TSUKUBA J. MATH.
Vol. 12 No. 1 (1988). 65–96

TAME TWO-POINT ALGEBRAS

(Dedicated to Professor Tosiro Tsuzuku on his sixtieth's birthday)

By

Mitsuo HOSHINO and Jun-ichi MIYACHI

Introduction.

Throughout this paper, we will work over a fixed algebraically closed field k. Let Λ be a finite dimensional basic algebra. We may consider Λ as a locally bounded k-category. As well known, any locally bounded k-category Λ is given by a quiver with relations, that is, there is a locally finite quiver Q such that $\Lambda \cong kQ/I$, where kQ is the path-category and I is an ideal of kQ generated by linear combinations of paths of length $\geqq 2$ (see [3] for details). A module over a locally bounded k-category Λ is a k-linear functor from Λ to the category of k-vector spaces, nemely, a representation of the quiver satisfying the relations if Λ is given by a quiver with relations. We will denote by $\mathrm{mod}\,\Lambda$ the category of all finite dimensional left Λ-modules.

In the present paper, we are interested in two-point algebras, namely, algebras which have just two non-isomorphic simple left modules. Our aim is to classify two-point algebras of certain classes according to their representation types. An algebra Λ is said to be representation-finite if there are only a finite number of pairwise non-isomorphic indecomposable objects in $\mathrm{mod}\,\Lambda$, to be wild if there is an exact embedding $\mathrm{mod}\,k\Omega \rightarrow \mathrm{mod}\,\Lambda$, where $k\Omega$ is the path-algebra of the quiver $\Omega : \circlearrowleft \cdot \circlearrowright$, which is a representation equivalence with the corresponding full subcategory of $\mathrm{mod}\,\Lambda$, and to be tame if Λ is neither representation-finite nor wild. There has been given the complete list of the maximal representation-finite two-point algebras [3].

Covering techniques ([1], [3], [5] and [6]) will play an indispensable role in deciding the representation type of a given algebra. For a certain class of algebras, by taking appropriate Galois coverings, the problem can be reduced to the calculation of vector space categories, which have been classified in [12] (see also [9]). On the other hand, we will come across an algebra which can be obtained as a quotient of a suitable Galois covering of the tame local algebra $\tau \circlearrowleft \cdot \circlearrowright \sigma$ with $\sigma^2 = \tau^2 = 0$ [11], thus is tame. The similar argument will also apply to the situation that there is a Galois covering of a given algebra which has a wild algebra as a finite quotient.

Received August 6, 1986.

1. Main Results.

In the present paper, we will consider two classes of two-point algebras, one is the class of the triangular matrix algebras, namely, the algebras of the form $\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ with A, B local, and the other is that of the distributive algebras of ordinary quiver $\cdot \leftrightarrows \cdot \circlearrowright$ (cf. [1]).

THEOREM A. *Let Λ be a finite dimensional two-point triangular matrix algebra. Then Λ is tame if and only if Λ is isomorphic to one of the algebras* (0)-(4_q) *in Table T or their duals.*

THEOREM B. *Let A be a finite dimensional basic distributive algebra of ordinary quiver $\cdot \rightleftarrows \cdot \circlearrowright$. Then Λ is tame if and only if Λ is isomorphic to one of the algebras* (5_q)-(11_q) *in Table T or their duals.*

During the preparation of the paper, the authors noticed that the algebra (4_q) in Table T is shown to be tame [15].

Table T

0)	$\cdot \rightrightarrows \cdot$		
1)	$\cdot \xrightarrow{\mu} \cdot \circlearrowright \alpha$	with	$\alpha^2\mu=\alpha^6=0$
2)	$\beta \circlearrowleft \cdot \xrightarrow{\mu} \cdot \circlearrowright \alpha$	with	$\alpha^2=\beta^2=0$
2′)		with	$\alpha\mu\beta=\alpha^2=\beta^2=0$
3_q)		with	$\alpha\mu-\mu\beta=\alpha^q\mu=\alpha^6=\beta^3=0, \quad q=2, 3$
4_q)		with	$\alpha\mu-\mu\beta=\alpha^q\mu=\alpha^4=\beta^4=0, \quad q=2, 3, 4$
5_q)	$\cdot \underset{\nu}{\overset{u}{\rightleftarrows}} \cdot \circlearrowright \alpha$	with	$\mu\nu-\alpha^2=\nu\alpha\mu=\alpha^q=0, \quad q=3, 4, 5$
$5'_q$)		with	$\mu\nu-\alpha^2=\nu\alpha\mu=\nu\alpha^2\mu=\alpha^q=0, \quad q=3, 4, 5$
$5''_q$)		with	$\mu\nu-\alpha^2=\nu\alpha\mu=\alpha^2\mu=\alpha^q=0, \quad q=3, 4$
$5'''_q$)		with	$\mu\nu-\alpha^2=\nu\alpha\mu=\alpha^2\mu=\nu\alpha^2=\alpha^q=0, \quad q=3, 4$
6_q)		with	$\mu\nu-\alpha^2=\nu\alpha\mu-\nu\alpha^2\mu=\alpha^q=0, \quad q=4, 5$ (only if char $k=3$)
7)		with	$\mu\nu-\alpha^3=\nu\mu=\alpha\mu=0$
8_q)		with	$\mu\nu-\alpha^3=\nu\mu=\alpha^2\mu=\nu\alpha^2=\alpha^q=0, \quad q=3, 4, 5$
$8'_q$)		with	$\mu\nu-\alpha^3=\nu\mu=\nu\alpha\mu=\alpha^2\mu=\nu\alpha^2=\alpha^q=0, \quad q=3, 4, 5$
9)		with	$\mu\nu-\alpha^3=\nu\mu-\nu\alpha\mu=\alpha^2\mu=\nu\alpha^2=\alpha^5=0$ (only if char $k=3$)

10) with $\mu\nu=\nu\mu=\nu\alpha\mu=\alpha^3=0$

10′) with $\mu\nu=\nu\mu=\nu\alpha\mu=\nu\alpha^2\mu=\alpha^3=0$

10″) with $\mu\nu=\nu\mu=\nu\alpha\mu=\alpha^2\mu=\alpha^3=0$

11_q) with $\mu\nu-\alpha^4=\alpha\mu=\nu\alpha^2=\alpha^q=0, \quad q=4, 5$

EXAMPLES. In Section 5, as an example, we will show that the following is tame [12]:

12) $\bullet \underset{\nu}{\overset{\mu}{\rightleftarrows}} \bullet \circlearrowright \alpha$ with $\alpha^2=\mu\nu=0$,

this is not distributive (cf. (W-11)).

There are several other algebras which have been known to be tame [4] (see also [7] and [11]):

13) $\bullet \underset{\nu}{\overset{\mu}{\rightleftarrows}} \bullet \circlearrowright \alpha$ with $\alpha^2=\nu\mu=0$

14) $\beta \circlearrowleft \bullet \underset{\nu}{\overset{\mu}{\rightleftarrows}} \bullet \circlearrowright \alpha$ with $\alpha^2=\beta^2=\mu\nu=\nu\mu=0$

15) with $\alpha\mu=\mu\beta=\nu\alpha=\beta\nu=0$

16) $$\begin{tikzcd} \bullet \arrow[r, bend left=50, "\mu_1"] \arrow[r, shift left, "\mu_2"] & \bullet \arrow[l, shift left, "\nu_2"] \arrow[l, bend left=50, "\nu_1"] \end{tikzcd}$$ with $\mu_1\nu_1=\mu_2\nu_2=\nu_1\mu_1=\nu_2\mu_2=0$

17) with $\mu_1\nu_1=\mu_2\nu_2=\nu_1\mu_2=\nu_2\mu_1=0$.

In particular, one can easily determine the finite dimensional basic tame algebras of ordinary quiver $$\begin{tikzcd} \bullet \arrow[r, shift left=2] \arrow[r] & \bullet \arrow[l, shift left=2] \end{tikzcd}$$.

In order to prove the "only if" parts of the theorems, we need the list of minimal wild algebras and that of maximal representation-finite algebras.

PROPOSITION 1. *The algebras* (0)-(18) *in Table W are wild. They are minimal, with the possible exception of* (11), *in the sense that no proper quotient of them is wild.*

PROPOSITION 2 (see [3]). *The algebras* (1)-(15_q) *in Table F are representation-finite. They are maximal in the sense that any finite dimensional basic representation-finite two-point algebra can be obtained as a quotient of one of them or their duals.*

Table W

0) $$\begin{tikzcd} \bullet \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \bullet \end{tikzcd}$$

1) $\bullet \xrightarrow{\mu} \bullet \, \begin{smallmatrix} \circlearrowright \alpha \\ \circlearrowright \beta \end{smallmatrix}$ with $\alpha\mu=\beta\mu=\alpha\beta=\beta\alpha=\alpha^2=\beta^2=0$

2) $\bullet \overset{\mu}{\underset{\nu}{\rightrightarrows}} \bullet \circlearrowright \alpha$ with $\alpha\mu=\alpha\nu=\alpha^2=0$
3) $\bullet \xrightarrow{\mu} \bullet \circlearrowright \alpha$ with $\alpha^2\mu=\alpha^7=0$
4) with $\alpha^3\mu=\alpha^4=0$
5) $\beta \circlearrowleft \bullet \xrightarrow{\mu} \bullet \circlearrowright \alpha$ with $\mu\beta=\alpha^3=\beta^2=0$
6) with $\alpha^2\mu=\mu\beta=\alpha^3=\beta^3=0$
7) with $\alpha^2\mu=\mu\beta=\alpha^4=\beta^2=0$
8) with $\alpha^2\mu=\alpha\mu\beta=\alpha^3=\beta^2=0$
9) with $\alpha\mu-\mu\beta=\alpha^2\mu=\alpha^5=\beta^4=0$
10) with $\alpha\mu-\mu\beta=\alpha^2\mu=\alpha^7=\beta^3=0$
11) $\bullet \overset{\mu}{\underset{\nu}{\rightleftarrows}} \bullet \circlearrowright \alpha$ with $\mu\nu-\alpha^2=\alpha^2\mu=\alpha^3=0$
12) with $\mu\nu-\alpha^3=\nu\mu=\nu\alpha\mu=\alpha^2\mu=\alpha^4=0$
13) with $\mu\nu=\alpha\mu=\alpha^3=0$
14) with $\mu\nu=\nu\mu=\alpha^2\mu=\alpha^3=0$
15) with $\mu\nu=\nu\alpha\mu=\alpha^2\mu=\nu\alpha^2=\alpha^3=0$
16) with $\mu\nu=\nu\mu=\alpha\mu=\nu\alpha^3=\alpha^4=0$
17) with $\mu\nu=\nu\mu=\nu\alpha\mu=\alpha^2\mu=\nu\alpha^2=\alpha^4=0$
18) with $\mu\nu=\nu\mu=\alpha\mu=\nu\alpha^2=\alpha^5=0$

Table F

1) $\bullet \xrightarrow{\mu} \bullet \circlearrowright \alpha$ with $\alpha^2\mu=\alpha^5=0$
2_q) $\beta \circlearrowleft \bullet \xrightarrow{\mu} \bullet \circlearrowright \alpha$ with $\alpha\mu=\mu\beta=\alpha^q=\beta^q=0$, $q\geqq 2$
3_q) with $\mu\beta=\alpha^2=\beta^q=0$, $q\geqq 2$
4_q) with $\alpha\mu-\mu\beta=\alpha^q=\beta^2=0$, $q\geqq 2$
5) with $\alpha\mu-\mu\beta=\alpha^5=\beta^3=0$
6) with $\alpha^2\mu-\mu\beta=\alpha^3=\beta^2=0$
7_q) $\bullet \overset{\mu}{\underset{\nu}{\rightleftarrows}} \bullet$ with $(\mu\nu)^q=(\nu\mu)^q=0$, $q\geqq 1$
8) $\bullet \overset{\mu}{\underset{\nu}{\rightleftarrows}} \bullet \circlearrowright \alpha$ with $\mu\nu-\alpha^2=\alpha\mu=0$
9) with $\mu\nu-\alpha^2=\nu\mu=0$
10) with $\mu\nu-\alpha^2=\nu\mu-\nu\alpha\mu=\alpha^4=0$ (only if char $k=2$)
11) with $\mu\nu=\nu\alpha\mu=\alpha^2=0$

12) with $\mu\nu-\alpha^3=\alpha\mu=\nu\alpha^2=0$

13) with $\mu\nu=\nu\mu=\alpha\mu=\alpha^3=0$

14) with $\mu\nu-\alpha^4=\nu\mu=\alpha\mu=\nu\alpha^2=0$

15_q) with $\mu\nu-\alpha^q=\alpha\mu=\nu\alpha=0,\ q\geqq 4$.

