{u}(s, \cdot)\|^2 ds \qquad (0 \leqq \varepsilon \leqq 1).$$

Noting the assumptions: $\vec{u}(0, x) = \partial_t \vec{u}(0, x) = 0$ in Ω and using Lemma 3.1 and Schwarz's inequality, from (6.28) we obtain

(6.29) $$\|\partial_t \vec{u}(t, \cdot)\|^2 + \|\vec{u}(t, \cdot)\|_{\mathcal{G}(t)}^2 \leqq \int_0^t \{\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2$$

$$+ \langle Q(s)[\vec{u}(s, \cdot)]\rangle_{1/2}^2\} ds + C\int_0^t \{\|\bar{D}^1 \vec{u}(s, \cdot)\|^2 + \langle \partial_s \vec{u}(s, \cdot)\rangle_{-1/2}^2\} ds$$

where $C = C(\lambda_2, \mathcal{M}(1))$. Substituting (6.27) into (6.29) and noting (3.31.b), we have the theorem.

§7. Proof of main results.

First, we shall prove the estimate (a) of Theorem 2.2-2°.

LEMMA 7.1. *Let* $T>0$. *Assume that* (A.1)-(A.6) *are valid. Then, there exist constants* $C_1=C(\delta_1, \delta_2, \mathcal{M}(1), \Gamma)$ *and* $C_2=C(\delta_1, \delta_2, \mathcal{M}(1+\mu), \Gamma)$ *such that*

$$(7.1)\qquad \|\partial_t\vec{u}(t, \cdot)\|^2+\|\vec{u}(t, \cdot)\|^2_{\mathcal{J}(t)}\leqq 2(\exp C_1t)\{\|\partial_t\vec{u}(0, \cdot)\|^2+\|\vec{u}(0, \cdot)\|^2_{\mathcal{J}(0)} +\int_0^t\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2ds\}+C_2(\exp C_2t)\int_0^t\langle Q(s)[\vec{u}(s, \cdot)]\rangle^2_{1/2}ds$$

for any $t\in[0, T]$, $\vec{u}\in E^2([0, T])$ *and* $0\leqq\varepsilon\leqq 1$.

Proof. First, we assume that

$$(7.2)\qquad Q(t)[\vec{u}]=0 \quad \text{on} \quad [0, T]\times\Gamma.$$

By (6.28) we have

$$(7.3)\qquad \|\partial_t\vec{u}(t, \cdot)\|^2+\|\vec{u}(t, \cdot)\|^2_{\mathcal{J}(t)}\leqq\|\partial_t\vec{u}(0, \cdot)\|^2+\|\vec{u}(0, \cdot)\|^2_{\mathcal{J}(0)} +2\int_0^t|(P_\varepsilon(s)[\vec{u}(s, \cdot)], \partial_s\vec{u}(s, \cdot))|ds+C(\mathcal{M}(1), \lambda_2)\int_0^t\|\bar{D}^1\vec{u}(s, \cdot)\|^2ds$$

for any $\vec{u}\in E^2([0, T])$ satisfying (7.2). By Schwarz's inequality and (3.31.b),

$$(7.4)\qquad \text{the right-hand side of (7.3)} \leqq\|\partial_t\vec{u}(0, \cdot)\|^2+\|\vec{u}(0, \cdot)\|^2_{\mathcal{J}(0)}+\int_0^t\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2ds +C(\Gamma, \mathcal{M}(1), \delta_1, \delta_2)\int_0^t\{\|\partial_s\vec{u}(s, \cdot)\|^2+\|\vec{u}(s, \cdot)\|^2_{\mathcal{J}(s)}\}ds.$$

Substituting (7.4) into (7.3) and using Gronwall's inequality, we have

$$(7.5)\qquad \|\partial_t\vec{u}(t, \cdot)\|^2+\|\vec{u}(t, \cdot)\|^2_{\mathcal{J}(t)}\leqq(\exp C_1t)\{\|\partial_t\vec{u}(0, \cdot)\|^2 +\|\vec{u}(0, \cdot)\|^2_{\mathcal{J}(0)}+\int_0^t\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2ds\}, \qquad 0\leqq t\leqq T,$$

for any $\vec{u}\in E^2([0, T])$ satisfying (7.2).

Now, we shall prove (7.1) without (7.2). Let $\vec{u}\in E^2([0, T])$ and put $\vec{u}_0(x)=\vec{u}(0, x)$, $\vec{u}_1(x)=\partial_t\vec{u}(0, x)$, $\vec{f}(t, x)=P_\varepsilon(t)[\vec{u}(t, x)]$ and $\vec{g}(t, x)=Q(t)[\vec{u}(t, x)]$. By Lemma 3.9 we know that there exist $\vec{u}_0^k\in H^2(\Omega)$ and $\vec{u}_1^k\in H^1(\Omega)$ such that

$$(7.6.\text{a})\qquad \vec{u}_0^k\longrightarrow\vec{u}_0 \text{ in } H^1(\Omega) \quad\text{and}\quad \vec{u}_1^k\longrightarrow\vec{u}_1 \text{ in } L^2(\Omega) \text{ as } k\to\infty,$$

$$(7.6.\text{b})\qquad B(0, \bar{\partial}^1)\vec{u}_0^k(x)+H^0(0, x)\vec{u}_1^k(x)=0 \text{ on } \Omega.$$

By the same arguments as in the proof of Lemma 6.7, we see that there exist

$\vec{g}^k \in C^\infty([0, T]); H^{1/2}(\Omega))$ such that

$$\vec{g}^k(0, x)=0 \text{ on } \Gamma, \tag{7.6.c}$$

$$\int_0^T \langle \vec{g}^k(s, \cdot)-\vec{g}(s, \cdot)\rangle_{1/2}^2 ds \longrightarrow 0 \text{ as } k\to\infty, \tag{7.6.d}$$

Since $C^1([0, T]; L^2(\Omega))$ is dense in $C^0([0, T]; L^2(\Omega))$, there exist $\vec{f}^k \in C^1([0, T]; L^2(\Omega))$ such that

$$\vec{f}^k \longrightarrow \vec{f} \text{ in } C^0([0, T]; L^2(\Omega)) \text{ as } k\to\infty. \tag{7.6.e}$$

For a moment, we assume that $0<\varepsilon\leqq 1$. In view of (7.6.b), by Lemma 6.5 we know that there exist $\vec{v}^k \in E^2([0, T])$ satisfying the equations:

$$P_\varepsilon(t)[\vec{v}^k]=\vec{f}^k \text{ in } [0, T]\times\Omega, \quad Q(t)[\vec{v}^k]=0 \text{ on } [0, T]\times\Gamma, \tag{7.7.a}$$

$$\vec{v}^k(0, x)=\vec{u}_0^k(x), \quad \partial_t\vec{v}^k(0, x)=\vec{u}_1^k(x) \text{ in } \Omega. \tag{7.7.b}$$

Applying (7.5) we have

$$\|\partial_t\vec{v}^k(t, \cdot)\|^2+\|\vec{v}^k(t, \cdot)\|_{\mathcal{J}(t)}^2 \leqq (\exp C_1 t)\{\|\vec{u}_1^k\|^2+\|\vec{u}_0^k\|_{\mathcal{J}(0)}^2 + \int_0^t \|\vec{f}^k(s, \cdot)\|^2 ds\}. \tag{7.8.a}$$

On the other hand, noting (7.6.c), by Lemma 6.5 we know that there exist $\vec{w}^k \in E^2([0, T])$ satisfying the equations:

$$P_\varepsilon(t)[\vec{w}^k]=0 \text{ in } [0, T]\times\Omega, \quad Q(t)[\vec{w}^k]=\vec{g}^k \text{ on } [0, T]\times\Gamma, \tag{7.7.c}$$

$$\vec{w}^k(0, x)=\partial_t\vec{w}^k(0, x)=0 \text{ in } \Omega. \tag{7.7.d}$$

By Theorem 6.8 and (3.31.b) we have

$$\|\partial_t\vec{w}^k(t, \cdot)\|^2+\|\vec{w}^k(t, \cdot)\|_{\mathcal{J}(t)}^2 \leqq C_2(\exp C_2 t)\int_0^t \langle \vec{g}^k(s, \cdot)\rangle_{1/2}^2 ds, \quad 0\leqq t\leqq T. \tag{7.8.b}$$

