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A CHARACTERIZATION OF CLOSED s-IMAGES OF METRIC SPACES

By

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Throughout the present note, we assume that all spaces are regular topological spaces and all mappings are continuous. Let N denote the set of all natural numbers.

Recall from [7] a collection $\mathcal{P}$ of subsets of a space X is called a *k-network* for X if for every compact subset K of X and every open set U of X with $K\subset U$, there is a finite subcollection $\mathcal{P}'$ of $\mathcal{P}$ such that $K\subset \cup\{P: P\in\mathcal{P}'\}\subset U$. A collection $\mathcal{P}$ of subsets of a space X is called a *cs-network* for X if for every sequence $\{x_n: n\in N\}$ converging to a point $x\in X$ and every neighborhood U of x, there is an element $P\in\mathcal{P}$ such that $P\subset U$ and $\{x_n: n\in N\}$ is eventually in P ([4]). A space is said to be an $\aleph$-space if it has a σ-locally finite k-network ([6]). A mapping f from a space X to a space Y is called an *s-mapping* if $f^{-1}(y)$ has a countable base for each $y\in Y$.

Recently, L. Foged [2] proved an interesting characterization of Lašnev spaces: A space X is Lašnev space (i. e. X is a closed image of a metric space) if and only if X is a Fréchet space with a σ-hereditarily closure preserving k-network. On the other hand, Y. Tanaka showed that every closed s-image X of a metric space is an $\aleph$-space if any closed metrizable subset of X is locally compact ([9, Lemma 4.1]). (Using this result, he gave a characterization for the product space $X\times Y$ of closed s-images X and Y of metric spaces to be a k-space (see [9, Theorem 4.3]).) He asked in the same paper whether every closed s-image of a metric space is an $\aleph$-space. The purpose of this note is to answer the above question and simultaneously to get a characterization of Fréchet $\aleph$-spaces.

Our result is the following.

THEOREM. *For a regular space X, the following are equivalent.*

(a) *X is a Fréchet $\aleph$-space.*

(b) *X is a closed s-image of a metric space.*

(c) *X is a Fréchet space with a point countable, σ-closure preserving,*

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closed k-network.

PROOF. The implication (a)→(b) can be shown by an argument similar to that of [2, Proposition 5]. To prove the implication (b)→(c), let X be an image of a metric space Y under a closed s-mapping f. Let $\mathcal{B}$ be a σ-locally finite base for Y. Then it is obvious that $\mathcal{P}=\{f(\bar{B}):B\in\mathcal{B}\}$ is a point countable and σ-closure preserving family of closed sets of X. Furthermore, $\mathcal{P}$ is a k-network for X. Indeed, let K be a compact subset of X and U an open set of X with $K\subset U$. By [5, Corollary 1.2], there is a compact subset C of Y with $C\subset f^{-1}(U)$ and $f(C)=K$. Let $B_1,\cdots, B_n$ be elements of $\mathcal{B}$ such that $C\subset B_1\cup\cdots\cup B_n\subset\bar{B}_1\cup\cdots\cup\bar{B}_n\subset f^{-1}(U)$. Then $K\subset f(\bar{B}_1)\cup\cdots\cup f(\bar{B}_n)\subset U$. Thus $\mathcal{P}$ is a k-network for X and hence the implication (b)→(c) is proved. To prove the implication (c)→(a), by [1, Theorem 4], it is sufficient to show that X has a σ-discrete cs-network. Now, let $\mathcal{P}=\cup\{\mathcal{P}_n:n\in N\}$ be a point countable, σ-closure preserving and closed k-network for X, where each $\mathcal{P}_n$ is closure preserving. Without loss of generality, we can assume that each $\mathcal{P}_n$ is closed under finite intersections and $\mathcal{P}_n\subset\mathcal{P}_{n+1}$ for each $n\in N$. Since $\mathcal{P}_n$ is locally countable, there is an open cover $\mathcal{U}_n$ of X such that each member U of $\mathcal{U}_n$ intersects at most countably many members of $\mathcal{P}_n$. Since X is a σ-space (see [8]), there is a σ-discrete closed refinement $\mathcal{F}_n=\cup\{\mathcal{F}_{nm}:m\in N\}$ of $\mathcal{U}_n$, where $\mathcal{F}_{nm}=\{F_{nm\alpha}:\alpha\in A_n\}$ is discrete in X. For each n, $m\in N$ and each $\alpha\in A_n$, we put

$$\mathcal{P}_{nm\alpha}=\{P\in\mathcal{P}_n:P\cap F_{nm\alpha}\neq\phi\}.$$

Since $\mathcal{P}_{nm\alpha}$ is countable, let $\mathcal{P}^*_{nm\alpha}=\{P^k_{nm\alpha}:k\in N\}$ be the family of all finite unions of $\mathcal{P}_{nm\alpha}$. For each n, $m\in N$ and each $\alpha\in A_n$, we put

$$W_{nm\alpha}=\cup_{i=1}^{\infty}[(\cup\{P\in\mathcal{P}_i:P\cap(\cup\{F_{nm\beta}:\beta\in A_n \text{ with } \beta\neq\alpha\})=\phi\}) \\ -(\cup\{P\in\mathcal{P}_i:P\cap F_{nm\alpha}=\phi\})].$$

We have the following.