REMARK 1. For an integer $m\geqq 1$, denote by P^m the linearly ordered set with m elements, and for $m, n\geqq 1$, consider $P^m\times P^n$ as a partially ordered set by componentwise order. Then the representation type of $P^m\times P^n$ is finite if $1/m+1/n>1/2$, tame if $1/m+1/n=1/2$ and wild if $1/m+1/n<1/2$ (cf. [14]). This is also the case with the algebra $\beta\circlearrowright\cdot\xrightarrow{\mu}\cdot\circlearrowleft\alpha$ with $\alpha\mu-\mu\beta=\alpha^m=\beta^n=0$.

REMARK 2. Let Λ be a local Nakayama algebra of length $q\geqq 2$. Then the triangular matrix algebra $\begin{bmatrix}\Lambda & \Lambda\\ 0 & \Lambda\end{bmatrix}$ can be given by the quiver $\beta\circlearrowright\cdot\xrightarrow{\mu}\cdot\circlearrowleft\alpha$ with relations $\alpha\mu-\mu\beta=\alpha^q=\beta^q=0$. This is tame if and only if $q=4$. In general, for a connected self-injective algebra Λ, the triangular matrix algebra $\begin{bmatrix}\Lambda & \Lambda\\ 0 & \Lambda\end{bmatrix}$ is tame if and only if Λ is representation-finite of Dynkin class A_3 (see [8] and [15]).

2. Preliminaries.

In this section, we will recall some basic definitions and results (see [1], [3], [5], [6], [10], [12] and [13]).

2.1. Locally Bounded Categories.

A locally bounded category Λ is a k-category such that: a) distinct objects are not isomorphic; b) for each $x\in\Lambda$, the algebra $\Lambda(x, x)$ is local; c) for each $x\in\Lambda$, $\sum_{y\in\Lambda}[\Lambda(x, y): k]$ and $\sum_{y\in\Lambda}[\Lambda(y, x): k]$ are finite [1]. The support supp M of a Λ-module M is the full subcategory of Λ consisting of the objects $x\in\Lambda$ such that $M(x)\neq 0$. The dimension vector of a Λ-module M is the family $\underline{\dim}\, M=\{[M(x): k]\}_{x\in\Lambda}$. Let Γ_i $(i\in I)$ be a family of full subcategories of Λ. Denote by $\bigcup_{i\in I}\Gamma_i$ the full subcategory of Λ consisting of the objects of the Γ_i. For a family of objects $x_i\in\Lambda$ $(i\in I)$, we denote by $\{x_i\}_{i\in I}$ the full subcategory consisting of the objects x_i. Λ is said to be locally support-finite if for each $x\in\Lambda$, $\bigcup_{\substack{M(x)\neq 0\\ M\in\operatorname{ind}\Lambda}}\operatorname{supp} M$ is finite [5].

2.2. Galois Coverings.

Let Λ be a connected locally bounded category and G a group of k-linear automorphisms of Λ. Then G acts naturally on mod Λ by the left. We assume that the action of G on Λ is free, namely, $gx\neq x$ for any $g\in G\setminus\{1\}$ and any $x\in\Lambda$. Following [6], we can consider the quotient Λ/G and the Galois covering $F:\Lambda\rightarrow\Lambda/G$. Then we have the push down functor F_λ : mod $\Lambda\rightarrow$mod Λ/G which is left adjoint to the induced functor $F.$: mod $\Lambda/G\rightarrow$mod Λ. If G acts freely on ind Λ, namey, ${}^gM\neq M$ for any $g\in G\setminus\{1\}$ and any $M\in$ind Λ, then F_λ preserves the Auslander-Reiten sequences. We will freely use the following results.

PROPOSITION 3 (see [6]). *Let S be a quotient category of Λ with the natural embedding* mod $S\rightarrow$mod Λ, *and* $L=\{M\in\text{ind } S\mid {}^gM\notin\text{ind } S$ *for any* $g\in G\setminus\{1\}\}$. *Then there exists a set-theoretic injection* $L\rightarrow$ind Λ/G. *In particular, in case L is cofinite in* ind S, *the following hold.*

(1) *If Λ/G is tame, so is S unless it is representation-finite.*

(2) *If S is wild, so is Λ/G.*

PROPOSITION 4 ([5]). *If Λ is locally support-finite and if G acts freely on* ind Λ, *then the push down functor* F_λ : mod $\Lambda\rightarrow$mod Λ/G *is dense. In particular, if Λ is tame, so is Λ/G.*

In what follows, we will deal only with a full subcategory A of a Galois covering U which is in fact a quotient category, thus we may consider mod A as a full subcategory of mod U by the natural embedding.

2.3. Vector Space Categories.

A vector space category K is an additive k-category together with a faithful functor $|\ |:K\rightarrow$mod k such that every idempotent in K splits. Given a vector space category K, its subspace category $\mathcal{U}(K)$ is defined as follows: its objects are triples of the form (U, X, ϕ), where U is a k-space, X is an object in K and $\phi: U\rightarrow|X|$ is a k-linear map. A homomorphism from (U, X, ϕ) to (U', X', ϕ') is given by a pair (α, β), where $\alpha: U\rightarrow U'$ is k-linear, $\beta: X\rightarrow X'$ is a morphism in K such that $|\beta|\phi=\phi'\alpha$. Given a poset S, considered as a category, addkS is a vector space category. Conversely, assume that K is a vector space category consisting only of 1-dimensional indecomposable objects, then K is of the form add kS for some poset S.

Let Λ be a one-point extension algebra of R by M, then a Λ-module is given by a triple $({}_kU, {}_RX, \phi: {}_RM\otimes_kU\rightarrow{}_RX)$. It is well known that $\mathcal{U}$(Hom $(M$, mod $R))$ is representation equivalent to the full subcategory of mod Λ consisting of the Λ-modules without non-zero direct summands of the form $(k, 0, 0)$ or $(0, Y, 0)$ with Hom $(M, Y)=0$. In case R is tame, if the vector space category Hom $(M$, mod $R)$ is tame, so is Λ.

3. Classification.

In Section 4, we will prove that the algebras in Table W are wild, and in Section 5 we will prove that the algebras in Table T are tame. We have only to consider the algebras of ordinary quiver $\bullet \longrightarrow \bullet \circlearrowright$, $\circlearrowright \bullet \longrightarrow \bullet \circlearrowright$ or $\bullet \rightleftarrows \bullet \circlearrowright$. Given an algebra Λ of the classes stated in the theorems, we will show that one of the following cases occurs: 1) Λ is isomorphic to a quotient of one of the algebras in Table F or their duals; 2) Λ is isomorphic to one of the algebras in Table T or their duals; and 3) Λ has a quotient isomorphic to one of the algebras in Table W or their duals. These are clearly pairwse inconsistent.

I) Λ of ordinary quiver $\bullet \xrightarrow{\mu} \bullet \circlearrowright \alpha$.

Suppose $\alpha^m \mu = \alpha^n = 0$, $m \leqq n$.

i) If $m=1$, then Λ is a quotient of (F-2_n).
ii) If $m=2$ and $n \leqq 5$, then Λ is a quotient of (F-1).
iii) If $m=2$ and $n=6$, then $\Lambda \cong$(T-1).
iv) If $m=2$ and $n \geqq 7$, then Λ has (W-3) as a quotient.
v) If $m=n=3$, then Λ is a quotient of the dual of (F-13).
vi) If $m \geqq 3$ and $n \geqq 4$, then Λ has (W-4) as a quotient.

II) Λ of ordinary quiver $\beta \circlearrowright b \xrightarrow{\mu} a \circlearrowleft \alpha$.

Suppose $\alpha^m = \beta^n = 0$. Let $A = k[\alpha]$, $B = k[\beta]$ and $M = \Lambda(b, a)$. We may assume $\dim A\mu \geqq \dim \mu B$. Note that if $M \neq A\mu$ then $\mu\beta \notin A\mu$.

1) If $\dim M=1$, then Λ is a quotient of (F-2_q), $q=\max\{m, n\}$.

2) Suppose $\dim M=2$. Then $M=A\mu$, and $\mu\beta = x\alpha\mu$ for some $x \in k$.

2.1) The case $x=0$:
i) If $m=2$, then $\Lambda=$(F-3_n).
ii) If $m=3$ and $n=2$, then Λ is a quotient of (F-6).
iii) If $m=3$ and $n \geqq 3$, then Λ has (W-6) as a quotient.
iv) If $m \geqq 4$, then Λ has (W-7) as a quotient.

2.2) The case $x \neq 0$: Replacing α with $x\alpha$, we can assume $\alpha\mu - \mu\beta = 0$. We may also assume $m \geqq n$.
i) If $n=2$, then $\Lambda \cong$(F-4_m).
ii) If $n=3$ and $m \leqq 5$, then Λ is a quotient of (F-5).
iii) If $n=3$ and $m=6$, then $\Lambda \cong$(T-3_2).
iv) If $n=3$ and $m \geqq 7$, then Λ has (W-10) as a quotient.
v) If $n=m=4$, then $\Lambda \cong$(T-4_2).
vi) If $n \geqq 4$ and $m \geqq 5$, then Λ has (W-9) as a quotient.

3) Suppose dim $M=3$, $M\neq A\mu$ and $mn>4$. Then, as a quotient, Λ has either (W-8) or its dual.

4) Suppose dim $M\geqq 3$, $M\neq A\mu$ and $m=n=2$.
i) If dim $M=3$, then $\Lambda\cong$(T-2′).
ii) If dim $M=4$, then $\Lambda\cong$(T-2).

5) Suppose dim $M=3$ and $M=A\mu$. Then $\mu\beta=x\alpha\mu+y\alpha^2\mu$ for some $x, y\in k$.
5.1) The case $x=y=0$: Λ has (W-5) as a quotient.

5.2) The case $x=0$ but $y\neq 0$: Replacing β with $y^{-1}\beta$, we can assume $\alpha^2\mu-\mu\beta=0$.
i) If $m=3$ and $n=2$, then $\Lambda\cong$(F-6).
ii) If $m=3$ and $n\geqq 3$, then Λ has (W-6) as a quotient.
iii) If $m\geqq 4$, then Λ has (W-7) as a quotient.

5.3) The case $x\neq 0$: Replacing α with $x\alpha+y\alpha^2$, we can assume $\alpha\mu-\mu\beta=0$. We may also assume $m\geqq n$.
i) If $n=3$ and $m\leqq 5$, then Λ is a quotient of (F-5).
ii) If $n=3$ and $m=6$, then $\Lambda\cong$(T-3_3).
iii) If $n=3$ and $m\geqq 7$, then Λ has (W-10) as a quotient.
iv) If $n=m=4$, then $\Lambda\cong$(T-4_3).
v) If $n\geqq 4$ and $m\geqq 5$, then Λ has (W-9) as a quotient.

6) Suppose dim $M=4$ and $M=A\mu$. Then $\mu\beta=x\alpha\mu+y\alpha^2\mu+z\alpha^3\mu$ for some $x, y, z\in k$.
6.1) The case $x=0$: Λ has (W-7) as a quotient.

6.2) The case $x\neq 0$: Replacing α with $x\alpha+y\alpha^2+z\alpha^3$, we can assume $\alpha\mu-\mu\beta=0$. We may also assume $m\geqq n$.
i) If $m=n=4$, then $\Lambda\cong$(T-4_4).
ii) If $m\geqq 5$ and $n\geqq 4$, then Λ has (W-9) as a quotient.

7) Suppose dim $M\geqq 5$ and $M=A\mu$. Then $\mu\beta=\Sigma_{i=1}^{d}x_i\alpha^i\mu$, where $d=\dim M-1$, for some $x_i\in k$, $1\leqq i\leqq d$.
7.1) The case $x_1=0$: Λ has (W-7) as a quotient.

7.2) The case $x_1\neq 0$: Replacing α with $\Sigma_{i=1}^{d}x_i\alpha^i$, we can assume $\alpha\mu-\mu\beta=0$. Then Λ has (W-9) as a quotient.

III) Λ of ordinary quiver $b\underset{\nu}{\overset{\mu}{\rightleftarrows}}a\circlearrowleft\alpha$.

Suppose $\alpha^n=0$. Let $A=k[\alpha]$. We may restrict ourselves to the case $\Lambda(a, a)=A$. Thus $\Lambda(b, a)=A\mu$ and $\Lambda(a, b)=\nu A$. We may also assume dim $A\mu$

$\leqq \dim \nu A$. In case $\mu\nu \neq 0$, there are some $f(x) \in k[x]$ with $f(0) \neq 0$ and some $m \geqq 1$ such that $\mu\nu = f(\alpha)\alpha^m$. Replacing μ with $f(\alpha)^{-1}\mu$, we can assume $\mu\nu - \alpha^m = 0$.