Put $\vec{u}^k=\vec{v}^k+\vec{w}^k$. Combining (7.8.a) and (7.8.b), we have

$$\|\partial_t\vec{u}^k(t, \cdot)\|^2+\|\vec{u}^k(t, \cdot)\|_{\mathcal{J}(t)}^2 \leqq 2(\exp C_1 t)\{\|\vec{u}_0^k\|^2+\|\vec{u}_1^k\|_{\mathcal{J}(0)}^2 + \int_0^t \|\vec{f}^k(s, \cdot)\|^2 ds\} + 2C_2(\exp C_2 t)\int_0^t \langle \vec{g}^k(s, \cdot)\rangle_{1/2}^2 ds, \quad 0\leqq t\leqq T. \tag{7.8.c}$$

Applying Lemma 6.4-(i) to $\vec{u}^k-\vec{u}$, by (7.7) we have

$$\begin{aligned} &\|\partial_t(\vec{u}^k(t, \cdot)-\vec{u}(t, \cdot))\|^2+\|\vec{u}^k(t, \cdot)-\vec{u}(t, \cdot)\|_{\mathcal{J}(t)}^2 \\ &\leqq e^{Ct}\{\|\vec{u}_1^k-\vec{u}_1\|^2+\|\vec{u}_0^k-\vec{u}_0\|_{\mathcal{J}(0)}^2 \\ &\quad + \int_0^t \{\|\vec{f}^k(s, \cdot)-\vec{f}(s, \cdot)\|^2+\varepsilon^{-1}\langle \vec{g}^k(s, \cdot)-\vec{g}(s, \cdot)\rangle_{1/2}^2\} ds. \end{aligned} \tag{7.9}$$

By (7.6.a, d and e) and (7.9) we see

$$\|\partial_t \vec{u}^k(t, \cdot)\|^2 + \|\vec{u}^k(t, \cdot)\|^2_{\mathcal{J}(t)} \longrightarrow \|\partial_t \vec{u}(t, \cdot)\|^2 + \|\vec{u}(t, \cdot)\|^2_{\mathcal{J}(t)} \text{ as } k \to \infty.$$

Letting $k \to \infty$ in (7.8.c) and recalling the definitions of $\vec{u}_0$, $\vec{u}_1$, $\vec{f}$ and $\vec{g}$, we have that (7.1) is valid for $0 < \varepsilon \leqq 1$. Since C_1 and C_2 are independent of ε and $\vec{u} \in E^2([0, T])$, letting $\varepsilon \downarrow 0$, we have that (7.1) is also valid for $P_0(t) = P(t)$, which completes the proof.

Now, we shall prove the estimate (b) of Theorem 2.2-2°.

LEMMA 7.2. *Assume that* (A.1)-(A.6) *are valid. Then, there exists a* $C = C(\delta_1, \delta_2, \mathcal{M}(1+\mu), \Gamma)$ *such that*

$$\int_0^t \langle \partial_s \vec{u}(s, \cdot) \rangle^2_{-1/2} ds \leqq C e^{Ct} \{ \|\bar{D}^1 \vec{u}(0, \cdot)\|^2 + \int_0^t (\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2 + \langle Q(s)[\vec{u}(s, \cdot)] \rangle^2_{1/2}) ds \} \tag{7.10}$$

for any $t \in [0, T]$, $\vec{u} \in E^2([0, T])$ *and* $0 \leqq \varepsilon \leqq 1$.

Proof. As was seen in Lemma 7.1, we may assume that $0 < \varepsilon \leqq 1$. In the proof, we use the same letter C to denote various constants depending on δ_1, δ_2, Γ and $\mathcal{M}(1+\mu)$. Put

$$H_\varepsilon(t, x) = \nu_i(x) H^i(t, x) + \varepsilon I_m + H^0(t, x),$$

$$E(t, \vec{u}) = \|\bar{D}^1 \vec{u}(0, \cdot)\|^2 + \|\bar{D}^1 \vec{u}(t, \cdot)\|^2 + \int_0^t (\|\bar{D}^1 \vec{u}(s, \cdot)\|^2 + \|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2) ds + \int_0^t |\langle Q(s)[\vec{u}(s, \cdot)], \partial_s \vec{u}(s, \cdot) \rangle| ds.$$

First of all, we shall prove that

$$\int_0^t |\langle H_\varepsilon(s, \cdot) \partial_s \vec{u}(s, \cdot), \partial_s \vec{v}(s, \cdot) \rangle| ds \leqq C E(t, \vec{u})^{1/2} E(t, \vec{v})^{1/2} \tag{7.11}$$

for any $\vec{u}$ and $\vec{v} \in E^2([0, T])$. Since $H_\varepsilon(t, x) = {}^t H_\varepsilon(t, x)$ and $H_\varepsilon(t, x) \geqq \varepsilon I_m$, we have

$$|H_\varepsilon(t, x) \vec{w}_1 \cdot \vec{w}_2| \leqq (H_\varepsilon(t, x) \vec{w}_1 \cdot \vec{w}_1)^{1/2} (H_\varepsilon(t, x) \vec{w}_2 \cdot \vec{w}_2)^{1/2} \tag{7.12}$$

for any $(t, x) \in \boldsymbol{R} \times \Gamma$ and constant vectors $\vec{w}_1, \vec{w}_2 \in \boldsymbol{R}^m$. By (7.12) we have

(7.13) the left-hand side of (7.11)

$$\leqq \left\{ \int_0^t \langle H_\varepsilon(s, \cdot) \partial_s \vec{u}(s, \cdot), \partial_s \vec{u}(s, \cdot) \rangle ds \right\}^{1/2} \times \left\{ \int_0^t \langle H_\varepsilon(s, \cdot) \partial_s \vec{v}(s, \cdot), \partial_s \vec{v}(s, \cdot) \rangle ds \right\}^{1/2}.$$

Putting $\vec{u}=\vec{v}$ and $\lambda=\lambda_2$ in Lemma 6.3 and integrating the resulting identity from 0 to t, we have

$$(7.14)\qquad \int_0^t\langle H_\varepsilon(s,\cdot)\partial_s\vec{u}(s,\cdot),\ \partial_s\vec{u}(s,\cdot)\rangle ds\leqq CE(t,\vec{u}).$$

Here, we have used the fact that $H_\varepsilon(t,x)={}^tH_\varepsilon(t,x)$ and (3.31.b). Since the same estimate as in (7.14) holds for $\vec{v}$, substituting (7.14) into (7.13), we have (7.11).

Now, let $t\in(0,T]$ be fixed. Put $C^1_{(0)}([0,t];H^{1/2}(\Gamma))=\{G\in C^1([0,t];H^{1/2}(\Gamma))\mid G(0,x)=0$ for $x\in\Gamma\}$ and $L^2((0,t);H^\gamma(\Gamma))=\{\vec{g}(s,x)\mid\vec{g}(s,x)$ is a $H^\gamma(\Gamma)$-valued L^2 function in $s\in(0,t)\}$. Note that $C^1_{(0)}([0,t];H^{1/2}(\Gamma))$ is dense in $L^2((0,t);H^{1/2}(\Gamma))$. Given $G\in C^1_{(0)}([0,t];H^{1/2}(\Gamma))$, let $\vec{v}\in E^2([0,t])$ be a solution to the equations:

$$(7.15.\mathrm{a})\qquad P_\varepsilon(s)[\vec{v}]=0 \text{ in } [0,t]\times\Omega,\quad Q(s)[\vec{v}]=G \text{ on } [0,t]\times\Gamma,$$

$$(7.15.\mathrm{b})\qquad \vec{v}(0,x)=\partial_t\vec{v}(0,x)=0 \text{ in } \Omega.$$

The existence of $\vec{v}$ is assured by Lemma 6.5, because $G(0,x)=0$ for $x\in\Gamma$. By Theorem 6.8 we have

$$(7.16)\qquad E(t,\vec{v})\leqq Ce^{Ct}\int_0^t\langle G(s,\cdot)\rangle^2_{1/2}ds.$$

Integrating the identity of Lemma 6.3 and substituting (7.15), we have

$$(7.17)\quad \left|\int_0^t\langle\partial_s\vec{u}(s,\cdot),\ G(s,\cdot)\rangle ds\right|\leqq\int_0^t|\langle Q(s)[\vec{u}(s,\cdot)],\ \partial_s\vec{v}(s,\cdot)\rangle|\,ds$$
$$+\int_0^t|(P_\varepsilon(s)[\vec{u}(s,\cdot)],\ \partial_s\vec{v}(s,\cdot))|\,ds+2\int_0^t|\langle H_\varepsilon(s,\cdot)\partial_s\vec{u}(s,\cdot),\ \partial_s\vec{v}(s,\cdot)\rangle|\,ds$$
$$+C(\mathcal{M}(0))\|\bar{D}^1\vec{u}(t,\cdot)\|\,\|\bar{D}^1\vec{v}(t,\cdot)\|+C(\mathcal{M}(1),\lambda_2)\int_0^t\|\bar{D}^1\vec{u}(s,\cdot)\|\,\|\bar{D}^1\vec{v}(s,\cdot)\|ds.$$