(1) $F_{nm\alpha}\subset W_{nm\alpha}$ for each $\alpha\in A_n$ and $n\in N$.

(2) $W_{nm\alpha}\cap W_{nm\beta}=\phi$ for each α, $\beta\in A_n$ with $\alpha\neq\beta$.

For each n, m, k, $r\in N$ and each $\alpha\in A_n$, we put

$$Q^r_{nm\alpha}=\cup\{P\in\mathcal{P}_r:P\subset W_{nm\alpha}),$$

and

$$\mathcal{Q}^{kr}_{nm}=\{P^k_{nm\alpha}\cap Q^r_{nm\alpha}:\alpha\in A_n\}.$$

Finally, we put

$$\mathcal{Q}=\cup\{\mathcal{Q}^{kr}_{nm}:(n,m,k,r)\in N\times N\times N\times N\}.$$

By (2), it follows that $\mathcal{Q}^{kr}_{nm}$ is discrete in X. To show that $\mathcal{Q}$ is a cs-network for X, let $\{x_n : n\in N\}$ be a sequence in X which converges to a point $x\in X$ and U a neighborhood of x. Since $\mathcal{P}$ is a k-network for X, there are a number $n\in N$ and a finite subcollection $\mathcal{P}_n'$ of $\mathcal{P}_n$ such that $\{x_n : n\in N\}$ is eventually in $\cup\{P : P\in\mathcal{P}_n'\}$, $\cup\{P : P\in\mathcal{P}_n'\}\subset U$ and $x\in\cap\{P : P\in\mathcal{P}_n'\}$. Since $\mathcal{F}_n$ is a cover of X, there are a number $m\in N$ and an element $\alpha\in A_n$ such that $x\in F_{nm\alpha}$. Then $\cup\{P : P\in\mathcal{P}_n'\}\in\mathcal{P}^*_{nm\alpha}$. Let us put $\cup\{P : P\in\mathcal{P}_n'\}=P^k_{nm\alpha}$ for some $k\in N$. On the other hand, since every sequence converging to a point of $F_{nm\alpha}$ is eventually in $W_{nm\alpha}$, it follows that there are a number $r\in N$ with $r\geqslant n$ and a finite subcollection $\mathcal{P}_r'$ of $\mathcal{P}_r$ such that $\{x_n : n\in N\}$ is eventually in $\cup\{P : P\in\mathcal{P}_r'\}$ and $\cup\{P : P\in\mathcal{P}_r'\}\subset W_{nm\alpha}$ by [1, Lemma 3]. Therefore, $Q=P^k_{nm\alpha}\cap Q^r_{nm\alpha}$ $(\in\mathcal{Q}^{kr}_{nm}\subset\mathcal{Q})$ contains a tail of $\{x_n : n\in N\}$ and Q is contained in U. Hence $\mathcal{Q}$ is a cs-network for X. This completes the proof.

REMARK 1. (i) By the theorem, every closed s-image of a metric space is an $\aleph$-space. This is an affirmative answer to the Tanaka's question stated before.

(ii) The proof of the implication (c)$\rightarrow$(a) in the theorem showed that every regular space with a point countable, σ-closure preserving and closed k-network is an $\aleph$-space. This is an affirmative answer to a question in [11] whether every regular space with a point countable, σ-hereditarily closure preserving closed k-network is an $\aleph$-space.

REMARK 2. In the statement (c) of the theorem, the assumption of the "closedness" of the k-network can not be dropped. Indeed, let X be the discrete sum $\oplus\{I_\alpha : \alpha<\omega_1\}$ of the copies I_α, $\alpha<\omega_1$, of the unit closed interval $I=[0, 1]$. Let A be the subset of X consisting of all zero's. Let $Y=X/A$ be the quotient space. It is well known that Y has no point countable closed k-network (cf. [10] or [3]). On the other hand, L. Foged [2] pointed out that every Lašnev space has a σ-hereditarily closure preserving and point countable k-network. Hence Y has a point countable, σ-closure preserving k-network.

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