1) Suppose $\dim A\mu = \dim \nu A = 1$. Then Λ is a quotient of (F-8), (F-12) or (F-15_q).

In what follows, we assume $\dim \nu A \geqq 2$.

2) Suppose $\mu\nu - \alpha^2 = 0$.
2.1) If $\dim A\mu = 1$, then Λ is a quotient of (F-8).

2.2) Suppose $\dim A\mu = 2$.
2.2.1) The case $\dim \nu A\mu = 0$: Λ is a quotient of (F-9).
2.2.2) The case $\dim \nu A\mu = 1$: We have $x\nu\mu + y\nu\alpha\mu = 0$ for some $x, y \in k$ with with $(x, y) \neq 0$. In case $xy \neq 0$, by replacing α with $-x^{-1}y\alpha$ and μ with $(x^{-1}y)^2\mu$, we can assume $\nu\mu - \nu\alpha\mu = 0$.
i) If $\nu\mu = 0$, then Λ is a quotient of (F-9).
ii) If $\nu\alpha\mu = 0$ and $n = 2$, then $\Lambda \cong$ (F-11).
iii) If $\nu\alpha\mu = 0$ and $n \geqq 3$, then $\Lambda \cong$ (T-$5''_n$) or (T-$5'''_n$).
iv) If $\nu\mu - \nu\alpha\mu = 0$, then Λ is a quotient of (F-9) or (F-10).
2.2.3) The case $\dim \nu A\mu = 2$: Λ is not distributive. Notice however that Λ has (W-11) as a quotient if $\dim \nu A \geqq 3$.

2.3) Suppose $\dim A\mu = 3$. Then $\nu\mu \notin (k\nu\alpha\mu + k\nu\alpha^2\mu)$.
2.3.1) The case $\dim \nu A\mu = 1$: $\Lambda \cong$ (T-$5'_n$).
2.3.2) The case $\dim \nu A\mu = 2$: We have $x\nu\alpha\mu + y\nu\alpha^2\mu = 0$ for some $x, y \in k$ with $(x, y) \neq 0$. In case $xy \neq 0$, by replacing α with $-x^{-1}y\alpha$ and μ with $(x^{-1}y)^2\mu$, we can assume $\nu\alpha\mu - \nu\alpha^2\mu = 0$.
i) If $\nu\alpha\mu = 0$, then $\Lambda \cong$ (T-5_n).
ii) If $\nu\alpha^2\mu = 0$, then Λ has (W-11) as a quotient.
iii) If $\nu\alpha\mu - \nu\alpha^2\mu = 0$ and $n = 3$, then $\Lambda \cong$ (T-5_3).
iv) If $\nu\alpha\mu - \nu\alpha^2\mu = 0$ and $n = 4$, then $\Lambda \cong$ (T-5_4) or (T-6_4).
v) If $\nu\alpha\mu - \nu\alpha^2\mu = 0$ and $n = 5$, then $\Lambda \cong$ (T-5_5) or (T-6_5).
2.3.3) The case $\dim \nu A\mu = 3$: Λ has (W-11) as a quotient.

2.4) Suppose $\dim A\mu \geqq 4$. Then, $\dim(k\nu\mu + k\nu\alpha\mu) = 2$ and $(k\nu\mu + k\nu\alpha\mu) \cap (k\nu\alpha^2\mu + k\nu\alpha^3\mu + \cdots) = 0$. Thus, Λ has (W-11) as a quotient.

3) Suppose $\mu\nu - \alpha^3 = 0$ and $n \geqq 3$.
3.1) Suppose $\dim A\mu = 1$.
3.1.1) The case $\dim \nu A = 2$: Λ is a quotient of (F-12).

3.1.2) The case $\dim \nu A=3$:
i) If $n=3$ and $\dim \nu A\mu=0$, then $\Lambda\cong$(F-13).
ii) If $n=4$ and $\dim \nu A\mu=0$, then $\Lambda\cong$(T-7).
iii) If $\dim \nu A\mu=1$, then Λ has (W-13) as a quotient.

3.2) Suppose $\dim A\mu\geqq 2$, $\dim \nu A\geqq 3$ and $n\geqq 4$. Then Λ has (W-12) as a quotient.

3.3) Suppose $\dim A\mu=2$.
3.3.1) The case $\dim \nu A\mu=0$:
i) If $\dim \nu A=2$, then $\Lambda\cong$(T-$8'_n$).
ii) If $\dim \nu A=n=3$, then $\Lambda\cong$(T-10″).
3.3.2) The case $\dim \nu A\mu=1$: We have $x\nu\mu+y\nu\alpha\mu=0$ for some $x, y\in k$ with $(x, y)\neq 0$. In case $xy\neq 0$, by replacing α with $-x^{-1}y\alpha$ and μ with $-(x^{-1}y)^3\mu$, we can assume $\nu\mu-\nu\alpha\mu=0$.
i) If $\nu\mu=0$ and $\dim \nu A=2$, then $\Lambda\cong$(T-8_n).
ii) If $\nu\mu=0$ and $\dim \nu A=n=3$, then $\Lambda\cong$(W-14).
iii) If $\nu\alpha\mu=0$, then Λ has (W-15) as a quotient.
iv) If $\nu\mu-\nu\alpha\mu=0$ and $\dim \nu A=2$, then $\Lambda\cong$(T-8_n) or (T-9).
v) If $\nu\mu-\nu\alpha\mu=0$ and $\dim \nu A=n=3$, then $\Lambda\cong$(W-14).
3.3.3) The case $\dim \nu A\mu=2$: Λ has (W-15) as a quotient.

3.4) Suppose $\dim A\mu=\dim \nu A=n=3$.
3.4.1) The case $\dim \nu A\mu=0$: $\Lambda\cong$(T-10′).
3.4.2) The case $\dim \nu A\mu=1$: Consider first the case $\nu\mu=0$. Then $x\nu\alpha\mu+y\nu\alpha^2\mu=0$ for some $x, y\in k$ with $(x, y)\neq 0$. In case $x\neq 0$, by replacing α with $x\alpha+y\alpha^2$, we can assume $\nu\alpha\mu=0$. Next, suppose $\nu\mu\neq 0$. In case $\nu\alpha\mu\neq 0$, we have $\nu\mu+x\nu\alpha\mu=0$ for some $x\in k\setminus\{0\}$. Replacing μ with $\mu+x\alpha\mu$, we can reduce the case to $\nu\mu=0$. Also, in case $\nu\alpha^2\mu\neq 0$, we have $\nu\mu+x\nu\alpha^2\mu$ for some $x\in k\setminus\{0\}$, and by replacing μ with $\mu+x\alpha^2\mu$, we can reduce the case to $\nu\mu=0$.
i) If $\nu\mu=\nu\alpha\mu=0$, then $\Lambda\cong$(T-10).
ii) If $\nu\mu=\nu\alpha^2\mu=0$, then Λ has (W-14) as a quotient.
iii) If $\nu\alpha\mu=\nu\alpha^2\mu=0$, then Λ has (W-15) as a quotient.
3.4.3) The case $\dim \nu A\mu=2$: We have $x\nu\mu+y\nu\alpha\mu+z\nu\alpha^2\mu=0$ for some $x, y, z \in k$ with $(x, y, z)\neq 0$. In case $(xy, xz)\neq 0$, by replacing μ with $\mu+x^{-1}y\alpha\mu+x^{-1}z\alpha^2\mu$, we can assume $\nu\mu=0$.
i) If $\nu\mu=0$, then Λ has (W-14) as a quotient.
ii) If $\nu\mu\neq 0$, then Λ has (W-13) as a quotient.
3.4.4) The case $\dim \nu A\mu=3$: Λ has (W-13) as a quotient.

4) Suppose $\mu\nu-\alpha^4=0$ and $n\geqq 4$.

4.1) If $\dim \nu A \geqq 3$, then Λ has (W-16) as a quotient.

4.2) If $\dim A\mu = \dim \nu A = 2$, then Λ has (W-17) as a quotient.

4.3) Suppose $\dim A\mu = 1$ and $\dim \nu A = 2$.

4.3.1) The case $\dim \nu A\mu = 0$: Λ is a quotient of (F-14).

4.3.2) The case $\dim \nu A\mu = 1$: $\Lambda \cong (\text{T-11}_n)$.

5) Sppose $\mu\nu - \alpha^m = 0$ and $n \geqq m \geqq 5$. Then Λ has (W-18) as a quotient.

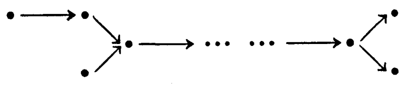
4. Wild Algebras.

To begin with, let us consider the following quivers without relations:

$\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet$;

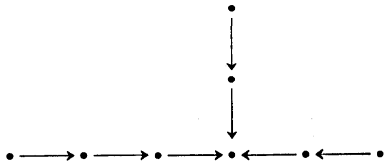
$\bullet \longrightarrow \bullet \rightrightarrows \bullet$;

$\tilde{\tilde{D}}_n$



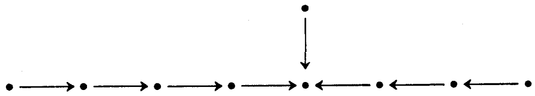
;

$\tilde{\tilde{E}}_6$



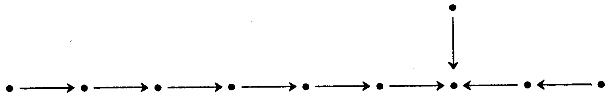
;

$\tilde{\tilde{E}}_7$



; and

$\tilde{\tilde{E}}_8$



.

These are well known to be wild. In fact, for a representation $\beta \circlearrowleft V \circlearrowright \alpha$ of the quiver Ω: $\circlearrowleft\bullet\circlearrowright$ by defining the representation

$$\begin{tikzcd} V \arrow[r, bend left, "\alpha"] \arrow[r, "\beta"] \arrow[r, bend right, "1"'] & V \end{tikzcd}$$

of the quiver $\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet$, we obtain a full exact embedding. Next, for a representation $\begin{tikzcd} U \arrow[r, bend left, "\alpha"] \arrow[r, "\beta"] \arrow[r, bend right, "\gamma"'] & V \end{tikzcd}$ of the quiver $\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet$, by defining the representationn

$$\begin{tikzcd} U \arrow[r, "\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}"] & V^3 \arrow[r, shift left, "\begin{bmatrix}E_3\\ 0\end{bmatrix}"] \arrow[r, shift right, "\begin{bmatrix}0\\ E_3\end{bmatrix}"'] & V^4 \end{tikzcd}$$

of the quiver $\bullet \longrightarrow \bullet \rightrightarrows \bullet$, we also obtain a full exact embedding. Finally, given a representation $U \xrightarrow{\alpha} V \overset{\beta}{\underset{\gamma}{\rightrightarrows}} W$ of the quiver $\bullet \longrightarrow \bullet \rightrightarrows \bullet$, following [2], let us construct the following representations:

$$\begin{tikzcd}
U \arrow[r, "\alpha"] & V \arrow[dr, "\begin{bmatrix}\beta\\ \gamma\end{bmatrix}"] & & & & & & W \\
& & W^2 \arrow[r, "E_2"] & W^2 \arrow[r, "E_2"] & \cdot\cdot & \cdot\cdot \arrow[r, "E_2"] & W^2 \arrow[ur, "{[1\ \ 0]}"] \arrow[dr, "{[0\ \ 1]}"'] & \\
& W \arrow[ur, "\begin{bmatrix}1\\ 1\end{bmatrix}"] & & & & & & W
\end{tikzcd} ;$$

$$\begin{tikzcd}
& & & W \arrow[d, "\begin{bmatrix}0\\ 1\end{bmatrix}"] & & \\
& & & W^2 \arrow[d, "\begin{bmatrix}1 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix}"] & & \\
U \arrow[r, "\alpha"'] & V \arrow[r, "\begin{bmatrix}\beta\\ \gamma\end{bmatrix}"'] & W^2 \arrow[r, "\begin{bmatrix}E_2\\ 0\end{bmatrix}"'] & W^3 & W^2 \arrow[l, "\begin{bmatrix}0\\ E_2\end{bmatrix}"] & W \arrow[l, "\begin{bmatrix}0\\ 1\end{bmatrix}"]
\end{tikzcd} ;$$

$$\begin{tikzcd}
& & & & W^2 \arrow[d, "\begin{pmatrix}1 & 1\\ 1 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}"] & & & \\
U \arrow[r, "\alpha"'] & V \arrow[r, "\begin{bmatrix}\beta\\ \gamma\end{bmatrix}"'] & W^2 \arrow[r, "\begin{bmatrix}E_2\\ 0\end{bmatrix}"'] & W^3 \arrow[r, "\begin{bmatrix}E_3\\ 0\end{bmatrix}"'] & W^4 & W^3 \arrow[l, "\begin{bmatrix}0\\ E_3\end{bmatrix}"] & W^2 \arrow[l, "\begin{bmatrix}0\\ E_2\end{bmatrix}"] & W \arrow[l, "\begin{bmatrix}0\\ 1\end{bmatrix}"]
\end{tikzcd} ; \text{ and}$$