By (7.11) and Schwarz's inequality, we have

$$(7.18)\qquad \text{the right-hand side of (7.17)}\leqq CE(t,\vec{u})^{1/2}E(t,\vec{v})^{1/2}.$$

Combining (7.16), (7.17) and (7.18), we have

$$(7.19)\qquad \left|\int_0^t\langle\partial_s\vec{u}(s,\cdot),\ G(s,\cdot)\rangle ds\right|\leqq CE(t,\vec{u})^{1/2}\left\{\int_0^t\langle G(s,\cdot)\rangle^2_{1/2}ds\right\}^{1/2}.$$

Noting that $L^2((0,t);H^{-1/2}(\Gamma))$ is the dual space of $L^2((0,t);H^{1/2}(\Gamma))$, by (7.19) and Lemma 7.1 we have

$$(7.20)\qquad \int_0^t\langle\partial_s\vec{u}(s,\cdot)\rangle^2_{-1/2}ds\leqq Ce^{Ct}\{\|\bar{D}^1\vec{u}(0,\cdot)\|^2+\int_0^t(\|P_\varepsilon(s)[\vec{u}(s,\cdot)]\|^2$$
$$+\langle Q(s)[\vec{u}(s,\cdot)]\rangle^2_{1/2})ds\}$$
$$+C\left\{\int_0^t\langle Q(s)[\vec{u}(s,\cdot)]\rangle^2_{1/2}ds\right\}^{1/2}\left\{\int_0^t\langle\partial_s\vec{u}(s,\cdot)\rangle^2_{-1/2}ds\right\}^{1/2}.$$

Here, we have used (3.31.b) and the fact that $1+t\leqq e^{Ct}$ $(C\geqq 1)$. From (7.20) the lemma follows immediately, which completes the proof.

Now, we shall prove the estimate (c) of Theorem 2.2-2°.

LEMMA 7.3. *Assume that* (A.1)-(A.6) *are valid. Then, there exist constants* $C_1=C(\delta_1, \delta_2, \mathcal{M}(1, \Gamma))$ *and* $C_2=C(\delta_1, \delta_2, \mathcal{M}(1+\mu), \Gamma))$ *such that*

$$\|\partial_t\vec{u}(t, \cdot)\|^2+\|\vec{u}(t, \cdot)\|^2_{\mathcal{J}(t)}\leqq(\exp C_1t)\{\|\partial_t\vec{u}(0, \cdot)\|^2+\|\vec{u}(0, \cdot)\|^2_{\mathcal{J}(0)}\}$$
$$+C_2(\exp C_2t)\{\|\bar{D}^1\vec{u}(0, \cdot)\|^2+\int_0^t(\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2+\langle Q(s)[\vec{u}(s, \cdot)]\rangle^2_{1/2})ds\}^{1/2}$$
$$\times\left\{\int_0^t(\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2+\langle Q(s)[\vec{u}(s, \cdot)]\rangle^2_{1/2})ds\right\}^{1/2}, \qquad 0\leqq t\leqq T,$$

for any $\vec{u}\in E^2([0, T])$ *and* $0\leqq\varepsilon\leqq 1$.

Proof. Putting $\vec{u}=\vec{v}$ and $\lambda=\lambda_2$ in Lemma 6.3 and integrating the resulting identity, we have

$$\|\partial_t\vec{u}(t, \cdot)\|^2+\|\vec{u}(t, \cdot)\|^2_{\mathcal{J}(t)}\leqq\|\partial_t\vec{u}(0, \cdot)\|^2+\|\vec{u}(0, \cdot)\|^2_{\mathcal{J}(0)}$$
$$+\left\{\int_0^t\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2ds\right\}^{1/2}\left\{\int_0^t\|\partial_s\vec{u}(s, \cdot)\|^2ds\right\}^{1/2}$$
$$+\left\{\int_0^t\langle Q(s)[\vec{u}(s, \cdot)]\rangle^2_{1/2}\right\}^{1/2}\left\{\int_0^t\langle\partial_s\vec{u}(s, \cdot)\rangle^2_{-1/2}ds\right\}^{1/2}$$
$$+C(\delta_1, \delta_2, \mathcal{M}(1), \Gamma)\int_0^t\{\|\partial_s\vec{u}(s, \cdot)\|^2+\|\vec{u}(s, \cdot)\|^2_{\mathcal{J}(s)}\}ds.$$

Here, we have used (3.31.b). Substituting the estimates of Lemmas 7.1 and 7.2 and using Gronwall's inequality, we have the lemma.

Now, we shall give a

Proof of Theorem 2.2-1°. We may assume that $\vec{f}\in C^1([0, 2T]\,;\,L^2(\Omega))$ and $\vec{g}\in C^1([0, 2T]\,;\,H^{1/2}(\Gamma))$. Let $\vec{u}_\varepsilon\in E^2([0, 2T])$, $0<\varepsilon\leqq 1$, be solutions of the equations:

(7.21.a) $\qquad P_\varepsilon(t)[\vec{u}_\varepsilon]=\vec{f}$ in $[0, 2T]\times\Omega$, $Q(t)[\vec{u}_\varepsilon]=\vec{g}$ on $[0, 2T]\times\Gamma$,

(7.21.b) $\qquad \vec{u}_\varepsilon(0, x)=\vec{u}_0(x)$, $\partial_t\vec{u}_\varepsilon(0, x)=\vec{u}_1(x)$ in Ω.

Since $\vec{u}_0$, $\vec{u}_1$ and $\vec{g}$ satisfy (4.41), the existence of such $\vec{u}_\varepsilon$ is assured by Lemma 6.5. In the proof, we use the same letter C to denote various constants depending essentially on δ_1, δ_2, $\mathcal{M}(2)$, Γ and T. By Lemmas 4.2 and 7.1, we have

(7.22) $\qquad \|\bar{D}^2\vec{u}_\varepsilon(t, \cdot)\|^2\leqq C\cdot C(\vec{u}_0, \vec{u}_1, \vec{f}, \vec{g})$ for any $t\in[0, 2T]$, where

$$C(\vec{u}_0, \vec{u}_1, \vec{f}, \vec{g})=\|\bar{\partial}^2\vec{u}_0\|^2+\|\bar{\partial}^1\vec{u}_1\|^2+\|\vec{f}(0, \cdot)\|^2+\langle\vec{g}(0, \cdot)\rangle_{1/2}^2$$
$$+\sum_{k=0}^{1}\sup_{0<s<2T}\{\|\partial_s^k\vec{f}(s, \cdot)\|^2+\langle\partial_s^k\vec{g}(s, \cdot)\rangle_{1/2}^2\}.$$

If we assume that $\partial_t\vec{u}_\varepsilon\in E^2([0, 2T])$, differentiating (7.21) in t, we have that

$$\text{(7.23)}\quad \begin{aligned}&P_\varepsilon(\cdot)[\partial_t\vec{u}_\varepsilon]=\partial_t\vec{f}-H^{(1)}(\cdot, \bar{\partial}^1)\partial_t\vec{u}_\varepsilon-A^{(1)}(\cdot, \bar{\partial}^2)\vec{u}_\varepsilon,\\ &Q(\cdot)[\partial_t\vec{u}_\varepsilon]=\partial_t\vec{g}-B^{(1)}(\cdot, \bar{\partial}^1)\vec{u}_\varepsilon-\partial_tH^0\partial_t\vec{u}_\varepsilon,\\ &\partial_t\vec{u}_\varepsilon(0, x)=\vec{u}_1(x) \text{ and } \partial_t^2\vec{u}_\varepsilon(0, x)=\vec{u}_2(x), \text{ where}\\ &\vec{u}_2(x)=\vec{f}(0, x)-H(0, \bar{\partial}^1)\vec{u}_1(x)-A(0, \bar{\partial}^2)\vec{u}_0(x).\end{aligned}$$