$$\begin{tikzcd}
& & & & & & W^3 \arrow[d, "\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}"] & & \\
U \arrow[r, "\alpha"'] & V \arrow[r, "\begin{bmatrix}\beta\\ \gamma\end{bmatrix}"'] & W^2 \arrow[r, "\begin{bmatrix}E_2\\ 0\end{bmatrix}"'] & W^3 \arrow[r, "\begin{bmatrix}E_3\\ 0\end{bmatrix}"'] & W^4 \arrow[r, "\begin{bmatrix}E_4\\ 0\end{bmatrix}"'] & W^5 \arrow[r, "\begin{bmatrix}E_5\\ 0\end{bmatrix}"'] & W^6 & W^4 \arrow[l, "\begin{bmatrix}0\\ E_4\end{bmatrix}"] & W^2 \arrow[l, "\begin{bmatrix}0\\ E_2\end{bmatrix}"]
\end{tikzcd}$$

of the quivers $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ respectively. It is not difficult to see that these definitions induce full exact embeddings respectively. It should be noted that the above constructions are due to the indecomposable representations of the corresponding Dynkin quivers whose dimension types are the maximal roots of the corresponding Dynkin diagrams. In fact, if $S=\begin{bmatrix} R & M \\ 0 & k \end{bmatrix}$ is the one-point extension algebra, and if $N\in\operatorname{ind} R$ has endomorphism ring k, let $\{\alpha_1, \cdots\cdots, \alpha_d\}$ be a fixed k-basis of $\operatorname{Hom}_R(M, N)$. Then, for a representation

$$\begin{tikzcd} U \arrow[r, shift left=3, "\phi_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\phi_d"'] & V \end{tikzcd}$$

of the quiver Γ: $\bullet \substack{\longrightarrow \\ \vdots \\ \longrightarrow} \bullet$ (d arrows), by defining the S-module

$$({}_kU,\ {}_RN\otimes_k V,\ \textstyle\sum_{i=1}^{d}\alpha_i\otimes\phi_i : {}_RM\otimes_k U \longrightarrow {}_RN\otimes_k V),$$

we obtain a full exact embedding $\operatorname{mod} k\Gamma \longrightarrow \operatorname{mod} S$ (cf. [13]).

In order to prove wildness of the algebras (W-1)-(W-18), we will show that for each of them there is some Galois covering U having one of the above wild algebras of their concealments as a finite quotient.

(W-1) $$\begin{tikzcd} b \arrow[r, "\mu"] & a \arrow[loop above, "\alpha"] \arrow[loop below, "\beta"] \end{tikzcd}$$ with $\alpha\mu=\beta\mu=\alpha\beta=\beta\alpha=\alpha^2=\beta^2=0$.

Take the following Galois covering U with Galois group $\cong Z$:

$$\begin{tikzcd} \cdots & b_{-1} \arrow[d, "\mu_{-1}"] & b_0 \arrow[d, "\mu_0"] & b_1 \arrow[d, "\mu_1"] & \cdots \\ {} \arrow[r, shift left] \arrow[r, shift right] & a_{-1} \arrow[r, shift left, "\alpha_{-1}"] \arrow[r, shift right, "\beta_{-1}"'] & a_0 \arrow[r, shift left, "\alpha_0"] \arrow[r, shift right, "\beta_0"'] & a_1 \arrow[r, shift left] \arrow[r, shift right] & {} \end{tikzcd}$$

with $\alpha_i\mu_i=\beta_i\mu_i=\alpha_{i+1}\beta_i=\beta_{i+1}\alpha_i=\alpha_{i+1}\alpha_i=\beta_{i+1}\beta_i=0$ for all $i\in Z$. Then, as a quotient, U has the following:

$$\begin{tikzcd} & b_1 \arrow[d, "\mu_1"] \\ a_0 \arrow[r, shift left, "\alpha_0"] \arrow[r, shift right, "\beta_0"'] & a_1 \end{tikzcd} .$$

(W-2) $$\begin{tikzcd} b \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\nu"'] & a \arrow[loop right, "\alpha"] \end{tikzcd}$$ with $\alpha\mu=\alpha\nu=\alpha^2=0$.

Take the following Galois covering U with Galois group $\cong Z$:

$$\begin{tikzcd} \cdots & b_{-1} \arrow[d, shift right, "\nu_{-1}"'] \arrow[d, shift left, "\mu_{-1}"] & b_0 \arrow[d, shift right, "\nu_0"'] \arrow[d, shift left, "\mu_0"] & b_1 \arrow[d, shift right, "\nu_1"'] \arrow[d, shift left, "\mu_1"] & \cdots \\ {} \arrow[r] & a_{-1} \arrow[r, "\alpha_{-1}"'] & a_0 \arrow[r, "\alpha_0"'] & a_1 \arrow[r] & {} \end{tikzcd}$$

with $\alpha_i\mu_i=\alpha_i\nu_i=\alpha_{i+1}\alpha_i=0$ for all $i\in Z$. Then, as a quotient, U has the following:

$$\begin{tikzcd}
 & b_1 \arrow[d, shift right, swap, "\nu_1"] \arrow[d, shift left, "\mu_1"] \\
a_0 \arrow[r, swap, "\alpha_0"] & a_1
\end{tikzcd} .$$

(W-3) $b \xrightarrow{\mu} a \circlearrowleft \alpha$ with $\alpha^2\mu=\alpha^7=0$.

The universal Galois covering with Galois group$\cong Z$ has as a quotient the following:

$$\begin{tikzcd}
 & & & & b_4 \arrow[d, "\mu_4"] & b_5 \arrow[d, "\mu_5"] & b_6 \arrow[d, "\mu_6"] \\
a_0 \arrow[r, swap, "\alpha_0"] & a_1 \arrow[r, swap, "\alpha_1"] & a_2 \arrow[r, swap, "\alpha_2"] & a_3 \arrow[r, swap, "\alpha_3"] & a_4 \arrow[r, swap, "\alpha_4"] & a_5 \arrow[r, swap, "\alpha_5"] & a_6
\end{tikzcd}$$

with $\alpha_5\alpha_4\mu_4=0$. This is a concealed hereditary algebra of type $\tilde{\tilde{E}}_8$.

(W-4) $b \xrightarrow{\mu} a \circlearrowleft \alpha$ with $\alpha^3\mu=\alpha^4=0$.

The universal Galois covering with Galois group $\cong Z$ has as a quotient the following:

$$\begin{tikzcd}
 & b_1 \arrow[d, "\mu_1"] & b_2 \arrow[d, "\mu_2"] & b_3 \arrow[d, "\mu_3"] \\
a_0 \arrow[r, swap, "\alpha_1"] & a_1 \arrow[r, swap, "\alpha_2"] & a_2 \arrow[r, swap, "\alpha_3"] & a_3
\end{tikzcd} .$$

(W-5) $\beta \circlearrowright b \xrightarrow{\mu} a \circlearrowleft \alpha$ with $\mu\beta=\alpha^3=\beta^2=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$, the free group on two generators, has as a quotient the following:

$$\begin{tikzcd}
 & & & b_y & & \\
 & & & b_e \arrow[u, "\beta_e"] \arrow[d, "\mu_e"] & & \\
b_{yx} & b_x \arrow[l, swap, "\beta_x"] \arrow[r, swap, "\mu_x"] & a_x & a_e \arrow[l, "\alpha_e"] & a_{x^{-1}} \arrow[l, "\alpha_{x^{-1}}"] & b_{x^{-1}} \arrow[l, "\mu_{x^{-1}}"]
\end{tikzcd} .$$

(W-6) $\beta \circlearrowright b \xrightarrow{\mu} a \circlearrowleft \alpha$ with $\alpha^2\mu=\mu\beta=\alpha^3=\beta^3=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$ has as a quotient the following:

$$\begin{tikzcd}
 & & & & a_{x-1} \arrow[d, "\alpha_{x-1}"] & & & \\
b_{y^2x} & b_{yx} \arrow[l, "\beta_{yx}"'] & b_x \arrow[l, "\beta_x"'] \arrow[r, "\mu_x"'] & a_x & a_e \arrow[l, "\alpha_e"'] & b_e \arrow[l, "\mu_e"'] \arrow[r, "\beta_e"'] & b_y \arrow[r, "\beta_y"'] & b_{y^2}\,.
\end{tikzcd}$$

(W-7) $\beta\circlearrowright b \xrightarrow{\mu} a\circlearrowleft\alpha$ with $\alpha^2\mu=\mu\beta=\alpha^4=\beta^2=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$ has as a quotient the following:

$$\begin{tikzcd}
 & & & b_y & & \\
 & & & b_e \arrow[u, "\beta_e"] \arrow[d, "\mu_e"] & & \\
b_{yx} & b_x \arrow[l, "\beta_x"'] \arrow[r, "\mu_x"'] & a_x & a_e \arrow[l, "\alpha_e"'] & a_{x-1} \arrow[l, "\alpha_{x-1}"'] & a_{x-2}\,. \arrow[l, "\alpha_{x-2}"']
\end{tikzcd}$$

(W-8) $\beta\circlearrowright b \xrightarrow{\mu} a\circlearrowleft\alpha$ with $\alpha^2\mu=\alpha\mu\beta=\alpha^3=\beta^2=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$ has as a quotient the following:

$$\begin{tikzcd}
 & & & & a_{x-1} \arrow[d, "\alpha_{x-1}"] & & & \\
a_{y^{-1}x} & b_{y^{-1}x} \arrow[l, "\mu_{y^{-1}x}"'] \arrow[r, "\beta_{y^{-1}x}"'] & b_x \arrow[r, "\mu_x"'] & a_x & a_e \arrow[l, "\alpha_e"'] & b_e \arrow[l, "\mu_e"'] \arrow[r, "\beta_e"'] & b_y \arrow[r, "\mu_y"'] & a_y\,.
\end{tikzcd}$$

(W-9) $\beta\circlearrowright b \xrightarrow{\mu} a\circlearrowleft\alpha$ with $\alpha\mu-\mu\beta=\alpha^2\mu=\alpha^5=\beta^4=0$.

The universal Galois covering with Galois group $\cong Z$ has as a quotient the following:

$$\begin{tikzcd}
 & & & b_3 \arrow[r, "\beta_3"] \arrow[d, "\mu_3"] & b_4 \arrow[r, "\beta_4"] \arrow[d, "\mu_4"] & b_5 \arrow[r, "\beta_5"] & b_6 \\
a_0 \arrow[r, "\alpha_0"'] & a_1 \arrow[r, "\alpha_1"'] & a_2 \arrow[r, "\alpha_2"'] & a_3 \arrow[r, "\alpha_3"'] & a_4 & &
\end{tikzcd}$$

with $\alpha_3\mu_3-\mu_4\beta_3=0$. This is a concealed hereditary algebra of type $\tilde{\tilde{E}}_7$.

(W-10) $\beta\circlearrowright b \xrightarrow{\mu} a\circlearrowleft\alpha$ with $\alpha\mu-\mu\beta=\alpha^7=\beta^3=0$.

The universal Galois covering with Galois group $\cong Z$ has as a quotient the following:

$$\begin{tikzcd}
 & & & & & b_5 \arrow[r, "\beta_5"] \arrow[d, "\mu_5"] & b_6 \arrow[r, "\beta_6"] \arrow[d, "\mu_6"] & b_7 \\
a_0 \arrow[r, "\alpha_0"'] & a_1 \arrow[r, "\alpha_1"'] & a_2 \arrow[r, "\alpha_2"'] & a_3 \arrow[r, "\alpha_3"'] & a_4 \arrow[r, "\alpha_4"'] & a_5 \arrow[r, "\alpha_5"'] & a_6 &
\end{tikzcd}$$

with $\alpha_5\mu_5-\mu_6\beta_5=0$. This is a concealed hereditary algebra of type $\tilde{\tilde{E}}_8$.

(W-11) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowright \alpha$ with $\mu\nu-\alpha^2=\alpha^2\mu=\alpha^3=0$.

The universal Galois covering with Galois group $\cong Z$ has as a quotient the following:

$$\begin{tikzcd}
b_0 \arrow[dr, "\mu_0"] & b_1 \arrow[dr, "\mu_1"] & b_2 & b_3 \\
a_0 \arrow[r, "\alpha_0"'] \arrow[ur, "\nu_0"] & a_1 \arrow[r, "\alpha_1"'] \arrow[ur, "\nu_1"] & a_2 \arrow[ur, "\nu_2"] &
\end{tikzcd}$$

with $\mu_1\nu_0-\alpha_1\alpha_0=0$. This is a concealed hereditary algebra of type $\tilde{\tilde{D}}_5$.

(W-12) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowright \alpha$ with $\mu\nu-\alpha^3=\nu\mu=\nu\alpha\mu=\alpha^2\mu=\alpha^4=0$.

The universal Galois covering with Galois group $\cong Z$ has as a quotient the following:

$$\begin{tikzcd}
b_0 \arrow[drr, "\mu_0"] & b_1 \arrow[drr, "\mu_1"] & b_2 & b_3 \\
a_0 \arrow[r, "\alpha_0"'] \arrow[ur, "\nu_0"] & a_1 \arrow[r, "\alpha_1"'] \arrow[ur, "\nu_1"] & a_2 \arrow[r, "\alpha_2"'] \arrow[ur, "\nu_2"] & a_3
\end{tikzcd}$$

with $\mu_1\nu_0-\alpha_2\alpha_1\alpha_0=\nu_2\mu_0=0$. This is a concealed hereditary algebra of type $\tilde{\tilde{E}}_6$.

(W-13) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowright \alpha$ with $\mu\nu=\alpha\mu=\alpha^3=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$ has as a quotient the following:

$$\begin{tikzcd}
 & & b_x \arrow[d, "\mu_x"] \\
a_{x^{-1}} \arrow[r, "\alpha_{x^{-1}}"] \arrow[d, "\nu_{x^{-1}}"] & a_e \arrow[r, "\alpha_e"] \arrow[d, "\nu_e"] & a_x \arrow[d, "\nu_x"] \\
b_{yx^{-1}} & b_y & b_{yx} .
\end{tikzcd}$$

(W-14) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu=\nu\mu=\alpha^2\mu=\alpha^3=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$ has as a quotient the following:

$$\begin{tikzcd}
 & b_e \arrow[d, "\mu_e"] & b_x \arrow[d, "\mu_x"] \\
a_{x^{-1}} \arrow[r, "\alpha_{x^{-1}}"] \arrow[d, "\nu_{x^{-1}}"] & a_e \arrow[r, "\alpha_e"] \arrow[d, "\nu_e"] & a_x \arrow[d, "\nu_x"] \\
b_{yx^{-1}} & b_y & b_{yx}
\end{tikzcd}$$

with $\nu_e\mu_e=\nu_x\mu_x=0$. This is a concealed hereditary algebra of type $\tilde{\tilde{E}}_6$.

(W-15) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu=\nu\alpha\mu=\alpha^2\mu=\nu\alpha^2=\alpha^3=0$.

The universal Galois covering with Galois Group $\cong\langle x, y\rangle$ has as a quotient the folltwing:

$$\begin{tikzcd}
 & b_e \arrow[d, "\mu_e"] & b_x \arrow[d, "\mu_x"] \\
a_{x^{-1}} \arrow[r, "\alpha_{x^{-1}}"] & a_e \arrow[r, "\alpha_e"] \arrow[d, "\nu_e"] & a_x \\
 & b_y & .
\end{tikzcd}$$

(W-16) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu=\nu\mu=\alpha\mu=\nu\alpha^3=\alpha^4=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$ has as a quotient the following:

$$\begin{tikzcd}
 & & & b_{x^2} \arrow[d, "\mu_{x^2}"] \\
a_{x^{-1}} \arrow[r, "\alpha_{x^{-1}}"] & a_e \arrow[r, "\alpha_e"] \arrow[d, "\nu_e"] & a_x \arrow[r, "\alpha_x"] \arrow[d, "\nu_x"] & a_{x^2} \\
 & b_y & b_{yx} & .
\end{tikzcd}$$

(W-17) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu=\nu\mu=\nu\alpha\mu=\alpha^2\mu=\nu\alpha^2=\alpha^4=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$ has as a quotient the following:

$$\begin{tikzcd}
 & & b_x \arrow[d, "\mu_x"] & b_{x^2} \arrow[d, "\mu_{x^2}"] \\
a_{x^{-1}} \arrow[r, "\alpha_{x^{-1}}"] & a_e \arrow[r, "\alpha_e"] \arrow[d, "\nu_e"] & a_x \arrow[r, "\alpha_x"] & a_{x^2} \\
 & b_y & &
\end{tikzcd} .$$

(W-18) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu=\nu\mu=\alpha\mu=\nu\alpha^2=\alpha^5=0$.

The universal Galois covering with Galois group $\cong\langle x, y\rangle$ has as a quotient the following:

$$\begin{tikzcd}
 & & b_x \arrow[d, "\mu_x"] & b_{x^2} \arrow[d, "\mu_{x^2}"] & b_{x^3} \arrow[d, "\mu_{x^3}"] \\
a_{x^{-1}} \arrow[r, "\alpha_{x^{-1}}"] \arrow[d, "\nu_{x^{-1}}"] & a_e \arrow[r, "\alpha_e"] \arrow[d, "\nu_e"] & a_x \arrow[r, "\alpha_x"] & a_{x^2} \arrow[r, "\alpha_{x^2}"] & a_{x^3} \\
b_{yx^{-1}} & b_y & & &
\end{tikzcd}$$

with $\alpha_x\mu_x=\alpha_{x^2}\mu_{x^2}=0$. This is a concealed hereditary algebra of type $\tilde{E}_8$.

5. Tame Algebras.

In this section, we will show that the algebras in Table T are tame. In dealing with extensions of algebras, we will always calculate vector space categories. In fact, we have to deal with extensions of algebras which are not tublar extensions.

It is easy to see that no algebra in Table T is representation-finite, and it is well known that the algebra (T-0) is tame. Thus, it suffices to prove the tameness of the algebras (T-1), (T-2), (T-3_3), (T-4_4), (T-5_5), (T-6_5), (T-7), (T-8_5), (T-9), (T-10) and (T-11_5), since any other algebra in Table T can be obtained as a quotient of one of them. As an example, we will show also that the algebra (T-12) is tame.

(T-1) $b \xrightarrow{\mu} a \circlearrowleft \alpha$ with $\alpha^2\mu=\alpha^6=0$.

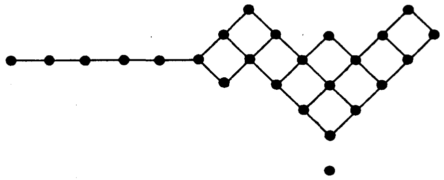
Take the universal Galois covering U with Galois group $\cong Z$:

$$\begin{tikzcd}
\cdots & b_{-1} \arrow[d, "\mu_{-1}"] & b_0 \arrow[d, "\mu_0"] & b_1 \arrow[d, "\mu_1"] & \cdots \\
{} \arrow[r] & a_{-1} \arrow[r, swap, "\alpha_{-1}"] & a_0 \arrow[r, swap, "\alpha_0"] & a_1 \arrow[r] & {}
\end{tikzcd}$$

with $\alpha_{i+1}\alpha_i\mu_i=\alpha_{i+5}\cdots\alpha_{i+1}\alpha_i=0$ for all $i\in Z$. For each $n\in Z$, let A_n be the following full subcategory of U:

$$\begin{tikzcd}
 & & & b_{n+3} \arrow[d] & b_{n+4} \arrow[d] & b_{n+5} \arrow[d] \\
a_n \arrow[r] & a_{n+1} \arrow[r] & a_{n+2} \arrow[r] & a_{n+3} \arrow[r] & a_{n+4} \arrow[r] & a_{n+5}
\end{tikzcd},$$

this is a concealed hereditary algebra of type $\tilde{E}_8$, and for $l, m\in Z$ with $l\leqq m$, let $A_{l,m}$ be the full sucategory of U consisting of the objects of the A_n, $l\leqq n\leqq m$. Notice that $A_{n,n+1}$ is the one-point coextension of $A_n\cup\{b_{n+6}\}$ by the module $D(N_n\oplus N'_n)$, where $N_n={}_{011111}^{0010}{}^{0}$ with restriction to A_n being preprojective and $N'_n={}_{000000}^{0001}{}^{1}$. The vector space category $\operatorname{Hom}(\operatorname{mod}(A_n\cup\{b_{n+6}\}), N_n\oplus N'_n)$ is of the following form:



.

This is a poset of tame type (see [9] and [12]). Thus, $A_{n,n+1}$ is tame. For $l, m\in Z$ with $l\leqq m$, $A_{l,m+1}$ is the one-point coextension of $A_{l,m}\cup\{b_{m+6}\}$ by the module with support in $A_m\cup\{b_{m+6}\}$ and with restriction to it being $D(N_m\oplus N'_m)$. The vector space category $\operatorname{Hom}(\operatorname{mod}(A_{l,m}\cup\{b_{m+6}\}), N_m\oplus N'_m)$ is isomorphic to $\operatorname{Hom}(\operatorname{mod}(A_m\cup\{b_{m+6}\}), N_m\oplus N'_m)$, and as a set $\operatorname{ind} A_{l,m+1}=\operatorname{ind} A_{l,m}\cup\operatorname{ind} A_{m,m+1}$. Therefore $\operatorname{ind} U=\bigcup_{n\in Z}\operatorname{ind} A_{n,n+1}$, in particular, U is locally support-finite and tame. Thus, (T-1) is tame by Proposition 4.

(T-2) $\beta\circlearrowright b\xrightarrow{\mu}a\circlearrowleft\alpha$ with $\alpha^2=\beta^2=0$.

This is a quotient of the following tame algebra: $\beta\circlearrowright b\underset{\nu}{\overset{\mu}{\rightleftarrows}}a\circlearrowleft\alpha$ with $\alpha^2=\beta^2=\nu\alpha\mu=\mu\beta\nu=\nu\mu=\mu\nu=0$ (see [4] for details). Thus, (T-2) is tame.

REMARK. Given a representation $\varphi\circlearrowright V\circlearrowleft\phi$ of the quiver $\tau\circlearrowright\bullet\circlearrowleft\sigma$ with relations $\sigma^2=\tau^2=0$, by defining the representation $\varphi\circlearrowright V\xrightarrow{1}V\circlearrowleft\phi$, we obtain a full exact embedding. Since the above algebra is a Galois covering of the algebra $\tau\circlearrowright\bullet\circlearrowleft\sigma$ with $\sigma^2=\tau^2=\tau\sigma\tau=0$, with Galois group $\cong Z/2Z$, by Proposition 3, the category of the finite dimensional representations of the quiver $\tau\circlearrowright\bullet\circlearrowleft\sigma$ with relations $\sigma^2=\tau^2=0$ is similar to that of the quiver $\tau\circlearrowright\bullet\circlearrowleft\sigma$ with relations $\sigma^2=\tau^2=\tau\sigma\tau=0$. Note that the latter is a finite dimensional algebra.

(T-3_3) $\beta\circlearrowright b\xrightarrow{\mu}a\circlearrowleft\alpha$ with $\alpha\mu-\mu\beta=\alpha^6=\beta^3=0$.

Take the universal Galois covering U with Galois group $\cong Z$:

$$\begin{tikzcd}
 & {} \arrow[r] & b_{-1} \arrow[r, "\beta_{-1}"] \arrow[d, "\mu_{-1}"] & b_0 \arrow[r, "\beta_0"] \arrow[d, "\mu_0"] & b_1 \arrow[r] \arrow[d, "\mu_1"] & {} & \\
\cdots & & & & & & \cdots \\
 & {} \arrow[r] & a_{-1} \arrow[r, "\alpha_{-1}"'] & a_0 \arrow[r, "\alpha_0"'] & a_1 \arrow[r] & {} &
\end{tikzcd}$$

with $\alpha_i\mu_i-\mu_{i+1}\beta_i=\alpha_{i+5}\cdots\alpha_{i+1}\alpha_i=\beta_{i+2}\beta_{i+1}\beta_i=0$ for all $i\in Z$. For each $n\in Z$, let A_{2n} be the following full subcategory of U:

$$\begin{tikzcd}
 & & & & b_{n+4} \arrow[r] \arrow[d] & b_{n+5} \arrow[r] \arrow[d] & b_{n+6} \\
a_n \arrow[r] & a_{n+1} \arrow[r] & a_{n+2} \arrow[r] & a_{n+3} \arrow[r] & a_{n+4} \arrow[r] & a_{n+5} &
\end{tikzcd}$$

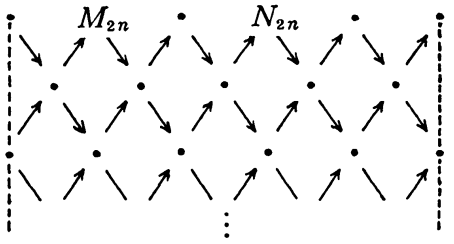
and let A_{2n-1} be the following full subcategory of U:

$$\begin{tikzcd}
 & & & b_{n+3} \arrow[r] \arrow[d] & b_{n+4} \arrow[r] \arrow[d] & b_{n+5} \arrow[d] \\
a_n \arrow[r] & a_{n+1} \arrow[r] & a_{n+2} \arrow[r] & a_{n+3} \arrow[r] & a_{n+4} \arrow[r] & a_{n+5}\ ,
\end{tikzcd}$$

these are concealed hereditary algebras of type $\tilde{E}_8$, and for l, $m\in Z$ with $l\leqq m$, as before, let $A_{l,m}$ be the full subcategory of U consisting of the objects of the A_n, $l\leqq n\leqq m$. Then, as an algebra, $A_{2n-1,2n+1}$ is isomorphic to

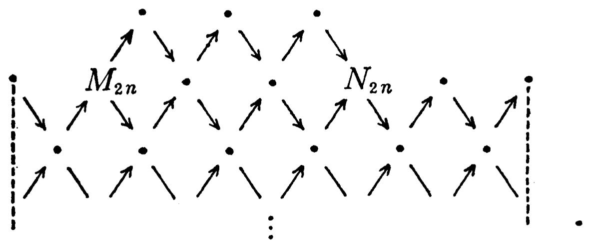
$$\begin{bmatrix} k & DN_{2n} & 0 \\ 0 & A_{2n} & M_{2n} \\ 0 & 0 & k \end{bmatrix}$$

where $M_{2n}={}_{000111}^{110}$ and $N_{2n}={}_{011111}^{111}$ are regular modules belonging to the same tube:



.