Applying Lemma 7.3, we have

$$\text{(7.24)}\quad \|\partial_t^2\vec{u}_\varepsilon(t, \cdot)\|^2+\|\partial_t\vec{u}_\varepsilon(t, \cdot)\|_{\mathcal{J}(t)}^2\leqq(\exp Ct)\{\|\vec{u}_2\|^2+\|\vec{u}_1\|_{\mathcal{J}(0)}^2$$
$$+C\{\|\bar{\partial}^2\vec{u}_0\|^2+\|\bar{\partial}^1\vec{u}_1\|^2+\|\vec{f}(0, \cdot)\|^2+\int_0^t(\|\partial_s\vec{f}(s, \cdot)\|^2+\langle\partial_s\vec{g}(s, \cdot)\rangle_{1/2}^2$$
$$+\|\bar{D}^2\vec{u}_\varepsilon(s, \cdot)\|^2)ds\}^{1/2}$$
$$\times\left\{\int_0^t(\|\partial_s\vec{f}(s, \cdot)\|^2+\langle\partial_s\vec{g}(s, \cdot)\rangle_{1/2}^2+\|\bar{D}^2\vec{u}_\varepsilon(s, \cdot)\|^2)ds\right\}^{1/2}\}.$$

Employing the same arguments as in the proof of Lemma 4.2, by using the mollifier with respect to t we can remove the additional assumption: $\partial_t\vec{u}_\varepsilon\in E^2([0, 2T])$. Thus, (7.24) is actually valid, and then it follows from (7.22) and (7.24) that

$$\text{(7.25)}\quad \|\partial_t^2\vec{u}_\varepsilon(t, \cdot)\|^2+\|\partial_t\vec{u}_\varepsilon(t, \cdot)\|_{\mathcal{J}(t)}^2\leqq(\exp Ct)\{\|\vec{u}_2\|^2+\|\vec{u}_1\|_{\mathcal{J}(0)}^2$$
$$+C\cdot C(\vec{u}_0, \vec{u}_1, \vec{f}, \vec{g})t^{1/2}\}.$$

(7.22) implies that $\{\vec{u}_\varepsilon\}_{0<\varepsilon<1}$ is a bounded set in $H^2(\Omega_{2T})$ ($\Omega_{2T}=(0, 2T)\times\Omega$). By passing to a subsequence if necessary, we may assume that the sequence $\{\vec{u}_\varepsilon\}_{0<\varepsilon<1}$ converges to $\vec{u}\in H^2(\Omega_{2T})$ weakly as $\varepsilon\downarrow 0$. In the similar manner to the proof of Lemma 4.4, we see that

$$\text{(7.26)}\quad \begin{aligned}&P(t)[\vec{u}]=\vec{f} \text{ in the sense of } L^2(\Omega_{2T}),\\ &Q(t)[\vec{u}]=\vec{g} \text{ in the sense of } L^2((0, 2T)\times\Gamma).\end{aligned}$$

Our task is to prove that $\vec{u}\in E^2([0, T])$. To do this, we use the similar arguments to the final part of the proof of Lemma 4.4. Let $\rho_\delta*$ be the same as in the proof of Lemma 4.2 and put $\vec{u}_\delta=\rho_\delta*\vec{u}$. Note that $\vec{u}_\delta\in C^\infty([0, T]; H^2(\Omega))$ provided that $0<\delta<T/2$. By (7.26) we see that (4.39) is also valid in the present case. From this point of view, to prove that $\{\vec{u}_\delta\}$ is a Cauchy sequence in $E^2([0, T])$, it suffices to show that

(7.27) $$\|\partial_t^2(\vec{u}_\delta-\vec{u}_{\delta'})(0,\cdot)\|^2+\|\bar{\partial}^1\partial_t(\vec{u}_\delta-\vec{u}_{\delta'})(0,\cdot)\|^2\longrightarrow 0 \text{ as } \delta, \delta'\downarrow 0.$$

Applying Lemma 7.1 to $\vec{u}_\varepsilon-\vec{u}_{\varepsilon'}$, we have

$$\sup_{0<t<T}\|\bar{D}^1(\vec{u}_\varepsilon(t,\cdot)-\vec{u}_{\varepsilon'}(t,\cdot))\|^2\leqq C|\varepsilon-\varepsilon'|\int_0^T\|\bar{D}^2\vec{u}_\varepsilon(s,\cdot)\|^2ds.$$

Combining this and (7.22) implies that $\{\vec{u}_\varepsilon\}_{0<\varepsilon<1}$ is a Cauchy sequence in $E^1([0, T])$, and then $\vec{u}\in E^1([0, T])$ and

(7.28) $$\sup_{0<t<T}\|\bar{D}^1(\vec{u}_\varepsilon(t,\cdot)-\vec{u}(t,\cdot))\|\longrightarrow 0 \text{ as } \varepsilon\downarrow 0.$$

Let α and β be multi-indices such that $|\alpha|=2$ and $|\beta|=1$. Then, we have that $\partial_x^\alpha\vec{u}(t, x)\in L^2(\Omega)$, $\partial_x^\beta\partial_t\vec{u}(t, x)\in L^2(\Omega)$ and

(7.29) $$\|\bar{\partial}^2\vec{u}(t,\cdot)\|^2+\|\bar{\partial}^1\partial_t\vec{u}(t,\cdot)\|^2\leqq C\cdot C(\vec{u}_0, \vec{u}_1, \vec{f}, \vec{g})$$

for any $t\in[0, T]$. In fact, given $\psi\in C_0^\infty(\Omega)$, by (7.22) we have

$$|(\partial_x^\alpha\vec{u}(t,\cdot), \psi)|\leqq|(\vec{u}(t,\cdot)-\vec{u}_\varepsilon(t,\cdot), (-\partial_x)^\alpha\psi)|+C\cdot C(\vec{u}_0, \vec{u}_1, \vec{f}, \vec{g})\|\psi\|.$$

Applying (7.28), we have (7.29) immediately. Put $\vec{v}(t, x)=\vec{f}(t, x)-H(t, \bar{\partial}^1)\partial_t\vec{u}(t, x)-A(t, \bar{\partial}^2)\vec{u}(t, x)$. Then, $\vec{v}(t, x)\in L^2(\Omega)$ for each $t\in[0, T]$ and $\partial_t^2\vec{u}(t, x)=\vec{v}(t, x)$ in the sense of $L^2(\Omega_T)$ ($\Omega_T=(0, T)\times\Omega$) as follows from (7.26). Given $\phi\in L^2(\Omega)$, we have

(7.30) $$(\vec{v}(t,\cdot), \phi)\longrightarrow(\vec{u}_2, \phi),\ (\partial_x^\beta\partial_t\vec{u}(t,\cdot), \phi)\longrightarrow(\partial_x^\beta\vec{u}_1, \phi) \text{ as } t\downarrow 0.$$

In fact, given $\sigma>0$, let us choose $\psi\in C_0^\infty(\Omega)$ so that $\|\phi-\psi\|<\sigma$. Then, by the definition of $\vec{v}$, (7.23), (7.29) and integration by parts, we have

$$\begin{aligned}|(\vec{v}(t,\cdot)-\vec{u}_2, \phi)|&\leqq|(\vec{v}(t,\cdot)-\vec{u}_2, \phi-\psi)|+|(\vec{f}(t,\cdot)-\vec{f}(0,\cdot), \psi)|\\&+C\sum_{|\beta|\leq 1}|(\partial_t\vec{u}(t,\cdot)-\vec{u}_1, \partial_x^\beta\psi)|+C\sum_{|\alpha|\leq 2}|(\vec{u}(t,\cdot)-\vec{u}_0, \partial_x^\alpha\psi)|.\end{aligned}$$

Since $|(\vec{v}(t,\cdot)-\vec{u}_2, \phi-\psi)|\leqq(C\cdot C(\vec{u}_0, \vec{u}_1, \vec{f}, \vec{g})+\|\vec{u}_2\|^2)^{1/2}\sigma$ as follows from Schwarz's inequality and (7.29) and since $\|\partial_t\vec{u}(t,\cdot)-\vec{u}_1\|\to 0$ and $\|\vec{u}(t,\cdot)-\vec{u}_0\|\to 0$ as $t\downarrow 0$ as follows from (7.21.b) and (7.28), we have

$$\limsup_{t\downarrow 0}|(\vec{v}(t,\cdot)-\vec{u}_2, \phi)|\leqq(C\cdot C(\vec{u}_0, \vec{u}_1, \vec{f}, \vec{g})+\|\vec{u}_2\|^2)^{1/2}\sigma.$$

The arbitrariness of the choice of σ implies the first part of (7.30). In the same way, we have the second part of (7.30). Since $H^1(\Omega)\times L^2(\Omega)$ is a Hilbert space equipped with scalar product $(\ ,\ )_{\mathcal{H}(0)}$ (cf. (3.30.b) and (3.31.a)), it follows from (7.30) that

(7.31) $$\|\vec{u}_2\|^2+\|\vec{u}_1\|_{\mathcal{J}(0)}^2\leqq\liminf_{t\downarrow 0}(\|\vec{v}(t,\cdot)\|^2+\|\partial_t\vec{u}(t,\cdot)\|_{\mathcal{J}(0)}^2).$$

Now, we shall prove that

(7.32) $$\lim_{t\downarrow 0}(\|\vec{v}(t,\ \cdot)\|^2+\|\partial_t\vec{u}(t,\ \cdot)\|^2_{\mathcal{J}(0)})=\|\vec{u}_2\|^2+\|\vec{u}_1\|^2_{\mathcal{J}(0)}.$$

If we get (7.32), by (3.31.b) we see that

(7.33) $$\lim_{t\downarrow 0}\|\vec{v}(t,\ \cdot)-\vec{u}_2\|=0,\quad \lim_{t\downarrow 0}\|\bar{\partial}^1(\partial_t\vec{u}(t,\ \cdot)-\vec{u}_1)\|=0.$$

In view of (7.31), to prove (7.32) it suffices to show that

(7.34) $$\limsup_{t\downarrow 0}(\|\vec{v}(t,\ \cdot)\|^2+\|\partial_t\vec{u}(t,\ \cdot)\|^2_{\mathcal{J}(0)})\leqq\|\vec{u}_2\|^2+\|\vec{u}_1\|^2_{\mathcal{J}(0)}.$$

The idea of proving (7.34) is essentially due to Majda [5, p. 44-46]. By (7.22) and (7.25) we have

(7.35) $$\|\partial_t^2\vec{u}_\varepsilon(t,\ \cdot)\|^2+\|\partial_t\vec{u}_\varepsilon(t,\ \cdot)\|^2_{\mathcal{J}(0)}\leqq(\exp C_1t)\{\|\vec{u}_2\|^2+\|\vec{u}_1\|^2_{\mathcal{J}(0)}+R(t)\},$$

where $R(t)$ is independent of ε and $R(t)\to 0$ as $t\downarrow 0$. In the same manner as (7.30) from (7.22) and (7.28), noting the definition of $\vec{v}$ and (7.21.a), we can prove that

(7.36) $$(\partial_t^2\vec{u}_\varepsilon(t,\ \cdot),\ \phi)\longrightarrow(\vec{v}(t,\ \cdot),\ \phi),\quad (\partial_x^\beta\partial_t\vec{u}_\varepsilon(t,\ \cdot),\ \phi)\longrightarrow(\partial_x^\beta\partial_t\vec{u}(t,\ \cdot),\ \phi)$$

for any $\phi\in L^2(\Omega)$ as $\varepsilon\downarrow 0$. Combining (7.35) and (7.36) implies that

$$\begin{aligned}\|\vec{v}(t,\ \cdot)\|^2+\|\partial_t\vec{u}(t,\ \cdot)\|^2_{\mathcal{J}(0)}&\leqq\liminf_{\varepsilon\downarrow 0}(\|\partial_t^2\vec{u}_\varepsilon(t,\ \cdot)\|^2+\|\partial_t\vec{u}_\varepsilon(t,\ \cdot)\|^2_{\mathcal{J}(0)})\\&\leqq(\exp C_1t)\{\|\vec{u}_2\|^2+\|\vec{u}_1\|^2_{\mathcal{J}(0)}+R(t)\}.\end{aligned}$$

Accordingly, (7.34) follows from the fact that $R(t)\to 0$ and $\exp C_1t\to 1$ as $t\downarrow 0$, and then we get (7.33).

Recall that to complete the proof we only prove (7.27). To do this, it suffices to prove that

(7.37) $$\|(\partial_t^2u_\delta)(0,\ \cdot)-u_2\|\longrightarrow 0,\quad \|\bar{\partial}^1((\partial_t\vec{u}_\delta)(0,\ \cdot)-\vec{u}_1)\|\longrightarrow 0 \text{ as } \delta\downarrow 0.$$

Since $\partial_t^2\vec{u}(t,\ x)=\vec{v}(t,\ x)$ in the sense of $L^2(\Omega_T)$, we have that $\partial_t^2\vec{u}_\delta(t,\ x)=\vec{v}_\delta(t,\ x)$ $(=\rho_\delta*\vec{v})$ in the sense of $L^2(\Omega)$ for each $t\in[0,\ T]$. Then, we have

$$\|(\partial_t^2\vec{u}_\delta)(0,\ \cdot)-\vec{u}_2\|\leqq\int\rho(-s)\|\vec{v}(\delta s,\ \cdot)-\vec{u}_2\|ds.$$

Recall that $\operatorname{supp}\rho\subset[-2,\ -1]$. By this and the first part of (7.33) we have the first part of (7.37). The second part of (7.37) also follows from the second part of (7.33). Accordingly, we have proved that (N) admits a solution $\vec{u}\in E^2([0,\ T])$. The uniqueness of solutions follows from Theorem 2.2-2°, which completes the proof of Theorem 2.2.

In the same manner as in the proof of Theorem 2 of Ikawa [3, p. 364-367] (cf. also Ikawa [2, p. 604-607]), we can prove Theorem 2.3. So, we may omit the proof of Theorem 2.3.

§ 8. The unique existence theorem and energy inequalities for $n=1$.

When $n=1$, we may assume that $\Omega=(\alpha, \beta)$ $(-\infty<\alpha<\beta<\infty)$ or $=\boldsymbol{R}_+$ $(=\{x\in\boldsymbol{R} \mid x>0\})$. The treatment of the case that $\Omega=(\alpha, \beta)$ is essentially the same as $\Omega=\boldsymbol{R}_+$. Thus, we consider the case that $\Omega=\boldsymbol{R}_+$, below. The problem (N) can be written as follows:

$$\text{(N)}\quad \begin{cases} P(t)[\vec{u}]=\partial_t^2\vec{u}+H(t, \bar{\partial}^1)\partial_t\vec{u}+A(t, \bar{\partial}^2)\vec{u}=\vec{f}(t, x) & \text{in } [0, T]\times\boldsymbol{R}_+, \\ Q(t)[\vec{u}]\,|_{x=0}=\{B(t, \bar{\partial}_1)\vec{u}+H^0(t)\partial_t\vec{u}\}\,|_{x=0}=\vec{g}(t) & \text{on } [0, T], \\ \vec{u}(0, x)=\vec{u}_0(x), \quad \partial_t\vec{u}(0, x)=\vec{u}_1(x) & \text{in } \boldsymbol{R}_+, \end{cases}$$

where $H(t, \bar{\partial}^1)$, $A(t, \bar{\partial}^2)$ and $B(t, \bar{\partial}^1)$ are $m\times m$ matrices of differential operators of the forms:

(8.1.a) $H(t, \bar{\partial}^1)\vec{v}=2H^1(t, x)\partial_x\vec{v}+H^2(t, x)\vec{v}$,

(8.1.b) $A(t, \bar{\partial}^2)\vec{v}=-\partial_x(A^0(t, x)\partial_x\vec{v})+A^1(t, x)\partial_x\vec{v}+A^2(t, x)\vec{v}$,

(8.1.c) $B(t, \bar{\partial}^1)\vec{v}=-A^0(t, x)\partial_x\vec{v}+B(t)\vec{v}$,

(8.1.d) $H^l(t, x)=(H^l{}_a{}^b(t, x))$, $l=1, 2$, $A^k(t, x)=(A^k{}_a{}^b(t, x))$, $k=0, 1, 2$,

$B(t)=(B_a{}^b(t))$, $H^0(t)=(H^0{}_a{}^b(t))$.

Now, we introduce the assumptions for $n=1$.

(A.8.1) The $A^0{}_a{}^b$ and $H^1{}_a{}^b$ are in $\mathscr{B}^2([0, T]\times\overline{\boldsymbol{R}_+})$, $H^2{}_a{}^b$ and $A^k{}_a{}^b$, $k=1, 2$, in $\mathscr{B}^1([0, T]\times\overline{\boldsymbol{R}_+})$, and $B_a{}^b$ and $H^0{}_a{}^b$ in $\mathscr{B}^2([0, T])$.

(A.8.2) ${}^tA^0(t, x)=A^0(t, x)$, ${}^tH^1(t, x)=H^1(t, x)$, ${}^tH^0(t)=H^0(t)$ for any $t\in[0, T]$ and $x\in\overline{\boldsymbol{R}_+}$.

(A.8.3) There exists a constant $\delta_3>0$ such that $A^0(t, x)\geqq\delta_3 \mathrm{I}_m$ for any $(t, x)\in[0, T]\times\overline{\boldsymbol{R}_+}$.