The vector space categories $\mathrm{Hom}(M_{2n}, \mathrm{mod}\, A_{2n})$ and $\mathrm{Hom}(\mathrm{mod}\, A_{2n}, N_{2n})$ belong to the pattern $(\tilde{E}_8, 5)$, and $\mathrm{ind}\, A_{2n-1,2n+1}=P_{2n}\cup R_{2n}\cup Q_{2n}$, where P_{2n} consists of the objects of $\mathrm{ind}\, A_{2n,2n+1}$ with restriction to A_{2n} being preprojective, Q_{2n} consists of the objects of $\mathrm{ind}\, A_{2n-1,2n}$ with restriction to A_{2n} being preinjective and R_{2n} consists of the regular objects of $\mathrm{ind}\, A_{2n}$ except that the above tube changes to the following:

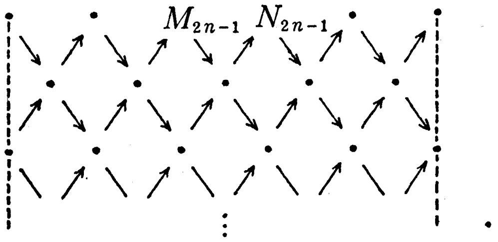


Thus, $A_{2n-1,2n+1}$ is tame.

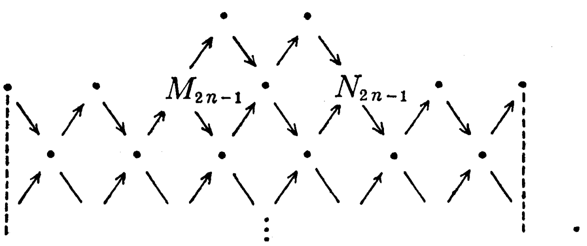
Similarly, $A_{2n-2,2n}$ is isomorphic to

$$\begin{bmatrix} k & DN_{2n-1} & 0 \\ 0 & A_{2n-1} & M_{2n-1} \\ 0 & 0 & k \end{bmatrix},$$

where $M_{2n-1}={}_{111110}^{000}$ and $N_{2n-1}={}_{000000}^{011}$ are regular modules:



The vector space categories Hom $(M_{2n-1}, \operatorname{mod} A_{2n-1})$ and Hom $(\operatorname{mod} A_{2n-1}, N_{2n-1})$ belong to the pattern $(\tilde{E}_8, 5)$, and ind $A_{2n-2,2n}=P_{2n-1}\cup R_{2n-1}\cup Q_{2n-1}$, where P_{2n-1} consists of the objects of ind $A_{2n-1,2n}$ with restriction to A_{2n-1} being preprojective, Q_{2n-1} consists of the objects of ind $A_{2n-2,2n-1}$ with restriction to A_{2n-1} being preinjective and R_{2n-1} consists of the regular objects of ind A_{2n-1} except that the above tube changes to the following:



Thus, $A_{2n-2,2n}$ is tame.

For $l, m\in Z$ with $l\leqq m$, $A_{l-1,m+1}$ is the one-point extension of $A_{l,m+1}$ by the module with support in A_l and with restriction to it being M_l. The vector space category Hom $(M_l, \operatorname{mod} A_{l,m+1})$ is isomorphic to Hom $(M_l, \operatorname{mod} A_l)$, and ind $A_{l-1,m+1}=$ind $A_{l-1,l+1}\cup$ind $A_{l,m+1}$. Therefore ind $U=\bigcup_{n\in Z}$ ind $A_{n-1,n+1}$, in particular, U is locally support-finite and tame. Thus, (T-3_3) is tame.

(T-4_4) $\beta \circlearrowright b \xrightarrow{\mu} a \circlearrowleft \alpha$ with $\alpha\mu-\mu\beta=\alpha^4=\beta^4=0$.

This has been proved to be tame [15], so we omit the proof.

(T-5_5) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu-\alpha^2=\nu\alpha\mu=0$.

Take the universal Galois covering $\tilde{U}$ with Galois group $\cong Z$:

$$\begin{tikzcd}
\cdots \arrow[dr] & b_{-1} \arrow[dr, "\mu_{-1}"] & b_0 \arrow[dr, "\mu_0"] & b_1 \arrow[dr] & {} \\
{} \arrow[r] \arrow[ur] & a_{-1} \arrow[r, "\alpha_{-1}"'] \arrow[ur, "\nu_{-1}"] & a_0 \arrow[r, "\alpha_0"'] \arrow[ur, "\nu_0"] & a_1 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

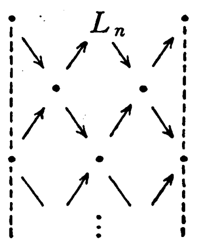
with $\mu_{i+1}\nu_i-\alpha_{i+1}\alpha_i=\nu_{i+1}\alpha_i\mu_{i-1}=0$ for all $i\in Z$. For each $n\in Z$, let A_n be the following full subcategory of $\tilde{U}$:

$$\begin{tikzcd}
b_n \arrow[dr] & b_{n+1} \arrow[dr] & b_{n+2} \\
a_n \arrow[r] \arrow[ur] & a_{n+1} \arrow[r] \arrow[ur] & a_{n+2}
\end{tikzcd},$$

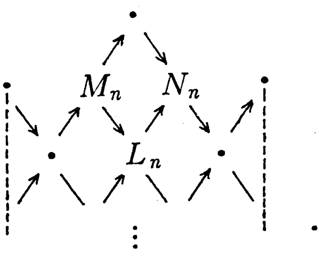
this is a concealed hereditary algebra of type $\tilde{D}_5$, let B_n and B_n^* be the full subcategories of $\tilde{U}$ obtained from A_n by adding a_{n-1} and a_{n+3} respectively, these are tilted algebras of type $\tilde{E}_6$, and let C_n be the full subcategory of $\tilde{U}$ consisting of the objects of B_n and B_n^*, this is isomorphic to

$$\begin{bmatrix} k & DL_n & k \\ 0 & A_n & L_n \\ 0 & 0 & k \end{bmatrix},$$

where $L_n={}^{111}_{111}$ is a regular module:



The vector space categories Hom$(L_n, \text{mod } A_n)$ and Hom$(\text{mod } A_n, L_n)$ belong to the pattern $(\tilde{D}_5, 2)$, and ind $C_n=P_n \cup R_n \cup Q_n$, where P_n consists of the objects of ind B_n^* with restriction to A_n being preprojective, Q_n consists of the objects of ind B_n with restriction to A_n being preinjective and R_n consists of the regular objects of ind A_n except that the above tube changes to the following:

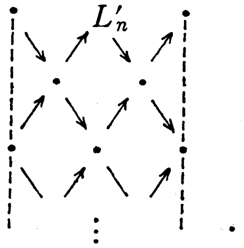


Thus, C_n is tame.

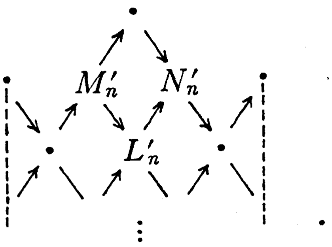
For $l, m \in Z$ with $l \leqq m$, define $A_{l,m}$ and $B_{l,m}$ as before. Then, $A_{n-1,n+1}$ is isomorphic to

$$\begin{bmatrix} k & DL'_n & k \\ 0 & C_n & L'_n \\ 0 & 0 & k \end{bmatrix},$$

where $L'_n = \begin{smallmatrix} 010 \\ 01110 \end{smallmatrix}$ is a regular module:



The vector space categories Hom$(L'_n, \text{mod}\, C_n)$ and Hom$(\text{mod}\, C_n, L'_n)$ belong to the pattern $(\tilde{E}_6, 2)$, and ind $A_{n-1,n+1} = P'_n \cup R'_n \cup Q'_n$, where P'_n consists of the objects of ind $A_{n,n+1}$ with restriction to C_n lying in P_n, Q'_n consists of the objects of ind $A_{n-1,n}$ with restriction to C_n lying in Q_n and R'_n coincides with R_n except that the above tube changes to the following:



Thus, $A_{n-1,n+1}$ is tame.

For $l, m \in Z$ with $l \leqq m$, $B_{l,m+1}$ is the one-point extension of $A_{l,m+1}$ by the module with support in B^*_l and with restriction to it being M_l. The vector space category Hom$(M_l, \text{mod}\, A_{l,m+1})$ is isomorphic to Hom$(M_l, \text{mod}\, B^*_l)$ and belongs to the pattern $(\tilde{D}_5, 2)$. Next, $A_{l-1,m+1}$ is the one-point extension of $B_{l,m+1}$ by the module with support in $B_{l,l+1}$ and with restriction to it being M'_l. The vector space category Hom$(M'_l, \text{mod}\, B_{l,m+1})$ is isomorphic to Hom$(M'_l, B_{l,l+1})$

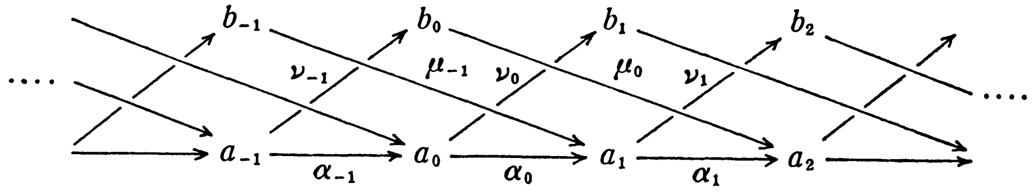
and belongs to the pattern $(\tilde{E}_6, 2)$, and $\operatorname{ind} A_{l-1,m+1} = \operatorname{ind} A_{l-1,l+1} \cup \operatorname{ind} A_{l,m+1}$. Therefore $\operatorname{ind} U = \bigcup_{n\in Z} \operatorname{ind} A_{n-1,n+1}$, in particular, U is locally support-finite and tame. Thus, (T-5_5) is tame.

(T-6_5) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu - \alpha^2 = \nu\alpha\mu - \nu\alpha^2\mu = \alpha^5 = 0$.

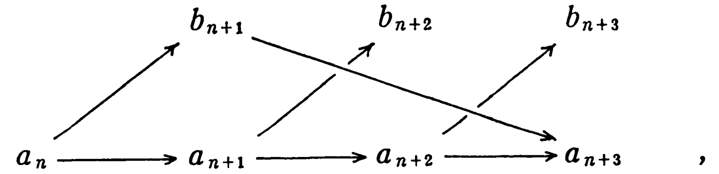
This is self-injective and the quotient by the socle is isomorphic to (T-$5'_4$).

(T-7) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu - \alpha^3 = \nu\mu = \alpha\mu = 0$.