(A.8.4) $-H^1(t, 0)+H^0(t)\geqq 0$ for any $t\in[0, T]$.

Put

$$M_T(1)=|A^0|_{\infty,1,[0,T]\times\bar{\Omega}}+|H^1|_{\infty,1,[0,T]\times\bar{\Omega}}+|H^2|_{\infty,0,[0,T]\times\bar{\Omega}}$$
$$+\sum_{k=1}^{2}|A^k|_{\infty,0,[0,T]\times\bar{\Omega}}+\sum_{l=0}^{1}\{\sup_{[0,T]}|d^lB(t)/dt^l|+\sup_{[0,T]}|d^lH^0(t)/dt^l|\}.$$

The following is our main result for $n=1$.

THEOREM 8.1. *Let* $T>0$ *and* $n=1$. *Assume that* (A.8.1)-(A.8.4) *are valid.*
1° *If* $\vec{u}_0\in H^2(\Omega)$, $\vec{u}_1\in H^1(\Omega)$, $\vec{f}\in C^1([0, T]; L^2(\Omega))$, $\vec{g}\in C^1([0, T])$ *and the compatibility condition of order* 0 *is satisfied, then there exists a unique solution*

$\vec{u}\in E^2([0, T])$ *to* (N) *with initial data* $\vec{u}_0$, $\vec{u}_1$, *right member* $\vec{f}$ *and boundary data* $\vec{g}$.

2° *Put* $\mathcal{E}(t, \vec{u})=\int_0^t\{\|P(s)[\vec{u}(s, \cdot)]\|^2+|Q(s)[\vec{u}(s, \cdot)]|_{x=0}|^2\}ds$. *Then, there exists a constant* $C=C(\delta_3, M_T(1))>0$ *such that for any* $t\in[0, T]$ *and* $\vec{u}\in E^2([0, T])$ *the following three estimates hold*:

(a) $\|\partial_t\vec{u}(t, \cdot)\|^2+\|\vec{u}(t, \cdot)\|^2_{\mathcal{J}(t)}\leqq 2e^{Ct}\{\|\partial_t\vec{u}(0, \cdot)\|^2+\|\vec{u}(0, \cdot)\|^2_{\mathcal{J}(0)}+C\mathcal{E}(t, \vec{u})\}$,

(b) $\int_0^t|\vec{u}(s, 0)|^2ds\leqq Ce^{Ct}\{\|\bar{D}^1\vec{u}(0, \cdot)\|^2+\mathcal{E}(t, \vec{u})\}$,

(c) $\|\partial_t\vec{u}(t, \cdot)\|^2+\|\vec{u}(t, \cdot)\|^2_{\mathcal{J}(t)}\leqq e^{Ct}\{\partial_t\vec{u}(0, \cdot)\|^2+\|\vec{u}(0, \cdot)\|^2_{\mathcal{J}(0)}\}+Ce^{Ct}\{\|\bar{D}^1\vec{u}(0, \cdot)\|^2$
$+\mathcal{E}(t, \vec{u})\}^{1/2}\mathcal{E}(t, \vec{u})^{1/2}$.

Here, the norm $\|\cdot\|_{\mathcal{J}(t)}$ *is defined by*:

$$\|\vec{u}\|^2_{\mathcal{J}(t)}=(A^0(t, \cdot)\partial_x\vec{u}, \partial_x\vec{u})+\lambda(\vec{u}, \vec{u})-2\int_0^\infty B(t)\vec{u}(x)\cdot\partial_x\vec{u}(x)dx,$$

where $\lambda=(\delta_3/2)+(2/\delta_3)(\sup_{[0,T]}|B(t)|)^2$.

Note that by (A.8.3) we have

$$\|\vec{u}\|^2_{\mathcal{J}(t)}\geqq(\delta_3/2)\|\bar{\partial}^1\vec{u}\|^2. \tag{8.2}$$

Recall that Theorem 6.8 plays an essential role to prove Theorem 2.2. To get Theorem 8.1, it suffices to prove the following lemma corresponding to Theorem 6.8.

LEMMA 8.2. *Let* $T>0$ *and* $n=1$. *Assume that* (A.8.1)-(A.8.4) *are valid. Then, there exists a constant* $C=C(\delta_3, M_T(1))$ *such that*

$$\|\bar{D}^1\vec{u}(t, \cdot)\|^2+\int_0^t(|\partial_s\vec{u}(s, \cdot)|^2+\|\bar{D}^1\vec{u}(s, \cdot)\|^2)ds$$

$$\leqq Ce^{Ct}\int_0^t\{\|P_\varepsilon(s)[\vec{u}(s, \cdot)]\|^2+|Q(s)[\vec{u}(s, \cdot)]|_{x=0}|^2\}ds$$

for any $\vec{u}\in E^2([0, T])$ *satisfying*: $\vec{u}(0, x)=\partial_t\vec{u}(0, x)=0$. *Here*, $P_\varepsilon(t)[\vec{u}]=P(t)[\vec{u}]-2\varepsilon\partial_x\partial_t\vec{u}$, $0\leqq\varepsilon\leqq 1$, *and* $P_0(t)[\vec{u}]=P(t)[\vec{u}]$.

Proof. First of all, note that

$$B(t)\vec{u}(t, 0)\cdot\partial_t\vec{u}(t, 0)=-\frac{d}{dt}\int_0^\infty B(t)\vec{u}(t, x)\cdot\partial_x\vec{u}(t, x)dx$$

$$+\int_0^\infty(dB(t)/dt)\vec{u}(t, x)\cdot\partial_x\vec{u}(t, x)dx+\int_0^\infty B(t)\partial_t\vec{u}(t, x)\cdot\partial_x\vec{u}(t, x)dx$$

$$-\int_0^\infty B(t)\partial_x\vec{u}(t, x)\cdot\partial_t\vec{u}(t, x)dx.$$

By integration by parts and (A.8.2) we have

$$(P(t)[\vec{u}(t,\cdot)]+\lambda\vec{u}(t,\cdot),\ \partial_t\vec{u}(t,\cdot))=\frac{1}{2}\frac{d}{dt}\{\|\partial_t\vec{u}(t,\cdot)\|^2+\|\vec{u}(t,\cdot)\|^2_{\mathcal{J}(t)}\}$$

$$-\frac{1}{2}(\partial_x H^1(t,\cdot)\partial_t\vec{u}(t,\cdot),\ \partial_t\vec{u}(t,\cdot))+(H^2(t,\cdot)\partial_t\vec{u}(t,\cdot),\ \partial_t\vec{u}(t,\cdot))$$

$$-Q(t)[\vec{u}(t,\cdot)]|_{x=0}\cdot\partial_t\vec{u}(t,0)-\frac{1}{2}(\partial_t A^2(t,\cdot)\vec{u}(t,\cdot),\ \partial_t\vec{u}(t,\cdot))$$

$$-\int_0^\infty B(t)\partial_x\vec{u}(t,x)\cdot\partial_t\vec{u}(t,x)dx+\int_0^\infty(dB/dt)(t)\vec{u}(t,x)\cdot\partial_x\vec{u}(t,x)dx$$

$$+\int_0^\infty B(t)\partial_t\vec{u}(t,x)\cdot\partial_x\vec{u}(t,x)dx+(-H^1(t,0)+H^0(t))\partial_t\vec{u}(t,0)\cdot\partial_t\vec{u}(t,0).$$

Hence, by (A.8.4) and the assumptions: $\vec{u}(0,x)=\partial_t\vec{u}(0,x)=0$, we have

$$\text{(8.3)}\qquad \|\partial_t\vec{u}(t,\cdot)\|^2+\|\vec{u}(t,\cdot)\|^2_{\mathcal{J}(t)}\leqq\int_0^t\|P(s)[\vec{u}(s,\cdot)]\|^2ds$$

$$+C(\delta_3, M_T(1))\int_0^t(\|\partial_s\vec{u}(s,\cdot)\|^2+\|\vec{u}(s,\cdot)\|^2_{\mathcal{J}(s)})ds$$

$$+\left\{\int_0^t|Q(s)[\vec{u}(s,\cdot)]|_{x=0}|^2ds\right\}^{1/2}\left\{\int_0^t|\partial_s\vec{u}(s,0)|^2ds\right\}^{1/2}.$$

By integration by parts we have also

$$(P(t)[\vec{u}(t,\cdot)],\ \partial_x\vec{u}(t,\cdot))=\frac{d}{dt}[(\partial_t\vec{u}(t,\cdot),\ \partial_x\vec{u}(t,\cdot))+\frac{1}{2}(H^1(t,\cdot)\partial_x\vec{u}(t,\cdot),\ \partial_x\vec{u}(t,\cdot))]$$

$$+\frac{1}{2}(|\partial_t\vec{u}(t,0)|^2+A^0(t,0)\partial_x\vec{u}(t,0)\cdot\partial_x\vec{u}(t,0))-\frac{1}{2}(\partial_t H^1(t,\cdot)\partial_x\vec{u}(t,\cdot),\ \partial_x\vec{u}(t,\cdot))$$

$$+\frac{1}{2}(\partial_x A^0(t,\cdot)\partial_x\vec{u}(t,\cdot),\ \partial_x\vec{u}(t,\cdot))+(H^2(t,\cdot)\partial_t\vec{u}(t,\cdot),\ \partial_x\vec{u}(t,\cdot))$$

$$+(A^1(t,\cdot)\partial_x\vec{u}(t,\cdot)+A^0(t,\cdot)\vec{u}(t,\cdot),\ \partial_x\vec{u}(t,\cdot)).$$

Hence, we have

$$\text{(8.4)}\qquad \int_0^t|\partial_s\vec{u}(s,0)|^2ds\leqq C(\delta_3, M_T(1))\int_0^t(\|\partial_s\vec{u}(s,\cdot)\|^2+\|\vec{u}(s,\cdot)\|^2_{\mathcal{J}(s)})ds$$

$$+C(\delta_3)(\|\partial_t\vec{u}(t,\cdot)\|^2+\|\vec{u}(t,\cdot)\|^2_{\mathcal{J}(t)})+\int_0^t\|P(s)[\vec{u}(s,\cdot)]\|^2ds.$$

Substituting (8.4) into (8.3) and noting that

$$3^{1/2}\left[\int_0^t|Q(s)[\vec{u}(s,\cdot)]|_{x=0}|^2ds\right]^{1/2}[C(\delta_3)(\|\partial_t\vec{u}(t,\cdot)\|^2+\|\vec{u}(t,\cdot)\|^2_{\mathcal{J}(t)})]^{1/2}$$

$$\leqq\frac{1}{2}(\|\partial_t\vec{u}(t,\cdot)\|^2+\|\vec{u}(t,\cdot)\|^2_{\mathcal{J}(t)})+(3C(\delta_3)/2)\int_0^t|Q(s)[\vec{u}(s,\cdot)]|_{x=0}|^2ds,$$

we have

$$\|\partial_t\vec{u}(t,\ \cdot)\|^2+\|\vec{u}(t,\ \cdot)\|^2_{\mathcal{J}(t)}\leqq C(\delta_3,\ M_T(1))\int_0^t(\|\partial_t\vec{u}(s,\ \cdot)\|^2+\|\vec{u}(s,\ \cdot)\|^2_{\mathcal{J}(s)})ds$$

$$+C(\delta_3)\int_0^t(\|P(s)[\vec{u}(s,\ \cdot)]\|^2+|Q(s)[\vec{u}(s,\ \cdot)]|_{x=0}|^2)ds.$$

Here, we have also used the inequality: $(a+b+c)^{1/2}\leqq 3^{1/2}\ (a^{1/2}+b^{1/2}+c^{1/2})$ for $a,\ b,\ c\geqq 0$. Applying Gronwall's inequality implies that

$$\|\partial_t\vec{u}(t,\ \cdot)\|^2+\|\vec{u}(t,\ \cdot)\|^2_{\mathcal{J}(t)}$$

$$\leqq C(\delta_3)(\exp C(\delta_3,\ M_T(1))t)\int_0^t(\|P(s)[\vec{u}(s,\ \cdot)]\|^2+|Q(s)[\vec{u}(s,\ \cdot)]|_{x=0}|^2)ds.$$

Combining this and (8.4), we have the lemma in the case that $\varepsilon=0$. When $0<\varepsilon\leqq 1$, by the same arguments we can prove the lemma. So, we may omit the proof.

Replacing Theorem 6.8 by Lemma 8.2, we can prove Theorem 8.1 by the completely same arguments as in §7. Thus, we may omit the proof.

In the same manner as in the proof of Theorem 2 of Ikawa [3, p. 364-367], we can prove the following theorem.

THEOREM 8.3. *Let $T>0$, $n=1$ and L be an integer $\geqq 3$. In addition to* (A.8.1)-(A.8.4), *we assume that $A^0{}_a{}^b\in\mathcal{B}^L([0,\ T]\times\bar{\Omega})$, $H^k{}_a{}^b$ and $A^k{}_a{}^b\in\mathcal{B}^{L-1}([0,\ T]\times\bar{\Omega})$, $k=1,\ 2$, and $H^0{}_a{}^b$ and $B_a{}^b\in\mathcal{B}^L([0,\ T])$. If $\vec{u}_0\in H^L(\Omega)$, $\vec{u}_1\in H^{L-1}(\Omega)$, $\vec{f}\in C^{L-1}([0,\ T]\ ;\ L^2(\Omega))\cap E^{L-2}([0,\ T])$, $\vec{g}\in C^{L-1}([0,\ T])$ and the compatibility condition of order $L-2$ is satisfied, then the solution $\vec{u}$ of* (N) *belongs to $E^L([0,\ T])$.*

Appendix On some L^2-boundedness theorem for commutators

By using the Kumano-go, Muramatsu and Nagase theory on pseudo-differential operators, we derive L^2-estimates for some kind of commutators, which are used in §5. First, we quote fundamental results from Muramatsu [7, Part II, Chapter 2]. In what follows, we use the following notations. By $X,\ Y$, we denote points of $\boldsymbol{R}^n$. By Ξ, we denote the dual variables with respect to innerproduct: $X\cdot\Xi=\sum X_j\Xi_j$. For differentiations, we use the symbols: $D_X=-\sqrt{-1}(\partial/\partial X_1,\ \cdots,\ \partial/\partial X_n)$ and $D_\Xi=\sqrt{-1}(\partial/\partial\Xi_1,\ \cdots,\ \partial/\partial\Xi_n)$. Put $\lambda(\Xi)=(1+|\Xi|^2)^{1/2}$. Functions considered in this section are in general complex-valued. Let $\sigma\in\boldsymbol{R}^1$. If $a(X,\ \Xi,\ Y)\in\mathcal{B}^\infty(\boldsymbol{R}^{3n})$ satisfies

$$|D_X^\alpha D_\Xi^\beta D_Y^\delta a(X,\ \Xi,\ Y)|\leqq C(\alpha,\ \beta,\ \delta)\lambda(\Xi)^{\sigma-|\beta|}$$

for any multi-indices $\alpha,\ \beta,\ \delta$, we say that a is the symbol of order σ and write

$a\in S^{\sigma}$. For any symbol $a\in S^{\sigma}$, let us put

$$Au=a(X, D_X, Y)u=(2\pi)^{-n}Os-\iint e^{\sqrt{-1}(X\cdot\Xi-Y\cdot\Xi)}a(X, \Xi, Y)u(Y)d\Xi dY,$$

where $Os-\iint$ means the usual oscillatory integrals. We call A a pseudo-differential operator with symbol a. When $Au=Bu$ for any $u\in\mathcal{S}(\boldsymbol{R}^n)$, we shall write $a\cong b$, where A and B are pseudo-differential operators with symbols a and b, respectively.

THEOREM Ap. 1 (*expansion formula* [7, p. 311]). *Let* $\sigma\in\boldsymbol{R}$ *and* $a\in S^{\sigma}$. *Then,*

$$a\cong a(X, \Xi, X)+\sum_{|\alpha|=1}\sqrt{-1}\int_0^1 a_{\alpha}^{(\alpha)}(X, \Xi, X+\kappa(Y-X))d\kappa$$

where $a_{\alpha}^{(\alpha)}(X, \Xi, Y)=D_{\Xi}^{\alpha}D_Y^{\alpha}a(X, \Xi, Y)$.