Take the universal Galois covering U with Galois group $\cong Z$:



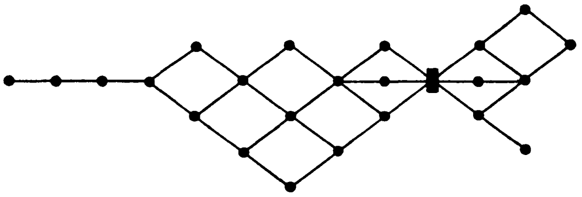
with $\mu_{i+1}\nu_i - \alpha_{i+2}\alpha_{i+1}\alpha_i = \nu_{i+1}\mu_{i-1} = \alpha_{i+2}\mu_i = 0$ for all $i \in Z$. For each $n \in Z$, let A_n be the following full subcategory of U:



,

this is a concealed hereditary algebra of type $\tilde{E}_6$, and let B_n be the full subcategory of U obtained from A_n by adding b_n, this is a tilted algebra of type $\tilde{E}_7$. Then B_n is the one-point extension of A_n by the regular module $M_n = {}^{\;000}_{0010}$, and the vector space category $\operatorname{Hom}(M_n, \operatorname{mod} A_n)$ belongs to the pattern $(\tilde{E}_6, 3)$.

For $l, m \in Z$ with $l \leqq m$, define $A_{l,m}$ and $B_{l,m}$ as before. Then $A_{n-1,n}$ is the one-point extension of B_n by the preinjective module $M'_n = {}^{1110}_{1110}$, and the vector space category $\operatorname{Hom}(M'_n, \operatorname{mod} B_n)$ is of the form:



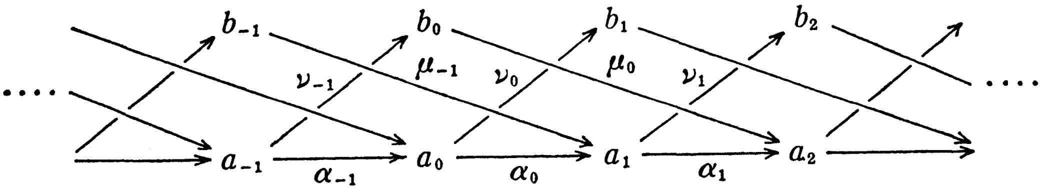
.

Thus, $A_{n-1,n}$ is tame.

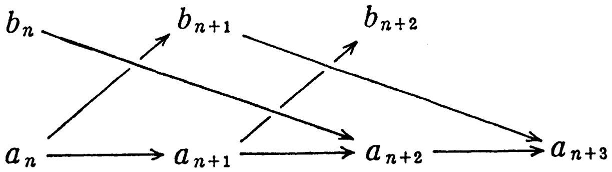
For $l, m \in Z$ with $l \leqq m$, $B_{l,m}$ is the one-point extension of $A_{l,m}$ by the module with support in A_l and with restriction to it being M_l. The vector space category $\operatorname{Hom}(M_l, \operatorname{mod} A_{l,m})$ is isomorphic to $\operatorname{Hom}(M_l, \operatorname{mod} A_l)$. Next, $A_{l-1,m}$ is the one-point extension of $B_{l,m}$ by the module with support in B_l and with restriction to it being M_l'. The vector space category $\mathrm{Hom}(M_l', \mathrm{mod}\, B_{l,m})$ is isomorphic to $\mathrm{Hom}(M_l', \mathrm{mod}\, B_l)$ and $\mathrm{ind}\, A_{l-1,m} = \mathrm{ind}\, A_{l-1,l} \cup \mathrm{ind}\, A_{l,m}$. Therefore $\mathrm{ind}\, U = \bigcup_{n\in Z} \mathrm{ind}\, A_{n-1,n}$, in particular, U is locally support-finite and tame. Thus, (T-7) is tame.

(T-8_5) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu - \alpha^3 = \nu\mu = \alpha^2\mu = \nu\alpha^2 = 0$.

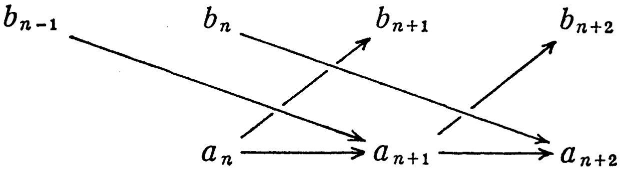
Take the universal Galois covering U with Galois group $\cong Z$:



with $\mu_{i+1}\nu_i - \alpha_{i+2}\alpha_{i+1}\alpha_i = \nu_{i+2}\mu_i = \alpha_{i+3}\alpha_{i+2}\mu_i = \nu_{i+2}\alpha_{i+1}\alpha_i = 0$ for all $i\in Z$. For each $n\in Z$, let A_{2n} be the following full subcategory of U:



and let A_{2n-1} be the following full subcategory of U:

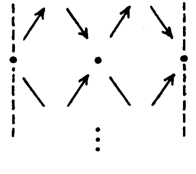


,

these are concealed hereditary algebras of type $\tilde{E}_6$, and for $l, m\in Z$ with $l \leqq m$, define $A_{l,m}$ as before. Then $A_{2n-1,2n+1}$ is isomorphic to

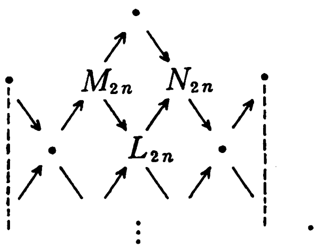
$$\begin{bmatrix} k & DL_{2n} & k \\ 0 & A_{2n} & L_{2n} \\ 0 & 0 & k \end{bmatrix},$$

where $L_{2n} = {}^{000}_{0110}$ is a regular module:



.

The vector space categories Hom$(L_{2n}$, mod $A_{2n})$ and Hom(mod A_{2n}, $L_{2n})$ belong to the pattern $(\tilde{E}_6, 2)$, and ind $A_{2n-1,2n+1}=P_{2n}\cup R_{2n}\cup Q_{2n}$, where P_{2n} consists of the objects of ind $A_{2n,2n+1}$ with restriction to A_{2n} being preprojective, Q_{2n} consists of the objects of ind $A_{2n-1,2n}$ with restriction to A_{2n} being preinjective and R_{2n} consists of the regular objects of ind A_{2n} except that the above tube changes to the following:

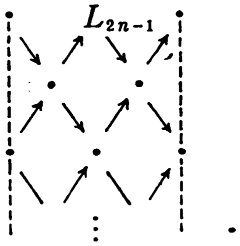


Thus, $A_{2n-1,2n+1}$ is tame.

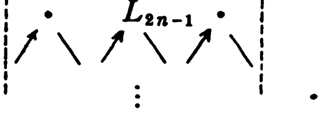
Similarly, $A_{2n-2,2n}$ is isomorphic to

$$\begin{bmatrix} k & DL_{2n-1} & k \\ 0 & A_{2n-1} & L_{2n-1} \\ 0 & 0 & k \end{bmatrix},$$

where $L_{2n-1}={}^{0110}_{\ 111}$ is a regular module:



The vector space categories Hom$(L_{2n-1}$, mod $A_{2n-1})$ and Hom(mod A_{2n-1}, $L_{2n-1})$ belong to the pattern $(\tilde{E}_6, 2)$, and ind $A_{2n-2,2n}=P_{2n-1}\cup R_{2n-1}\cup Q_{2n-1}$, where P_{2n-1} consists of the objects of ind $A_{2n-1,2n}$ with restriction to A_{2n-1} being preprojective, Q_{2n-1} consists of the objects of ind $A_{2n-2,2n-1}$ with restriction to A_{2n-1} being preinjective and R_{2n-1} consists of the regular objects of ind A_{2n-1} except that the above tube changes to the following:



Thus, $A_{2n-2,2n}$ is tame.

For $l, m\in Z$ with $l\leqq m$, $A_{l-1,m+1}$ is the one-point extension of $A_{l,m+1}$ by the module with support in $A_{l,l+1}$ and with restriction to it being M_l. The vector space category $\mathrm{Hom}(M_l, \mathrm{mod}\, A_{l,m+1})$ is isomorphic to $\mathrm{Hom}(M_l, \mathrm{mod}\, A_{l,l+1})$, thus belongs to the pattern $(\tilde{E}_6, 2)$, and $\mathrm{ind}\, A_{l-1,m+1}=\mathrm{ind}\, A_{l-1,l+1}\cup \mathrm{ind}\, A_{l,m+1}$. Therefore $\mathrm{ind}\, U=\cup_{n\in Z}\mathrm{ind}\, A_{n-1,n+1}$, in particular, U is locally support-finite and tame. Thus (T-8_5) is tame.

(T-9) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu-\alpha^3=\nu\mu-\nu\alpha\mu=\alpha^2\mu=\nu\alpha^2=0$.

This is self-injective and the quotient by the socle is isomorphic to (T-$8_4'$).

(T-10) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu=\nu\mu=\nu\alpha\mu=\alpha^3=0$.

The relation $\mu\nu=0$ is splitting-zero, thus it suffices to prove the tameness of the following algebra:

$$\begin{tikzcd} b \arrow[r, "\mu"] & a \arrow[loop above, "\alpha"] \arrow[r, "\nu"] & c \end{tikzcd} \quad \text{with } \nu\mu=\nu\alpha\mu=\alpha^3=0.$$

Take the universal Galois covering U with Galois group $\cong Z$:

$$\begin{tikzcd} & b_{-1} \arrow[d, "\mu_{-1}"] & b_0 \arrow[d, "\mu_0"] & b_1 \arrow[d, "\mu_1"] & \\ \cdots \arrow[r] & a_{-1} \arrow[r, "\alpha_{-1}"] \arrow[d, "\nu_{-1}"] & a_0 \arrow[r, "\alpha_0"] \arrow[d, "\nu_0"] & a_1 \arrow[r] \arrow[d, "\nu_1"] & \cdots \\ & c_{-1} & c_0 & c_1 & \end{tikzcd}$$

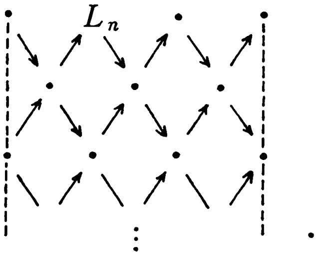
with $\nu_i\mu_i=\nu_{i+1}\alpha_i\mu_i=\alpha_{i+2}\alpha_{i+1}\alpha_i=0$ for all $i\in Z$. For each $n\in Z$, let A_n be the following full subcategory of U:

$$\begin{tikzcd} & b_{n+1} \arrow[d] & b_{n+2} \arrow[d] \\ a_n \arrow[r] \arrow[d] & a_{n+1} \arrow[r] \arrow[d] & a_{n+2}, \\ c_n & c_{n+1} & \end{tikzcd}$$

this is a concealed hereditary algebra of type $\tilde{E}_6$, let B_n and B_n^* be the full subcategories of U obtained from A_n by adding b_n and c_{n+2} respectively, these are tilted algebras of type $\tilde{E}_7$, and let C_n be the full subcategory of U consisting of the objects of B_n and B_n^*, this is isomorphic to

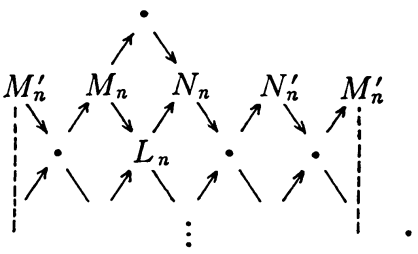
$$\begin{bmatrix} k & DL_n & k \\ 0 & A_n & L_n \\ 0 & 0 & k \end{bmatrix},$$

where $L_n=\begin{smallmatrix}00\\111\\00\end{smallmatrix}$ is a regular module:



.

The vector space categories Hom $(L_n, \operatorname{mod} A_n)$ and Hom $(\operatorname{mod} A_n, L_n)$ belong to the pattern $(\tilde{E}_6, 3)$, and ind $C_n=P_n \dot{\cup} R_n \dot{\cup} Q_n$, where P_n consists of the objects of ind B_n^* with restriction to A_n being preprojective, Q_n consists of the objects of ind B_n with restriction to A_n being preinjective and R_n consists of the regular objects of ind A_n except that the above tube changes to the following:



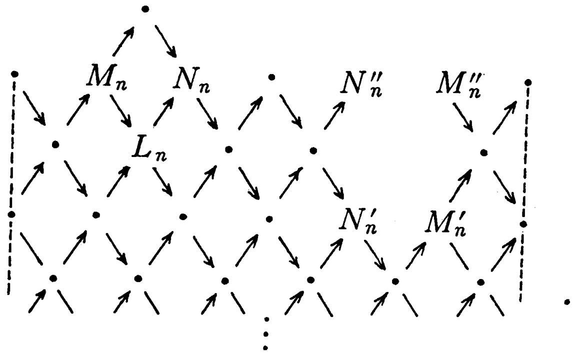
.

Thus, C_n is tame.