THEOREM Ap. 2 (L^2-*boundedness* [7, p. 320]). (i) *Let* $\sigma>0$. *For* $a\in S^{-\sigma}$, *put*

$$N_{\sigma}^{0}(a)=\sup_{X,Y,\Xi}\max_{|\alpha|\leq 2k_1}|a^{(\alpha)}(X, \Xi, Y)|\lambda(\Xi)^{\sigma+|\alpha|}.$$

Here k_1 *is the least integer such that* $2k_1>n$ *and* $a^{(\alpha)}=D_{\Xi}^{\alpha}a$. *Then,*

$$\|Au\|_{\boldsymbol{R}^n}\leqq CN_{\sigma}^{0}(a)\|u\|_{\boldsymbol{R}^n} \quad \textit{for any } u\in L^2(\boldsymbol{R}^n).$$

(ii) *For any* $a(X, \Xi, Y)\in S_0$ *and* $\theta\in(0, 1)$, *put*

$$N_{\theta}(a)=\sup_{X,Y,Z,\Xi}\max_{|\alpha|\leq 2k_2}\Big\{|a^{(\alpha)}(X, \Xi, Y)|\lambda(\Xi)^{|\alpha|},$$
$$\frac{|a^{(\alpha)}(X, \Xi, Y)-a^{(\alpha)}(X, \Xi, Z)|\lambda(\Xi)^{|\alpha|}}{|Y-Z|^{\theta}},$$
$$\frac{|a^{(\alpha)}(X, \Xi, Z)-a^{(\alpha)}(Y, \Xi, Z)|\lambda(\Xi)^{|\alpha|}}{|X-Y|^{\theta}}\Big\}.$$

Here, k_2 *is the least integer such that* $2k_2>n+\theta$. *Let* A *be a pseudo-differential operator with symbol* a. *Then,*

$$\|Au\|_{\boldsymbol{R}^n}\leqq CN_{\theta}(a)\|u\|_{\boldsymbol{R}^n} \quad \textit{for any } u\in L^2(\boldsymbol{R}^n).$$

Now, we shall derive the estimates used to evaluate L^2-bounds of some kinds of commutators in § 5. In what follows, we use the same notations as in § 5. For the sake of simplicity, we write $X=(t, x')$, $Y=(s, y')$, $\Xi=(\tau, \xi')$. Put $\lambda_{\gamma}(\Xi)=(|\Xi|^2+\gamma^2)^{1/2}=(\tau^2+|\xi'|^2+\gamma^2)^{1/2}$. We shall say that $a(X, \Xi, Y, \gamma)$ is a symbol in S_{γ}^{σ} (we write $a\in S_{\gamma}^{\sigma}$) if a is in C^{∞} with respect to all variables (X, Ξ, Y, γ) and for multi-indices α, β, δ

$$|D_X^\alpha D_\Xi^\beta D_Y^\delta a(X, \Xi, Y, \gamma)| \leqq C(\alpha, \beta, \delta)\lambda_\gamma(\Xi)^{\sigma-|\beta|}.$$

Let us define the weighted pseudo-differential operator A with symbol $a \in S_\gamma^\sigma$ by

$$A(X, D_X, Y, \gamma)u = (2\pi)^{-n} e^{t\gamma} Os-\iint e^{\sqrt{-1}(X\cdot\Xi - Y\cdot\Xi)} a(X, \Xi, Y, \gamma) e^{-s\gamma} u(Y) d\Xi dY$$

for any $u \in \mathcal{S}_\gamma(\boldsymbol{R}^n) = \{u \in \mathcal{D}'(\boldsymbol{R}^n) | e^{-t\gamma} u(X) \in \mathcal{S}(\boldsymbol{R}^n)\}$. If we put

$$a_\gamma(X, \Xi, Y) = a(\gamma^{-1}X, \gamma\Xi, \gamma^{-1}Y, \gamma), \quad u_\gamma(X) = e^{-t} u(\gamma^{-1}X),$$

by the change of variables we have that

$$A(X, D_X, Y, \gamma)u = e^{t\gamma}(A_\gamma u_\gamma)(\gamma X) \tag{Ap.1}$$

where A_γ is a pseudo-differential operator with symbol $a_\gamma(X, \Xi, Y)$. By Theorem Ap. 1 and (Ap. 1) we have

LEMMA Ap. 3. *Let $\sigma \in \boldsymbol{R}$ and $a(X, \Xi, Y, \gamma) \in S_\gamma^\sigma$. Put*

$$a_1(X, \Xi, Y, \gamma) = a(X, \Xi, X, \gamma) - \sqrt{-1} \sum_{|\alpha|=1} \int_0^1 (D_\Xi^\alpha D_Y^\alpha a)(X, \Xi, X + \kappa(Y-X), \gamma) d\kappa.$$

Then, $Au = A_1 u$ for any $u \in \mathcal{S}_\gamma(\boldsymbol{R}^n)$. Here, A and A_1 are weighted pseudo-differential operators with symbols a and a_1, respectively.

Now, we put

$$N_{\sigma,\gamma}^0(a) = \max_{|\alpha| \leq 2k_1} \sup_{X,Y,\Xi} |a^{(\alpha)}(X, \Xi, Y, \gamma)| \lambda(\Xi)^{\sigma+|\alpha|}$$

for $\sigma > 0$ where k_1 is the same as in Theorem Ap. 1-(i). And also, we put

$$\begin{aligned} N_{\theta,\gamma}(a) = \max_{|\alpha| \leq 2k_2} \Big\{ &\sup_{X,Y,\Xi} |a^{(\alpha)}(X, \Xi, Y, \gamma)| \lambda_\gamma(\Xi)^{|\alpha|}, \\ &\sup_{X,Y,Z,\Xi} |a^{(\alpha)}(X, \Xi, Y, \gamma) - a^{(\alpha)}(Z, \Xi, Y, \gamma)| \lambda_\gamma(\Xi)^{|\alpha|} |X-Z|^{-\theta}, \\ &\sup_{X,Y,Z,\Xi} |a^{(\alpha)}(X, \Xi, Y, \gamma) - a^{(\alpha)}(X, \Xi, Z, \gamma)| \lambda_\gamma(\Xi)^{|\alpha|} |Y-Z|^{-\theta} \Big\} \end{aligned}$$

for $\theta \in (0, 1)$ and $\gamma \geqq 1$ where k_2 is the same as in Theorem Ap. 1-(ii). Since

$$N_\theta(a_\gamma) \leqq N_{\theta,\gamma}(a), \quad N_\sigma^0(a_\gamma) \leqq N_{\sigma,\gamma}^0(a),$$

by Theorem Ap. 2 and (Ap. 1) we have

LEMMA Ap. 4. (i) *Let $a(X, \Xi, Y, \gamma) \in S_\gamma^0$, $\theta \in (0, 1)$ and A be a weighted pseudo-differential operator with symbol a. Then,*

$$|Au|_{0,\gamma} \leqq C(\theta) N_{\theta,\gamma}(a) |u|_{0,\gamma}.$$

(ii) *Let $a(X, \Xi, Y, \gamma) \in S^{-\sigma}$ for some $\sigma > 0$. Then,*

$$|Au|_{0,\gamma}\leqq C(\sigma)N^0_{\sigma,\gamma}(a)|u|_{0,\gamma}.$$

Under these preparations, we can easily prove

THEOREM Ap. 5. *Let $a(X)\in\mathscr{B}^\infty(\boldsymbol{R}^n)$, $0<s\leqq 1$ and $\phi(\Xi,\gamma)$ be a function in $C^\infty(\boldsymbol{R}^n\times\{\gamma\in\boldsymbol{R}|\gamma\geqq 1\})$ such that for any α*

$$|D^\alpha_\Xi\phi(\Xi,\gamma)|\leqq C(\alpha)\lambda_\gamma(\Xi)^{s-|\alpha|}.$$

Let Φ be a weighted pseudo-differential operator with symbol ϕ and put $[a,\Phi]=a\Phi u-\Phi(au)$. Then,

$$|[a,\Phi]u|_{0,\gamma}\leqq\begin{cases}C(s)|a|_{\infty,1,\boldsymbol{R}^n}|u|_{0,\gamma}, & 0<s<1,\\ C(\mu)|a|_{\infty,1+\mu,\boldsymbol{R}^n}|u|_{0,\gamma}, & s=1\end{cases}$$

Here, μ is any small positive number.

Proof. Put

$$b(X,\Xi,Y,\gamma)=-\sqrt{-1}\sum_{|\alpha|-1}\int_0^1(D^\alpha_Xa)(X+\kappa(Y-X))d\kappa D^\alpha_\Xi\phi(\Xi,\gamma).$$

Then, $b\in S^{s-1}_\gamma$ and by Lemma Ap. 3 we have

$$[a,\Phi]u=Bu\quad\text{for any } u\in\mathcal{S}_\gamma(\boldsymbol{R}^n),$$

where B is a weighted pseudo-differential operator with symbol a. Thus, applying Lemma Ap. 4 and using the fact that $\mathcal{S}_\gamma(\boldsymbol{R}^n)$ is dense in $\mathcal{L}^2_\gamma=\{u\in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)|e^{-\gamma t}u(X)\in L^2(\boldsymbol{R}^n)\}$, we have the lemma.

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Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki 305
Japan.