For $l, m \in Z$ with $l \leqq m$, define $A_{l,m}$ and $B_{l,m}$ as before. Let C_n' be the full subcategory of U obtained from C_n by adding c_{n-1} and b_{n+3} Then, $A_{n-1,n+1}$ is isomorphic to

$$\begin{bmatrix} k & D(N_n' \oplus N_n'') & 0 \\ 0 & C_n' & M_n' \oplus M_n'' \\ 0 & 0 & k \end{bmatrix},$$

where $M_n'=\begin{smallmatrix}0000\\110\\0110\end{smallmatrix}$, $M_n''=\begin{smallmatrix}0000\\000\\1000\end{smallmatrix}$, $N_n'=\begin{smallmatrix}0110\\011\\0000\end{smallmatrix}$ and $N_n''=\begin{smallmatrix}0001\\000\\0000\end{smallmatrix}$. The vector space categories Hom $(M_n' \oplus M_n'', \operatorname{mod} C_n')$ and Hom $(\operatorname{mod} C_n', N_n' \oplus N_n'')$ belong to the pattern $(\tilde{E}_7, 4) \dot{\cup} \{\cdot\}$, and ind $A_{n-1,n+1}=P_n' \dot{\cup} R_n' \dot{\cup} Q_n'$, where P_n' consists of the objects of ind $A_{n,n+1}$ with restriction to C_n lying in P_n, Q_n' consists of the objects of ind $A_{n-1,n}$ with restriction to C_n lying in Q_n and R_n' coincides with R_n except that the above tube changes to the following:

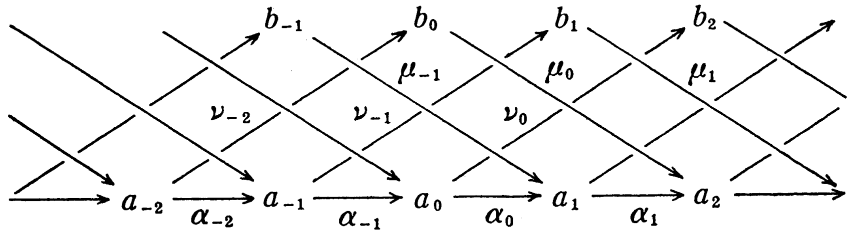


Thus, $A_{n-1,n+1}$ is tame.

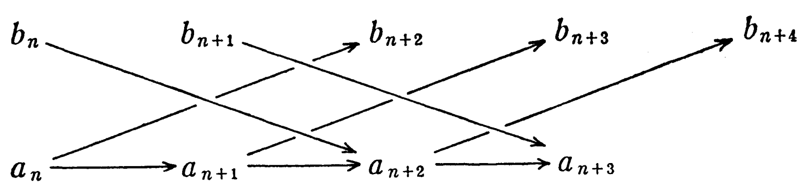
For l, $m\in Z$ with $l\leqq m$, $B_{l,m+1}$ is the one-point extension of $A_{l,m+1}$ by the module with support in B_l^* and with restriction to it being M_l. The vector space category $\mathrm{Hom}\,(M_l, \mathrm{mod}\, A_{l,m+1})$ is isomorphic to $\mathrm{Hom}\,(M_l, \mathrm{mod}\, B_l^*)$ and belongs to the pattern $(\tilde{E}_6, 3)$. Next, $A_{l-1,m+1}$ is the one-point extension of $B_{l,m+1}\cup\{c_{l-1}\}$ by the module with support in C_l' and with restriction to it being $M_l'\oplus M_l''$. The vector space category $\mathrm{Hom}\,(M_l'\oplus M_l'', \mathrm{mod}\,(B_{l,m+1}\cup\{c_{l-1}\}))$ is isomorphic to $\mathrm{Hom}\,(M_l'\oplus M_l'', \mathrm{mod}\, C_l')$, and $\mathrm{ind}\, A_{l-1,m+1}=\mathrm{ind}\, A_{l-1,l+1}\cup\mathrm{ind}\, A_{l,m+1}$. Therefore $\mathrm{ind}\, U=\bigcup_{n\in Z}\mathrm{ind}\, A_{n-1,n+1}$, in particular, U is locally support-finite and tame. We are done.

(T-11$_5$) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu-\alpha^4=\alpha\mu=\nu\alpha^2=0$.

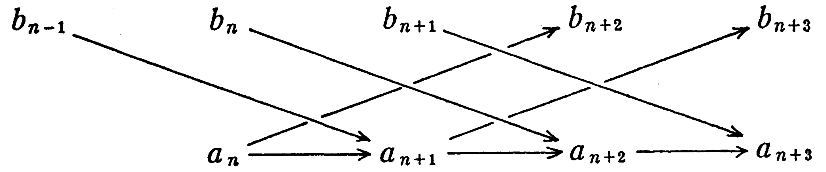
Take the universal Galois covering U with Galois group $\cong Z$:



with $\mu_{i+2}\nu_i-\alpha_{i+3}\alpha_{i+2}\alpha_{i+1}\alpha_i=\alpha_{i+2}\mu_i=\nu_{i+2}\alpha_{i+1}\alpha_i=0$ for all $i\in Z$. For each $n\in Z$, let A_{2n} be the following ful subcategory of U:



and let A_{2n-1} be the following full subcategory of U:

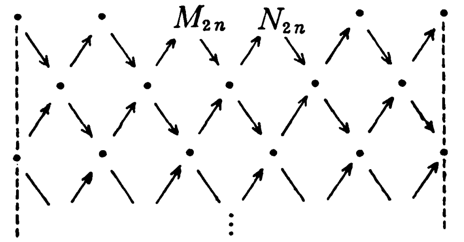


,

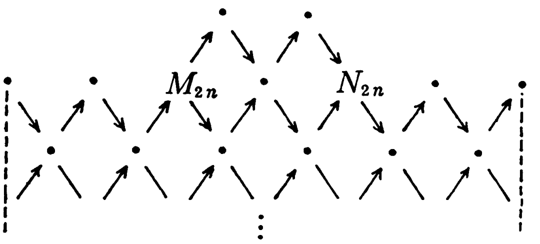
these are concealed hereditary algebras of type $\tilde{E}_8$, and for $l, m \in Z$ with $l \leqq m$, let $A_{l,m}$ be as before. Then, $A_{2n-1,2n+1}$ is isomorphic to

$$\begin{bmatrix} k & DN_{2n} & 0 \\ 0 & A_{2n} & M_{2n} \\ 0 & 0 & k \end{bmatrix},$$

where $M_{2n}=\begin{smallmatrix}00010\\0100\end{smallmatrix}$, $N_{2n}=\begin{smallmatrix}00100\\1111\end{smallmatrix}$ are regular modules:



The vector space categories Hom$(M_{2n}$, mod $A_{2n})$ and Hom(mod A_{2n}, $N_{2n})$ belong to the pattern $(\tilde{E}_8, 5)$, and ind $A_{2n-1,2n+1}=P_{2n} \cup R_{2n} \cup Q_{2n}$, where P_{2n} consists of the objects of ind $A_{2n,2n+1}$ with restriction to A_{2n} being preprojective, Q_{2n} consists of the objects of ind $A_{2n-1,2n}$ with restriction to A_{2n} being preinjective and R_{2n} consists of the regular objects of ind A_{2n} except that the above tube changes to the following:

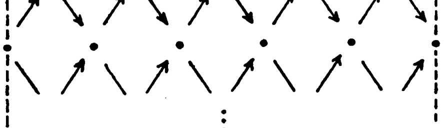


Thus, $A_{2n-1,2n+1}$ is tame.

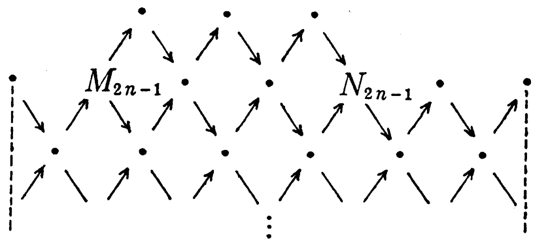
Similarly, $A_{2n-2,2n}$ is isomorphic to

$$\begin{bmatrix} k & DN_{2n-1} & 0 \\ 0 & A_{2n-1} & M_{2n-1} \\ 0 & 0 & k \end{bmatrix},$$

where $M_{2n-1}=\begin{smallmatrix}00110\\1111\end{smallmatrix}$ and $N_{2n-1}=\begin{smallmatrix}01000\\0110\end{smallmatrix}$ are regular modules:



The vector space categories Hom (M_{2n-1}, mod A_{2n-1}) and Hom (mod A_{2n-1}, N_{2n-1}) belong to the pattern ($\tilde{E}_8$, 5), and ind $A_{2n-2,2n}=P_{2n-1}\cup R_{2n-1}\cup Q_{2n-1}$, where P_{2n-1} consists of the objects of ind $A_{2n-1,2n}$ with restriction to A_{2n-1} being preprojective, Q_{2n-1} consists of the objects of ind $A_{2n-2,2n-1}$ with restriction to A_{2n-1} being preinjective and R_{2n-1} consists of the regular objects of ind A_{2n-1} except that the above tube changes to the following:



.

Thus, $A_{2n-2,2n}$ is tame.

For l, $m\in Z$ with $l\leqq m$, $A_{l-1,m+1}$ is the one-point extension of $A_{l,m+1}$ by the module with support in A_l and with restriction to it being M_l. The vector space category Hom (M_l, mod $A_{l,m+1}$) is isomorphic to Hom (M_l, mod A_l), and ind $A_{l-1,m+1}$=ind $A_{l-1,l+1}\cup$ind $A_{l,m+1}$. Therefore ind $U=\bigcup_{n\in Z}$ ind $A_{n-1,n+1}$, in particular, U is locally support-finite and tame. Thus, (T-11_5) is tame.

(T-12) $b \underset{\nu}{\overset{\mu}{\rightleftarrows}} a \circlearrowleft \alpha$ with $\mu\nu=\alpha^2=0$.

Since the relation $\mu\nu=0$ is splitting-zero, it suffices to consider the algebra

$$\begin{tikzcd} b \arrow[r, "\mu"] & a \arrow[loop above, "\alpha"] \arrow[r, "\nu"] & c \end{tikzcd} \quad \text{with } \alpha^2=0.$$

This can be considered as a full subcategory of the algebra obtained from the tame one-relation algebra [12]:

$$\begin{tikzcd} a' \arrow[r, "\alpha'"] \arrow[rr, bend left, "\beta"] & a \arrow[r, "\alpha''"] \arrow[dr, "\nu"] & a'' \\ b \arrow[ur, "\mu"] & & c \end{tikzcd} \quad \text{with } \alpha''\alpha'=0,$$

by shrinking the arrow β. Therefore (T-12) is tame.

References

[1] Bautista, R., Gabriel, P., Rojter, A. V. and Salmerón, L., Representation-finite algebras and mutiplicative bases, Invent. Math. **81** (1985), 217-285.

[2] Berman, S., Moody, R. and Wonenburger, M., Certain matrices with null roots and finite Cartan matrices, Indiana Univ. Math. J. **21** (1971/72), 1091-1099.

[3] Bongartz, K. and Gabriel, P., Covering spaces in representation-theory, Invent.

Math. **65** (1982), 331-378.
[4] Donovan, P. W. and Freislich, M. R., The indecomposable modular representations of certain groups with dihedral Sylow subgroup, Math. Ann. **238** (1978), 207-216.
[5] Dowber, P., Lenzing, H. and Skowroński, A., Galois coverings of algebras by locally support-finite categories, Proc. Puebla 1984, Springer LNM **1177**, 91-93.
[6] Gabriel, P., The universal cover of a representation-finite algebra, Proc. Puebla 1980, Springer LNM **903**, 68-105.
[7] Gelfand, I. M. and Ponomarev, V. A., Indecomposable representations of the Lorentz group, Usp. Mat. Nauk **23** (1968), 3-60.
[8] Hoshino, M. and Miyachi, J., Tame triangular matrix algebras over self-injective algebras, Tsukuba J. Math. **11**(2) (1987), 383-391.
[9] Nazarova, L. A., The representations of partially ordered sets of infinite type, Proc. Ottawa 1975, Springer LNM **488**, 244-252.
[10] Nazarova, L. A. and Rojter, A. V., Categorical matrix problems and the Brauer-Thrall conjectures, Mitt. Math. Sem. Giessen **115** (1975).
[11] Ringel, C. M., The indecomposable representations of the dihedral 2-groups, Math. Ann. **214** (1975), 19-34.
[12] Ringel, C. M., Tame algebras, Springer LNM **831** (1980), 137-297.
[13] Ringel, C. M., Tame algebras and integral quodratic forms, Springer LNM **1099** (1984).
[14] Rojter, A. V., The norm of a relation, Proc. Puebla 1984, Springer LNM **1177**, 269-271.
[15] Skowroński, A., Tame triangular matrix algebras over Nakayama algebras, J. London Math. Soc. (2)**34** (1986), 245-264.

Institute of Mathematics
University of Tsukuba
Ibaraki, 305
Japan

Department of Mathematics
Tokyo Gakugei University
Koganei-shi, Tokyo, 184
Japan